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STRONG RAINBOW COLORING OF UNICYCLIC GRAPHS

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Abstract. A path in an edge-colored graph is called a *rainbow path*, if no two edges of the path are colored the same. An edge-colored graph G, is *rainbow-connected* if any two vertices are connected by a rainbow path. A rainbow-connected graph is called strongly rainbow connected if for every two distinct vertices u and v of $V(G)$, there exists a rainbow path P from u to v that in the length of P is equal to $d(u, v)$. The notations $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ are the smallest number of colors that are needed in order to make G rainbow connected and strongly rainbow connected, respectively. In this paper, we find the exact value of $\mathrm{rc}(G)$, where G is a unicyclic graph. Moreover, we determine the upper and lower bounds for $\mathrm{src}(G)$, where G is a unicyclic graph, and we show that these bounds are sharp.

1. Introduction and preliminaries

Throughout this paper, all graphs are finite, undirected, and connected. We refer the reader to the book [1] for graph-theoretical notation and terminology not described here. A path P in an edge-colored graph G, where adjacent edges may be colored the same, is a *rainbow path* if no two edges of it are colored the same. An *edge-connected* graph G, whose adjacent edges may have the same color, is rainbow-connected if every two vertices are connected by a rainbow path. The *rainbow connection number* of a connected graph G, denoted $\mathrm{rc}(G)$, is the smallest number of colors that are needed in order to make G rainbow connected.

For any two vertices u and v of G, a rainbow u-v *geodesic* in G is a rainbow path from u to v by the length $d(u, v)$. A graph G is strongly rainbow connected if there exists a rainbow u-v geodesic for every pair of distinct vertices u and v

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in G. In this case, the edge coloring $c : E(G) \longrightarrow \{1, 2, \ldots, n\}, n \in \mathbb{N}$, is called a *strong rainbow coloring* of G. Similarly we define the strong rainbow connection number of a connected graph G, denoted by $\mathrm{src}(G)$, as the smallest number of colors that are needed in order to make G strong rainbow connected. A strong rainbow coloring of G using $\mathrm{src}(G)$ colors is called a *minimum strong rainbow coloring* of G.

Chartrand et al. [3] found that the numbers $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ are equal to the size of the graph G if and only if G is a tree. Therefore, this relevant question arises: What is the relation between $\mathrm{rc}(G)$, $\mathrm{src}(G)$, and the size of G, where G is not a tree? Li and Sun [7] showed that there is no graph G with $\mathrm{src}(G) = m-1$. Moreover, they characterized all graphs that satisfy $\mathrm{src}(G) = m - 2$ in which m is the size of graph.

Although the rainbow connection number is obtained for some specific graphs, it is proved that for a given graph G, deciding if $\mathrm{rc}(G) = 2$ is already NP-complete; see [6]. More generally, it was shown in [6] that for any fixed $k \geq 2$, deciding if $\mathrm{rc}(G) = k$ is NP-complete. Therefore giving the precise value of $\mathrm{rc}(G)$ and $\mathrm{src}(G)$ for a given arbitrary graph G is almost impossible. Nowadays, finding the upper bound for $\mathrm{rc}(G)$ and describing the relation between $\mathrm{rc}(G)$ and other parameters of graph, specially the k-connectivity [2, 4, 5], radius, bridge, minimum degree, and order of graph are investigated.
In this paper, we concentrate on determining $\mathrm{rc}(G_r)$ and $\mathrm{src}(G_r)$, where G_r is a *unicyclic* graph defined as follows.

Definition 1.1. Suppose that $T_1, T_2, \ldots, T_r$ are trees with the roots $w_1, \ldots, w_r$, respectively, and let C_n be a cycle with the vertices $v_1, v_2, \ldots, v_n$ $(n \geq r)$. We consider G_r as a unicyclic graph such that $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, n\}$ and w_j is identified to v_{i_j} for every $1 \leq j \leq r$. In addition, we define α_i as the number of leaves of T_i minus one. We show the sum of all $\alpha_i, 1 \leq i \leq r$, by α.

In Section 2, we find the exact value of G_1 and G_2. In Section 3, we obtain the sharp upper and lower bounds for $\mathrm{src}(G_r)$, where G_r has an odd cycle. Finally, in Section 4, we look at unicyclic graphs with even cycle, and we set the sharp upper and lower bounds for $\mathrm{src}(G_r)$, depending on the size of trees that are attached to the cycle C.

2. Rainbow connection number

For finding $\mathrm{rc}(G_1)$ and $\mathrm{rc}(G_2)$, first we need the following two lemmas, which show that if c is a rainbow coloring of G_r, then each color of trees is used on the cycle at most once and the maximum number of colors of T_i that we can use on the cycle, is α_i.

Lemma 2.1. *Let G_r be a unicyclic graph, and let $\{c_1, \ldots, c_t\}$ be the colors of a rainbow coloring of G_r. Then each color of $E(T_i)$, $1 \leq i \leq r$, is used at most twice on G_r.*

Proof. Let $e_1 = u_1v_1$ with color c_1 be an edge in $\mathrm{E}(T_j)$, and let $e_2 = u_2v_2$ and $e_3 = u_3v_3$ be another edges of G_r with color c_1. It is obvious that e_2 and e_3 are in the cycle of G_r. Now, delete the edges e_2 and e_3 from G_r (do not remove their vertices). Thus, we have $G_r \setminus \{e_2, e_3\} \simeq H_1 \cup H_2$, where H_1 and H_2 are connected and $w_j \in V(H_2)$. Without loss of generality, assume that $d(w_j, u_1) > d(w_j, v_1)$ on G_r. In this case, we do not have any rainbow path between u_1 and vertices of H_1, which is a contradiction. □

Remark 2.2. Let c be a rainbow coloring of G_r, and let $e \in E(T_i)$ be an edge that its color is used on the cycle. If P is the path $w_i - v_1 - \cdots - v_h$ such that $v_1, \ldots, v_h$ are in $V(T_i)$ and P includes the edge e, then none of the colors of $\mathrm{E}(P) \setminus \{e\}$ are used in the cycle.

Lemma 2.3. *The maximum number of colors of T_i, which can be used in the cycle of G_r, is equal to α_i $(1 \le i \le r)$.*

Proof. Let the edges of T_i be colored by c_j, $1 \le j \le m_i$, which corresponds to the edges e_j, respectively. Assume that the colors $c_{i_1}, \ldots, c_{i_t}$ are the maximum number of colors of T_i that are used in the cycle of G_r. It is enough to show that $t \le \alpha_i$.

If all edges $e_{i_1}, \ldots, e_{i_t}$ are leaves, then we have nothing to prove. Suppose that $e_{i_1} = u_1v_1$ is not a leaf of T_i and that $d(u_1, w_i) \le d(v_1, w_i)$. Now, remove the edge e_{i_1} from T_i, and so we have two trees T_{i_1} and T_{i_2} such that $w_i \in V(T_{i_2})$. Therefore, we can say that T_{i_1} is isomorphic to a path with root v_1. Otherwise, Remark 2.2 implies that T_{i_1} has at least two leaves like e_1 and e_2 that their colors are not in the set $\{c_{i_1}, \ldots, c_{i_t}\}$. Now, remove the color c_{i_1} from the set and add the colors c_1 and c_2 to it, which is a contradiction. Thus, each edge of e_{i_j}, $1 \le j \le t$, is either a leaf or corresponds to a leaf of tree, and we have $t \le \alpha_i$. □

Definition 2.4. Let k be a positive integer. Define the function H_k as follows:

$$H_k(s) = \begin{cases} 1, & k > s, \\ 0, & k \le s. \end{cases}$$

Theorem 2.5. *Let G_1 be a unicyclic graph. Then $\mathrm{rc}(G_1) = m_1 + \lceil \frac{k - \alpha_1}{2} \rceil H_k(\alpha_1)$, where k is the length of C.*

Proof. We know that m_1 colors are needed for coloring the tree. By Lemma 2.3, maximum colors of tree that we can use in the cycle, is α_1; on the other hand, each color of tree is used at most once in C. If $k \le \alpha_1$, then we can color the edges of cycle with the color of leaves and $\mathrm{rc}(G_1) = m_1$; otherwise if $k > \alpha_1$, then we can color α_1 edges of C. Thus, there are $k - \alpha_1$ edges without color. Since each color is used on G_1 at most twice, we need $\lceil \frac{k - \alpha_1}{2} \rceil$ new colors.

□

Remark 2.6. We must put the colors of leaves of T_1 in the cycle C (for example, an even cycle with length of n) in the last theorem by this method that we are going to illustrate. At first, we denominate the edges of cycle C, starting from w_1 in the cycle clockwise by e_i $(1 \le i \le \frac{n}{2})$, and the edges of C, starting anticlockwise from w_1 in the cycle C, are denoted by e'_i $(1 \le i \le \frac{n}{2})$. Set the priority order of putting the color of leaves in the color of edges in the cycle C, by finding the greater index of i $(1 \leq i \leq \frac{n}{2})$ in $\{e_i, e'_i\}$.

Theorem 2.7. *Let G_2 be a unicyclic graph, and let $\alpha_2 \geq \alpha_1$. Then*

$$\mathrm{rc}(G_2) = \min\{m_1 + m_2 + s + \lceil \frac{k-\alpha-2s}{2} \rceil H_k(\alpha+2s), m_1 + m_2 + \lceil \frac{k-\alpha_2}{2} \rceil H_k(\alpha_2)\},$$

where k is the length of cycle and s is the length of minimum path between w_1 and w_2.

Proof. For a rainbow coloring of G_2, we need at least $m_1 + m_2$ colors for $\mathrm{E}(T_1) \cup E(T_2)$. Now, consider the following cases:
Case1. There is at least one color of each tree in $\mathrm{E}(C)$. Since the maximum number of colors T_i, which we can use on C, is α_i $(i = 1, 2)$, we have at most α colors of T_i's in the cycle, by Lemma 2.3. Suppose that P_1 and P_2 are two paths between w_1 and w_2 in G_2 such that $s = l(P_1) \leq l(P_2)$. From the fact that, each tree has at least one color on the cycle, the colors of leaves, which are used in the cycle, have to be in P_1 or P_2. Since $l(P_1) \leq l(P_2)$, we assume that these colors are in P_2 (Note that if $l(P_2) < \alpha$, then we cannot use all colors of leaves). Therefore, we need s new colors for coloring the edges of P_1. It is not difficult to see that in this case we have $\mathrm{rc}(G) \leq m_1 + m_2 + s + \lceil \frac{k-\alpha-2s}{2} \rceil H_k(\alpha+2s)$. Now, we show that the inequality is sharp. Clearly, we need $m_1 + m_2 + s$ colors for the rainbow coloring of T_i's and P_1, $i = 1, 2$. First assume $k > \alpha + 2s$. Since each color of leaves or the edges P_1 is used twice, by Lemma 2.1, we need at least $\lceil \frac{k-\alpha-2s}{2} \rceil$ new colors for the edges P_2. Thus, $\mathrm{rc}(G) = m_1 + m_2 + s + \lceil \frac{k-\alpha-2s}{2} \rceil$. Second, assume that $k \leq \alpha + 2s$. In this case, we have $\mathrm{rc}(G) = m_1 + m_2 + s$.
Case 2. The colors of one tree are used in the cycle. Since $\alpha_1 \leq \alpha_2$, we use α_2 colors of tree T_2 on C. In this case, we have a unicyclic with one tree T_2. Thus, Theorem 2.5 implies that

$$\mathrm{rc}(G_2) = m_1 + m_2 + \frac{k-\alpha_2}{2} H_k(\alpha_2).$$

□

3. Unicyclic graph with odd cycle

Throughout this section, we assume that G_r is a connected unicyclic graph with r trees and odd cycle with order $2k+1$. Since we need $\sum_{i=1}^{r} m_i$ distinct colors for all edges of trees, we want to know how many colors of trees we can use on C and to find the strongly rainbow connection number.

Lemma 3.1. *If c is a strongly rainbow coloring of G_r, then at most one color of T_i is used on C.*

Proof. Suppose by contradiction that c_1 and c_2 are, respectively, two distinct colors of $e_1 = u_1v_1$ and $e_2 = u_2v_2$ of T_i, such that they are used on C. Without loss of generality, assume that $d(u_1, w_i) \leq d(v_1, w_i)$. It is not difficult to see that there is a unique edge $e = uv$ of C such that $d(u, w_i) = d(v, w_i) = k$. Since we use the colors c_1 and c_2 on C, there is a distinct edge from e on C like $e^{'} = u^{'}v^{'}$ with color c_1 or c_2. Without loss of generality, assume that $e^{'}$ is colored by c_1 and that $d(u_1, w_i) \leq d(v_1, w_i)$. From the fact that C has an odd length, we have either $d(u^{'}, w_i) < d(v^{'}, w_i)$ or $d(u^{'}, w_i) > d(v^{'}, w_i)$. If the former case happens, then there is no v_1-$v^{'}$ geodesic path. For the latter case, consider the vertices v_1 and $u^{'}$. □

From the proof of Lemma 3.1, if we use one color of $\mathrm{E}(T_i)$ in the cycle of G_r, then there is only one edge for that color, which is introduced above. Let the edges of C be colored by each specific color of T_i's for $1 \leq i \leq r$. Then we can extend C to a strongly rainbow connected graph, because, for every integers $i \neq j$, if $u \in V(T_i)$ and $v \in V(T_j)$, then there are no repetitive colors of trees on the shortest path between u and v. Thus, for an arbitrary graph G_r, to find $\mathrm{src}(G_r)$, it is enough to consider a cycle of length $2k+1$ with r different coloring edges that belong to colors of leaves originally, and to extend C by minimum new colors to a strongly rainbow connected graph.

Definition 3.2. Let H_r be a cycle of length $2k+1$ with r distinguished edge colors such that the induced subgraph on these colored edges forms a path of length $r(r > 0)$, and let $H_r^{'}$ be a cycle of length $2k+1$ with r different edge colors such that these colors decompose edges C_{2k+1} to four paths P_1, P_2, P_3, and P_4 such that the induced subgraph on colored edges is $P_1 \cup P_2$, $V(P_1) \cap V(P_2) = \emptyset$, and $|l(P_1) - l(P_2)| \leq 1$. Similarly the induced subgraph, which edges do not have any color, is $P_3 \cup P_4$, $V(P_3) \cap V(P_4) = \emptyset$, and $|l(P_3) - l(P_4)| \leq 1$.

Theorem 3.3. *Let H be the cycle C_{2k+1} with r distinct coloring edges $(1 \leq r \leq 2k+1)$. Then*

(i) $\mathrm{src}(H)$ *is maximum, if* $H \simeq H_r$;
(ii) $\mathrm{src}(H)$ *is minimum, if* $H \simeq H_r^{'}$.

Proof. First we find the maximum $\mathrm{src}(H)$. Since $\mathrm{src}(C_{2k+1}) = k+1$, then $\mathrm{src}(H) \leq k$. Suppose $r \leq k$. In this sense, H_r has a path with length k without any color; thus, we need k new colors for coloring the path. Thus, $\mathrm{src}(H_r) = k$ and the maximum case happens. If $r > k$, then H_r has a path with length $2k - r + 1$, which needs $2k - r + 1$ new colors for its own. In this case, the graph H_r is colored by $2k+1$ distinct colors, which is maximum.

Second, we show that $H_r^{'}$ needs minimum colors. If r is even, then $l(P_1) = l(P_2) = \frac{r}{2}$. Without loss of generality, assume that $l(P_3) = k - \frac{r}{2}$ and that $l(P_4) = k - \frac{r}{2} + 1$. Since $l(P_4) \leq k$, we need $k - \frac{r}{2} + 1$ new colors for $\mathrm{E}(P_4)$. Locate $k - \frac{r}{2} + 1$ colors on P_4, and put these colors for $\mathrm{E}(P_3)$ such that the result is a strongly rainbow coloring. Thus, the minimum $\mathrm{src}(H)$ happens and equals $\lceil \frac{2k-r+1}{2} \rceil$. If r is odd, then assume $l(P_1) = \frac{r-1}{2}$ and $l(P_2) = \frac{r+1}{2}$. Hence, $l(P_3) = l(P_4) = \frac{2k-r+1}{2}$, and similarly, $\mathrm{src}(H) = \frac{2k-r+1}{2}$. □

The following corollary is an immediate result of Theorem 3.3, which shows upper and lower bounds for a strongly rainbow connection of unicyclic graphs with odd cycle.

Corollary 3.4. *Let G_r be an arbitrary unicyclic graph with a cycle of length $2k+1$, and let $T_1, \ldots, T_r$ be r trees of it.*

- *If $r \leq k$, then*

$$\sum_{i=1}^{r} m_i + \lceil \frac{2k-r+1}{2} \rceil \leq \operatorname{src}(G_r) \leq \sum_{i=1}^{r} m_i + k. \tag{3.1}$$

- *If $k < r \leq 2k$, then*

$$\sum_{i=1}^{r} m_i + \lceil \frac{2k-r+1}{2} \rceil \leq \operatorname{src}(G_r) \leq \sum_{i=1}^{r} m_i + 2k - r + 1. \tag{3.2}$$

4. Unicyclic graph with even cycle

Throughout this section, we assume that G_r is a connected unicyclic graph with r trees that are attached to an even cycle C of order $2k$. Let T be the set of all trees in G_r, let $A = \{T_1, T_2, \ldots, T_z\}$ be the set of z trees from G_r that are not paths, and let B be the set of trees that are paths. Therefore $T = A \cup B$. In this paper, the variable z denotes the number of trees that are not paths, and $T_{z+1}, \ldots, T_r$ are paths.

Lemma 4.1. *Let c be a strongly rainbow coloring of G_r; then at most two colors of $E(T_i)$ are used on c.*

Proof. Since $l(C) = 2k$, for each $1 \leq i \leq r$, there is a vertex $u \in V(C)$ such that $d(u, w_i) = k$. Suppose that $N_i = \{uu', uu''\}$, where $u', u'' \in V(C)$. Similar to the argument of the proof of Lemma 3.1, the colors of $\mathrm{E}(T_i)$ have just two choices on C, which are uu' and uu''. If T_i is a path, then we can use just one color of $\mathrm{E}(T_i)$. Otherwise, we can use two colors of $\mathrm{E}(T_i)$. □

From the proof of Lemma 4.1, there are two fixed options on C for colors of each tree, but depending on the position of trees on the cycle, sometimes we can use at most one color of each tree on C.

Lemma 4.2. *Let r be a positive integer, and let $k \geq 3$. Then G_r cannot use maximum colors of $E(T_i)$ on C if and only if one of the following cases occurs:*
Case 1. *T_i is not a path and there is T_j $(j \neq i)$ such that $d(w_i, w_j) = k$.*
Case 2. *T_i is not a path and there is T_j $(j \neq i)$ such that T_j is not path and $d(w_i, w_j) = 1$.*

Proof. Let i be a fixed integer. Then $1 \leq d(w_i, w_j) \leq k$ for every $1 \leq j \leq r$ $(j \neq i)$. Now, Consider two cases:
Case 1. We have $2 \leq d(w_i, w_j) \leq k$ $(j \neq i)$ for every j. In this case, $N_i \cap N_j = \emptyset$. If $2 \leq d(w_i, w_j) \leq k-1$, then we can select maximum colors of $\mathrm{E}(T_i)$ for the edges of N_i. Moreover, if $d(w_i, w_j) = k$, then we can use just two colors of $\mathrm{E}(T_i) \cup \mathrm{E}(T_j)$ depending on the following cases:

(i) Both T_i and T_j are paths.
(ii) Both T_i and T_j are not paths.
(iii) T_i is not a path, but T_j is a path, or conversely.

For (i), we can use one color from each tree in C. Since there are two w_iw_j-paths with same length, two colors have to utilize in one of w_iw_j-paths to be a strongly rainbow coloring. If (ii) happens, then it should be used two colors of one tree or one color of each tree (The latter case is more efficient). The last item is similar to (ii).
Case 2. We have $d(w_i, w_j) = 1$ for some $j \neq i$, $1 \leq j \leq r$. Clearly, $|N_i \cap N_j| = 1$ and we can use maximum three colors of $\mathrm{E}(T_i) \cup \mathrm{E}(T_j)$. Thus, when both T_i and T_j are not paths, we miss one color of the four colors.

□

For achieving $\mathrm{src}(G_r)$, we need $\sum_{i=1}^{r} m_i$ colors for trees. Hence, it is important to find the position of trees, and then color the edges of C corresponding to each tree. The maximum number of colors of trees, which we can use on C, is denoted by $\lambda(G_r)$. In this case, finding $\mathrm{src}(G_r)$ is equivalent to considering a cycle of length $2k$ with $\lambda(G_r)$ colored edges corresponding to trees for $1 \leq \lambda(G_r) \leq 2r$.

Lemma 4.3. *Let $T = A$ and let $|A| = k$, in which k is an even positive integer and $l(C) = 2k$. Then $\lambda(G_r) \leq 2k - 1$.*

Proof. Since k is even, we cannot use all $T_1, \ldots, T_k$ on C such that $d(w_i, w_j) \neq 1, k$ $(1 \leq i, j \leq k)$. Define G_r^1 as follows: Assign $T_1, \ldots, T_{k-1}$ on C such that $d(w_i, w_j) \neq 1, k$. Now, put T_k on C such that $d(w_k, w_i) \neq k$. In this sense, $\lambda(G_r) = 2k - 1$. □

If $T = A$ and $|A| = k$, in which k is an even positive integer and $l(C) = 2k$, then we denote G_r by G'_k. In the following theorems, we find the maximum and minimum of $\lambda(G_r)$ for each G_r with r fixed trees, where G_r is not isomorphic to G'_k.

Lemma 4.4. *Suppose that $\mathcal{T}$ is the set of all unicyclic graphs G_r with r trees, in which G_r is not isomorphic to G'_k. Then $\max_{G_r \in \mathcal{T}}(\lambda(G_r)) = \min\{z + r, 2k\}$.*

Proof. It is straightforward to see that $\max_{G_r \in A}(\lambda(G_r)) = \min\{z + r, 2k\} \leq z + r$, by Lemma 4.2. We show that the sharp case happens. Define a unicyclic graph with r trees as follows: If $z < k$, then put z trees $T_1, T_2, \ldots, T_z$ such that $d(w_i, w_j) \neq 1, k$ for every $1 \leq i, j \leq m$. Since $z < k$, it is possible. If $e = uv \in E(C)$ is not colored by the first z trees, then there exist vertices $u', v' \in V(C)$ such that $d(u, u') = d(v, v') = k$ and at least one of u' and v' is distinct from $w_i, 1 \leq i \leq z$. Thus, put T_{z+1} on this vertex. By following this process, we put trees on C such that the maximum number of colors is used. If $z \geq k$ and k is odd, then we can put k trees alternatively on C such that $d(w_i, w_{i+1}) = 2$ and we use $2k$ colors on the cycle. If $z \geq k$ and k is even, then put $k - 1$ trees $T_1, \ldots, T_{k-1}$ on C such that $d(w_i, w_j) \neq 1, k$. Clearly, we can utilize $2k - 2$ colors of these trees on C. Since G_r is not isomorphic to G'_k, there exist T_k and T_{k+1}, yet. Similar to the above argument, there are two vertices on C that if we put the two trees there, then all edges are colored.

□

Lemma 4.5. *Let G_r be an arbitrary unicyclic graph with r trees. Then* $\min\{r+1, 2k\} \leq \lambda(G_r)$, *if $T = A$, and $|A|$ is an odd number. Otherwise,* $\min\{r, 2k\} \leq \lambda(G_r)$.

Proof. If $r = z$ and z is odd, then put z trees on C such that $d(w_i, w_{i+1}) = 1$, say G. In this case, $\lambda(G) = \min\{r+1, 2k\}$. Obviously, we cannot use less than $r+1$ colors, because by Lemma 4.2, each tree loses at most one color, and it happens if for each tree T_i, there is a tree T_j such that $d(w_i, w_j) = k$. Since z is odd, it does not occur. If z is even, then define the graph H by putting trees $T_1, T_2, \ldots, T_z$ on C such that $d(w_i, w_{\frac{z}{2}+i}) = k$ for $1 \leq i \leq \frac{z}{2}$. Now, assign $r-z$ paths $T_{z+1}, \ldots, T_r$ on vertices of C that are distinguished from $w_1, \ldots, w_z$. Thus, $\lambda(H) = \min\{r, 2k\}$. Since each tree does not lose all colors on C, this number is minimum. If $r \neq z$ and z is odd, then define the graph H as above by the difference that $d(w_i, w_{\frac{z+1}{2}+i}) = k$ for $1 \leq i \leq \frac{z+1}{2}$. □

The following theorem is similar to Theorem 3.3, and so we state it without proof.

Theorem 4.6. *Let H be cycle C_{2k} with r distinct coloring edges* $(1 \leq r \leq 2k)$. *Then*

(i) $\operatorname{src}(C_{2k})$ *is maximum, if* $H \simeq H_r$.
(ii) $\operatorname{src}(C_{2k})$ *is minimum, if* $H \simeq H_r'$.

Definition 4.7. Suppose that some edges of the graph G_r have colors. We call an edge of cycle C, where $l(C) = 2k$, as *independent edge*, if $e = uv$ does not have any color, and the edges u_1u_2, u_2v_2, and v_1v_2 have colors, where $d(u, v_2) = d(v, u_2) = k$. We denote an independent edge by IE. If every edge of path P is IE, then it is called *the IE path* or *IE* briefly.

Remark 4.8. Suppose that r edges of cycle C with length of $2k$ are colored by r different colors and that another $2k-r$ new colors are independent edges. In this case, we need $2k-r$ new colors for C to be a strongly rainbow coloring.

In the following theorems, we determine an upper bound for $\operatorname{src}(G_r)$, where r is a positive integer.

Theorem 4.9. *Let r be a positive integer, and let z be an even number. Then*

(i) *if $r \geq z \geq k$ and k is even number, then*
$$\operatorname{src}(G_r) \leq \sum_{i=1}^{r} m_i + 2k - r;$$
(ii) *if $r+2 \geq z \geq k$ and k is an odd number, then*
$$\operatorname{src}(G_r) \leq \sum_{i=1}^{r} m_i + 2k - r;$$
(iii) *otherwise,*
$$\operatorname{src}(G_r) \leq \sum_{i=1}^{r} m_i + \min\{k, 2k-r-1\}.$$

Proof. Let H be a graph such that the trees $T_1, \dots, T_r$ are located on C consecutively. In this case, $\mathrm{src}(H) = \sum_{i=1}^{r} m_i + 2k - r - 1$. It is obvious if $r < k$, then $\mathrm{src}(G_r) \leq \mathrm{src}(H)$. Suppose that $r \geq k$. We show that there exists a graph H such that $\mathrm{src}(H) = \sum_{i=1}^{r} m_i + 2k - r$. Theorem 4.2 and Remark 4.8 imply that H has to satisfy in the following conditions:

(i) For each $T_i \in A$, there is $T_j \in T$, $j \neq i$ such that $d(w_i, w_j) = k$.
(ii) After using the colors of trees on C, each uncolored edge is IE.

If we put r trees on C randomly and color the correspond edges of trees, then uncolored edges create a path P_i, $1 \leq i \leq t$, where t is minimum. Now we can say the following statements:

(1) If $l(P_i) = 1$, then the edge of P_i is not IE.
(2) If $l(P_i) = 2$, then $E(P_i)$ can be IE. Hence we have to put trees and their colorings like Figure 1.

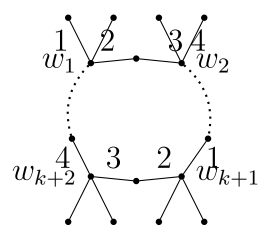

FIGURE 1. w_1w_2-path is IE.

(3) If $l(P_i) \geq 3$, then none of the edges $E(P_i)$ are IE.

Therefore, we have to locate r trees on C such that the number of P_i's with $l(P_i) = 2$ is maximum. Thus, we put trees like Figure 2.

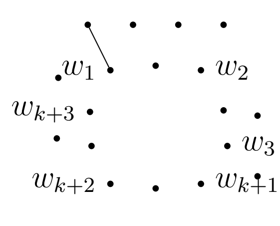

FIGURE 2. Graph H with $\mathrm{src}(H) = \sum_{i=1}^{r} m_i + 2k - r$

Since we are looking for the minimum colors from trees for G_r, we use two colors from one tree and no color from the opposite tree. Hence it creates $2k - r$ edges without any coloring consecutively and r edges with color consecutively. It is obvious that at most two edges of r colors could be IE. Since $r - z$ remaining trees are path, we put one path on C after w_{k+1} and one path on the opposite side on C before w_1. Therefore all of the coloring edges are IE. $\square$

By Lemma 4.5, we have the following theorem, which we omit its proof.

Theorem 4.10. *Let $T = A$, and let $|A|$ be an odd number. Then for the graph G_r, it follows that*

$$\mathrm{src}(G_r) \leq \sum_{i=1}^{r} m_i + \min\{k, 2k - r - 1\}.$$

Now we find a lower bound for G_r, where r is a positive number.

Theorem 4.11. *For a positive integer r,*

(i) *if $T = A$ and $|A|$ is an even number, then*

$$\sum_{i=1}^{r} m_i + \max\{0, k - z + 1\} \leq \mathrm{src}(G_r);$$

(ii) *if $T = A$ and $|A|$ is an odd number, then*

$$\sum_{i=1}^{r} m_i + \max\{0, k - z\} \leq \mathrm{src}(G_r);$$

(iii) *if $T \neq A$, then*

$$\sum_{i=1}^{r} m_i + \max\{0, k + \lceil\frac{-r - z}{2}\rceil\} \leq \mathrm{src}(G_r).$$

Proof. First assume that $r = z = k$ and that k is even. Now, define G_r^1 as the proof of Lemma 4.3. Then we need $\sum_{i=1}^{r} m_i$ colors for edges of trees and one new color for the cycle C. Thus, $\mathrm{src}(G_r^1) = \sum_{i=1}^{r} m_i + 1$. If $z > k$, then we can color the edges of C by colors of trees. If $z < k$, then we can use $2z$ colors of trees on C, by Lemma 4.4. Since we cannot construct the graph H_r' and all trees have at least two leaves, we have to define G_r as follows: Put $T_1, \ldots, T_z$ on C such that $d(w_i, w_{\frac{z}{2}+i}) = k - 1$, $d(w_2, w_3) = k - 1$, and $d(w_j, w_{j+1}) = 2$ for $1 \leq i \leq \frac{z}{2}, 1 \leq j \leq z, j \neq \dfrac{z}{2}$. In this case, we create a cycle with four disjoint paths P_1, P_2, P_3, and P_4 such that P_1 and P_2 are colored by tree's colors and edges of P_3 and P_4 are uncolored. Clearly $|l(P_1) - l(P_2)| = 2$, and we need $\lceil\frac{2k-2z+2}{2}\rceil$ new colors for the edges C. It is not difficult to see that there is no graph like G_r such that $\mathrm{src}(G_r) = k - z$.

For the second case, if $z \geq k$, then we can color the edges of C by trees. If $z < k$, then assign $T_1, \ldots, T_{z-1}$ as above and put T_z on C such that $d(w_z, w_{\frac{z-1}{2}}) = 2$. Hence, in this case, the strongly rainbow connection number is $\lceil\frac{2k-2z}{2}\rceil$, which is minimum.

For the last case, first put $T_1, T_2, \ldots, T_z$ like above. If z is an odd number, then put T_{z+1} on C such that $d(w_{z+1}, w_z) = 1$ and $d(w_{z+1}, w_{z-1}) = k - 2$. Put T_{z+2} on C such that $d(w_{z+2}, w_{z+1}) = k$. Put $T_{z+3}, T_{z+5}, \ldots$ consecutively on C after T_{z+1} and similarly $T_{z+4}, T_{z+6}, \ldots$ after T_{z+2}. If z is even, then put $T_1, T_2, \ldots, T_z$ as above. Now, put T_{z+1} and T_{z+2} continuously on C such that $d(w_{z+1}, w_{\frac{z}{2}}) = 2$

and $d(w_{z+2}, w_{\frac{z}{2}}) = 3$. Thus we have two disjoint uncolored paths on the cycle C, which have the equal length.

For other trees, put $T_{z+3}, T_{z+5}, \ldots$ consecutively after T_{z+2} and $T_{z+4}, T_{z+6}, \ldots$ such that $d(w_{2i-1}, w_{2i}) = k$ for each $\frac{z}{2} + 2 \leq i$. It is not difficult to see that we need $\max\{0, \lceil \frac{2k-r-z}{2} \rceil\}$ new colors for the uncolored edges of C, whether z is even or odd.

□

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