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DISTINGUISHING NUMBER (INDEX) AND DOMINATION NUMBER OF A GRAPH

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ABSTRACT. The distinguishing number (index) of a graph G is the least integer d such that G has a vertex labeling (edge labeling) with d labels that is preserved only by the trivial automorphism. A set S of vertices in G is a dominating set of G if every vertex of $V(G) \setminus S$ is adjacent to some vertex in S. The minimum cardinality of a dominating set of G is the domination number of G. In this paper, we obtain some upper bounds for the distinguishing number and the distinguishing index of a graph based on its domination number.

1. Introduction and definitions

Domination in graphs is very well studied in the graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [4, 5]. For notation and graph theory terminology, we in general follow [5]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is the set $N(v) = \{u \in V | \ uv \in E\}$ and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. For a set S of vertices, the open neighborhood of S is defined by $N(S) = \bigcup_{v\in S} N(v)$, and the closed neighborhood of S by $N[S] = N(S) \cup S$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$ while the graph $G - S$ is the graph obtained from G by deleting the vertices in S and all edges incident with S. We denote the minimum and maximum degrees among the vertices of G by $\delta(G)$ and $\Delta(G)$, respectively, or simply by δ and Δ if the graph G is clear from the context. The *girth*, $g(G)$, of a graph G is the length of a shortest cycle in G. As usual, we denote the

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complement of graph G by $\overline{G}$. A set S of vertices in a graph G is a *dominating set* of G if every vertex of $V(G) \setminus S$ is adjacent to some vertex in S, that is, $N[S] = V$. The minimum cardinality of a dominating set of G is the *domination number* of G and denoted by $\gamma(G)$ or simply γ. A γ-set of G is a set S that is a dominating set with cardinality γ. Here, we state some known results on the domination number, which are needed in the next section:

Theorem 1.1 ([5]). (i) *For any graph G of order n,* $\left\lceil \frac{n}{1+\Delta} \right\rceil \leq \gamma(G) \leq n-\Delta$.

(ii) *If a graph G has $\delta(G) \geq 2$ and $g(G) \geq 7$, then $\gamma(G) \geq \Delta$.*

(iii) *If a graph G is disconnected, then $\gamma(\overline{G}) \leq 2$.*

A labeling of a simple graph G, $\phi : V \to \{1, 2, \dots, r\}$, is said to be r-*distinguishing*, if no nontrivial automorphism of G preserves all of the vertex labels. The point of the labels on the vertices is to destroy the symmetries of the graph, that is, to make the automorphism group of the labeled graph trivial. Formally, ϕ is r-distinguishing if for every nonidentity $\sigma \in \mathrm{Aut}(G)$, there exists x in V such that $\phi(x) \neq \phi(\sigma(x))$. The *distinguishing number* of a graph G is defined by

$$D(G) = \min\{r | \ G \text{ has a labeling that is } r\text{-distinguishing}\}.$$

This number was defined in [1]. Similar to this definition, the *distinguishing index* $D'(G)$ of G, defined in [6], is the least integer d such that G has an edge coloring with d colors that is preserved only by the trivial automorphism. Obviously, this invariant is not defined for graphs having K_2 as a connected components. If a graph has no nontrivial automorphisms, its distinguishing number is 1. In other words, $D(G) = 1$ for the asymmetric graphs. The other extreme, $D(G) = |V(G)|$, occurs if and only if G is a complete graph. The distinguishing index of some examples of graphs was exhibited in [6]. For instance, $D(P_n) = D'(P_n) = 2$ for every $n \geq 3$, $D(C_n) = D'(C_n) = 3$ for $n = 3, 4, 5$, and $D(C_n) = D'(C_n) = 2$ for $n \geq 6$. It is easy to see that the value $|D(G) - D'(G)|$ can be large. For example, $D'(K_{p,p}) = 2$ and $D(K_{p,p}) = p + 1$, for $p \geq 4$. A graph and its complement, always have the same automorphism group, while their graph structure usually differs, hence $D(G) = D(\overline{G})$ for every simple graph G. In what follows, we need the following results.

Theorem 1.2 ([3, 7]). *If G is a connected graph with maximum degree Δ, then $D(G) \leq \Delta + 1$. Furthermore, the equality holds if and only if G is K_n, $K_{n,n}$, C_3, C_4, or C_5.*

Theorem 1.3 ([6]). *If G is a connected graph of order $n \geq 3$, then $D'(G) \leq \Delta(G)$, unless G is C_3, C_4, or C_5.*

Theorem 1.4 ([8]). *Let G be a connected graph that is neither a symmetric nor an asymmetric tree. If $\Delta(G) \geq 3$, then $D'(G) \leq \Delta(G) - 1$ unless G is K_4 or $K_{3,3}$.*

In the next section, we investigate the relationship between the distinguishing number (index) and the domination number of a graph G. For any two natural

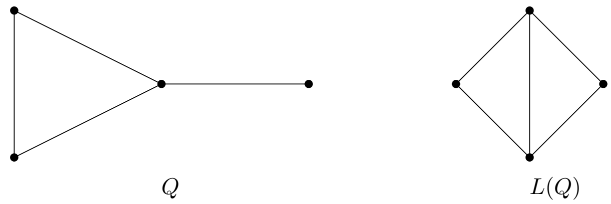

FIGURE 1. Graphs Q and $L(Q)$ of Theorem 2.3.

numbers γ and d, we present a connected graph G such that $\gamma(G) = \gamma$ and $D(G) = d$. In Section 3, we propose a problem and state a result about graphs G with $D'(G) \leq \gamma(G)$.

2. $\gamma(G)$ VERSUS $D(G)$ AND $D'(G)$

We begin this section by an observation, which is an immediate consequence of Theorems 1.1 and 1.2.

Observation 2.1. Let G be a simple graph of order n. We have

(i) $D(G) - \gamma(G) \leq \Delta + 1 - \left\lceil \dfrac{n}{1+\Delta} \right\rceil$.
(ii) $D(G) \leq n - \gamma(G)$, except for the complete graphs and C_4.
(iii) If $\delta(G) \geq 2$ and $g(G) \geq 7$, then $D(G) \leq \gamma(G)$.
(iv) If the graph G is a connected graph such that $\overline{G}$ is disconnected, then $\gamma(G) \leq D(G) + 1$.
(v) If the graph G is a connected graph of order n, then $\dfrac{1}{n-\Delta} \leq \dfrac{D(G)}{\gamma(G)} \leq \dfrac{(\Delta+1)^2}{n}$.

Remark 2.2. By Theorems 1.3 and 1.4, we can see that the Observation 2.1 is true for the distinguishing index of G, $D'(G)$, except for Part (ii). In this case, the distinguishing index version of Part (ii) of Observation 2.1 is true for all connected graphs, except K_3 and C_4.

For a simple graph G, the line graph $L(G)$ is the graph whose vertices are edges of G and two edges $e, e' \in V(L(G)) = E(G)$ are adjacent if they share an endpoint in common, (see Figure 1). To study the distinguishing number and the distinguishing index of $L(G)$, we need more information about the automorphism group of $L(G)$. Let $\Gamma_G : \mathrm{Aut}(G) \to \mathrm{Aut}(L(G))$ be given by $(\Gamma_G\phi)(\{u, v\}) = \{\phi(u), \phi(v)\}$ for every $\{u, v\} \in E(G)$. In [9], Sabidussi proved the following theorem, which we need it later.

Theorem 2.3 ([9])**.** *Suppose that G is a connected graph that is not P_2, Q or $L(Q)$ (see* Figure 1*). Then* $\mathrm{Aut}(G) \cong \mathrm{Aut}(L(G))$.

Theorem 2.4. *Suppose that G is a connected graph that is not P_2 and $L(Q)$* (Figure 1)*. Then $D(L(G)) = D'(G)$.*

Proof. If $G = Q$, then it is easy to see that $D'(Q) = D(L(Q)) = 2$. If $G \neq Q$, then first we show that $D(L(G)) \leq D'(G)$. Let $c : E(G) \to \{1, \ldots, D'(G)\}$ be an edge distinguishing labeling of G. We define $c' : V(L(G)) \to \{1, \ldots, D'(G)\}$ such that $c'(e) = c(e)$, where $e \in V(L(G)) = E(G)$. The vertex labeling c' is a distinguishing vertex labeling of $L(G)$, because if f is an automorphism of $L(G)$ preserving the labeling, then $c'(f(e)) = c'(e)$, and hence $c(f(e)) = c(e)$ for any $e \in E(G)$. On the other hand, by Theorem 2.3, $f = \Gamma_G \phi$ for some automorphism ϕ of G. Thus from $c(f(e)) = c(e)$ for any $e \in E(G)$, we can conclude that $c(\Gamma_G \phi(e)) = c(e)$ and so $c(\{\phi(u), \phi(v)\}) = c(\{u, v\})$ for every $\{u, v\} \in E(G)$. This means that ϕ is an automorphism of G preserving the labeling c, and so ϕ is the identity automorphism of G. Therefore f is the identity automorphism of $L(G)$, and hence $D(L(G)) \leq D'(G)$. For the converse, suppose that $c : V(L(G)) \to \{1, \ldots, D(L(G))\}$ is a vertex distinguishing labeling of $L(G)$. We define $c' : E(G) \to \{1, \ldots, D(L(G))\}$ such that $c'(e) = c(e)$, where $e \in E(G)$. The edge labeling c' is a distinguishing edge labeling of G. Because if f is an automorphism of G preserving the labeling, then $c'(f(e)) = c'(e)$, and hence $c(f(e)) = c(e)$ for any $e \in E(G)$. Then, there exists an automorphism $\Gamma_G f$ of $L(G)$ such that $\Gamma_G f(\{u, v\}) = \{f(u), f(v)\}$ for every $\{u, v\} \in E(G)$, by Theorem 2.3. Thus from $c(f(e)) = c(e)$ for any $e \in E(G)$, we can conclude that $c(\{u, v\}) = c(\{f(u), f(v)\}) = c(\Gamma_G f(\{u, v\}))$ for every $\{u, v\} \in E(G)$, which means that $\Gamma_G f$ preserves the distinguishing vertex labeling of $L(G)$, and hence $\Gamma_G f$ is the identity automorphism of $L(G)$. Therefore f is the identity automorphism of G, and so $D'(G) \leq D(L(G))$. □

By Observation 2.1(ii) and Theorem 2.4, we can obtain a new upper bound for the distinguishing index of a graph, using the domination number of its line graph.

Corollary 2.5. *Suppose that G is a connected graph of order at least three and size m. Then $D'(G) \leq m - \gamma(L(G))$, except for star graphs, K_3, and C_4.*

Now, we state and prove one of the main results of this paper.

Theorem 2.6. (i) *For any two natural numbers γ and d, there exists a connected graph G such that $\gamma(G) = \gamma$ and $D(G) = d$.*

(ii) *For any two natural numbers γ and d, there exists a connected graph G such that $\gamma(G) = \gamma$ and $D'(G) = d$.*

Proof. (i) We consider the following cases:

- If $2 \leq \gamma \leq d$, then we consider the graph G as a tree with central vertex x of degree γ, and $N(x) = \{x_1, \ldots, x_\gamma\}$. Suppose that each vertex x_i is adjacent to d pendant vertices. Since $\gamma \leq d$, so $D(G) = d$. Also, it is clear that the set $\Gamma = \{x_1, \ldots, x_\gamma\}$ is the minimum dominating set of G.
- If $\gamma = 1$ and $d \geq 2$, then it is sufficient to consider the graph G as the star graph $K_{1,d}$. Hence, $D(G) = d$ and $\gamma(G) = 1$.
- If $\gamma \geq 2$ and $d = 1$, then we consider the graph G as the asymmetric graph shown in Figure 2(a).

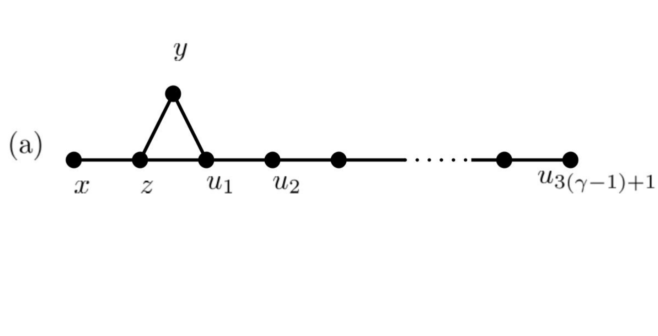

FIGURE 2. Graphs in Theorem 2.6.

- If $2 \leq d \leq \gamma$, then we consider the graph G as the star graph $K_{1,d}$ with a path of length $3\gamma - 3$ attached to the central vertex of $K_{1,d}$; (see Figure 2(b)).
- If $\gamma = d = 1$, then we consider the graph G as the join of an asymmetric graph H and K_1 such that the graph H has no vertex of degree $|V(H)|-1$. In this case, the graph G is an asymmetric graph with a vertex of degree $|V(G)| - 1$. Hence $D(G) = 1$ and $\gamma(G) = 1$.

The proof of Part (ii) is exactly the same as Part (i). □

The graph $G - v$ is a graph that is made by deleting the vertex v and all edges connected to v from the graph G. The following theorem examines the effects on $D(G)$ and $D'(G)$ when G is modified by deleting a vertex of G.

Theorem 2.7 ([2]). *Let G be a connected graph of order $n \geq 3$ and $v \in V(G)$. Then*

(i) $D(G) - 1 \leq D(G - v) \leq 2D(G)$.
(ii) $D'(G) - 1 \leq D'(G - v) \leq 2D'(G)$.

By Theorem 2.7, we can obtain the two following bounds for the distinguishing number and the distinguishing index of a graph based on its domination number.

Corollary 2.8. *Let S be a γ-set of a connected graph G of order $n \geq 3$. Then*

(i) $D(G) \leq D(G - S) + \gamma(G)$.
(ii) *If the induced subgraph $G - S$ does not have K_2 as its connected component, then* $D'(G) \leq D'(G - S) + \gamma(G)$.

Now by Observation 2.1(ii) and Corollary 2.8, we can prove the following result.

Corollary 2.9. *Let S be a γ-set of a connected graph G of order $n \geq 3$.*

(i) *Except for the complete graphs and C_4, we have* $D(G) \leq \frac{1}{2}(n + D(G - S))$.
(ii) *If the induced subgraph $G - S$ does not have K_2 as its connected component, then* $D'(G) \leq \frac{1}{2}(n + D'(G - S))$, *except for* K_3.

3. Graphs G with $D'(G) \leq \gamma(G)$

As we have seen in Observation 2.1, the distinguishing index of a graph G can be more or less than its domination number. With this motivation, we propose the following problem.

Problem 3.1. (i) Characterize the connected graphs G with $D(G) = \gamma(G)$ and $D'(G) = \gamma(G)$.
(ii) Characterize the connected graphs G and H with $D'(G) \leq \gamma(G)$ and $\gamma(H) \leq D'(H)$.

In the following we try to solve this problem at least in certain cases. To state our result, we need the following theorem, which is easy to obtain.

Theorem 3.2 ([10])**.** *Let G be a simple graph that is not a forest and have the girth at least 5. Then the complement of G is Hamiltonian.*

By Observation 2.1(iv), we can suppose that both of G and $\overline{G}$ are connected graphs.

Observation 3.3. Let G and $\overline{G}$ be connected graphs.

(i) If $\gamma(G) \geq \Delta$, then $D'(G) \leq \gamma(G)$, except for graphs C_3, C_4, and C_5.
(ii) If $\gamma(G) < \Delta$ and $g \geq 5$, then $D'(\overline{G}) \leq 2$.
(iii) If $\gamma(G) < \Delta$ and $g = 4$, then $D'(\overline{G}) \leq 3$.
(iv) If $\gamma(G) < \Delta$, $g \leq 3$, and $\gamma(G) < \Delta \leq n - \Delta$, then $D'(\overline{G}) \leq 2$.
(v) If $\gamma(G) < \Delta$, $g = 3$ and $\delta = 1$, then $\gamma(\overline{G}) = 2$.
(vi) If G is a tree, then $D'(\overline{G}) \leq 3$.

Proof. (i) It follows from Theorem 1.3.
(ii) Since $\overline{G}$ is Hamiltonian, by Theorem 3.2, we have $D'(\overline{G}) \leq 2$.
(iii) In this case, the graph G is triangle free, and so $\overline{G}$ is claw-free. Therefore $D'(\overline{G}) \leq 3$.
(iv) We have $\delta(\overline{G}) \geq n/2$, and so $\overline{G}$ is Hamiltonian. Therefore $D'(\overline{G}) \leq 2$.
(v) In this case, $\Delta(\overline{G}) = n - 2$, and so $\gamma(\overline{G}) = 2$.
(vi) Since the graph G is triangle free, the graph $\overline{G}$ is claw-free, and so $D'(\overline{G}) \leq 3$.

□

We end this paper with the following remark.

Remark 3.4. Observation 3.3 is a partial answer to Problem 3.1. The exact characterization for this problem remains as an open problem. Note that in Observation 3.3, we have no result for the case $g = 3$ and $\delta \geq 2$. Therefore this case also remains open.

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