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STABILITY RESULT OF THE BRESSE SYSTEM WITH DELAY AND BOUNDARY FEEDBACK

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Abstract. Our interest in this paper is to analyze the asymptotic behavior of a Bresse system together with three boundary controls, with delay terms in the first, second, and third equations. By using the semigroup method, we prove the global well-posedness of solutions. Assuming the weights of the delay are small, we establish the exponential decay of energy to the system by using an appropriate Lyapunov functional.

1. Introduction

Let $0 < T \leq \infty$ and let $L > 0$. We denote by $\varphi = \varphi(x,t) : (0,L) \times (0,T) \longrightarrow \mathbb{R}$, $\psi = \psi(x,t) : (0,L) \times (0,T) \longrightarrow \mathbb{R}$, and $\omega = \omega(x,t) : (0,L) \times (0,T) \longrightarrow \mathbb{R}$, the longitudinal, vertical, and shear angle displacements of the cross section at $x \in (0,L)$ and at time $t \in (0,t)$, respectively. The original Bresse system is given by the following equations (see [4]):

$$\begin{cases} \rho_1\varphi_{tt} &= Q_x + lN + F_1, \\ \rho_2\psi_{tt} &= M_x - Q + F_2, \\ \rho_1\omega_{tt} &= N_x - lQ + F_3, \end{cases}$$

where we use N, Q, and M to denote the axial force, the shear force, and the bending moment, respectively. These forces are stress-strain relations for elastic behavior and given by

$$N = K_0(\omega_x - l\varphi), \quad Q = K(\varphi_x + \psi + l\omega), \quad \text{and } M = b\psi_x,$$

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where K, K_0, and b are positive constants. Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $K_0 = EA$, $K = K'GA$, $b = EI$, and $l = R^{-1}$. Coefficients aforementioned, all assumed positive, represent as follows:

- ρ the density,
- E the modulus of elasticity,
- G the shear modulus,
- K' the shear factor,
- A the cross-sectional area,
- I the second moment of area of the cross section,
- R the radius of curvature,
- l the curvature $l = 1/R$.

Finally, by the terms F_i, we denote external forces. Therefore, the evolutive problem can be written as

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi + l\omega)_x - K_0 l(\omega_x - l\varphi) = 0 & \text{in } (0,L)\times(0,T), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi + l\omega) = 0 & \text{in } (0,L)\times(0,T), \\ \rho_1\omega_{tt} - K_0(\omega_x - l\varphi)_x + Kl(\varphi_x + \psi + l\omega) = 0 & \text{in } (0,L)\times(0,T), \end{cases} \tag{1.1}$$

when the external forces are null.
It is well known that system (1.1) for $l = 0$ is the standard Timoshenko system when $\omega = 0$:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) = 0. \end{cases}$$

Till now, there are so many works concerning the Timoshenko system in the literature, most of those results recover the global well-posedness, stability, and long-time dynamics by adding some kinds of damping. Generally speaking, if we add the damping term in one of the equations, then the system decays exponentially under the so-called equal wave speeds assumption:

$$\frac{\rho_1}{\rho 2} = \frac{K}{b}.$$

Indeed, if the damping terms are added in the two equations, the system is exponentially stable without the "equal wave speeds" assumption. See, for example, the literature [1–9, 13–15].

In this paper, we investigate the well-posedness and the boundary stabilization of the following linear Bresse system in bounded interval $(0, L)$.

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi + l\omega)_x - K_0 l(\omega_x - l\varphi) + a_1\varphi_t(x, t-\tau) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi + l\omega) + a_2\psi_t(x, t-\tau) = 0, \\ \rho_1\omega_{tt} - K_0(\omega_x - l\varphi)_x + Kl(\varphi_x + \psi + l\omega) + a_3\omega_t(x, t-\tau) = 0. \end{cases} \tag{1.2}$$

System (1.2) is subjected to the following boundary conditions:

$$\begin{cases} K(\varphi_x + \psi + l\omega)(L,t) = -\alpha\varphi_t(L,t), \\ b\psi_x(L,t) = -\mu\psi_t(L,t), \\ K_0(\omega_x - l\varphi)(L,t) = -\gamma\omega_t(L,t), \\ \varphi(0,t) = \psi(0,t) = \omega(0,t) = 0, \end{cases} \tag{1.3}$$

where $(x,t) \in (0,L)\times(0,+\infty)$, $L > 0$, and the parameters $a_1, a_2, a_3, \alpha, \mu$, and γ are positive constants. The system is completed with the following initial conditions:

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \\ \psi_t(x,0) = \psi_1(x), \quad \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \quad x \in (0,L), \\ \varphi_t(x,t-\tau) = f_1(x,t-\tau) \quad \text{in } (0,L)\times[0,\tau], \\ \psi_t(x,t-\tau) = f_2(x,t-\tau) \quad \text{in } (0,L)\times[0,\tau], \\ \omega_t(x,t-\tau) = f_3(x,t-\tau) \quad \text{in } (0,L)\times[0,\tau], \end{cases} \tag{1.4}$$

where $\tau > 0$ is the time delay. The initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, f_1, f_2, f_3)$ belong to a suitable Sobolev space. By ω, ψ, and φ we denote the longitudinal, vertical, and shear angle displacements.

In recent years, one very active area of mathematical control theory has been the investigation of the delay effect in the stabilization of hyperbolic systems, and many authors have shown that an arbitrary small delay may destabilize a system, which is asymptotically stable in the absence of delay (see [7] and [6, Example 3.5]).

The delay effects often appear in many practical problems, for instance, chemical, physical, thermal, and economic phenomena, and may turn a well-behaved system into a wild one. The time delay term can be regarded as a source of instability. If the coefficient of delay is very small, the system may stabilize when additional control terms have been added. We first recall two classes of second-order evolution equations with time delay. Nicaise and Pignotti [12] studied abstract evolution equations with constant time delay of the following form:

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + F(U(t)) + K\mathcal{B}U(t-\tau), \\ U(0) = U_0, \qquad \mathcal{B}U(t-\tau) = f(t), \end{cases}$$

where $\mathcal{B}$ is a bounded operator. They proved that the operator associated to the part without delay generates a strongly continuous semigroup, which is exponentially stable. In addition, under a smallness condition on the time delay feedback, they obtained that the model with delay is also exponentially stable. Nicaise and Pignotti [11] considered the following second-order evolution equations with time delay:

$$\begin{cases} u_{tt} + Au + B_1B_1^*u_t(t) + B_2B_2^*u_t(t-\tau) = 0, \quad t > 0, \\ u(0) = u_0, \qquad u_t(0) = u_1, \\ B_2^*u_t(t) = f^0(t), \quad t \in (-\tau, 0), \end{cases}$$

where the bounded operator B_2 is the delay feedback operator. For a system that is exponentially stable in absence of time delay, that is, for $B_2 = 0$, they proved that the exponential stability is preserved if $\|B_2^*\|$ is sufficiently small.

Recently, Ammari, Nicaise, and Pignotti (see [2]) treated the N-dimensional wave equation

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a u_t(x,t-\tau) = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ \frac{\partial u}{\partial \nu} = -K u_t(x,t), & x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t) = g(x,t), & x \in \Omega,\ t \in (-\tau, 0), \end{cases} \tag{1.5}$$

where Ω is an open bounded domain of $\mathbb{R}$, $N \geq 2$ with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. Under the usual geometric condition on the domain Ω, they showed an exponential stability result, provided that the delay coefficient a is sufficiently small. When both the damping and the delay in (1.5) are acting in the boundary, that is, if in (1.5), the third equation is replaced by

$$\frac{\partial u}{\partial \nu} = -Ku_t(x,t) - au_t(x,t-\tau), \qquad x \in \Gamma_1,\ t > 0, \tag{1.6}$$

Nicaise and Pignotti [10] investigated this problem and showed an exponential decay rate of the total energy under the assumption

$$a < k. \tag{1.7}$$

On the contrary, if (1.7) does not hold, then they found a sequence of delays for which the corresponding solution of (1.5) will be unstable. The analysis in [10] is based on an observability inequality obtained with a Carleman estimate.

We also would like to mention the contribution of Said-Houari and Soufyane [14] in which the authors studied a Timoshenko system with delay and boundary feedback. The authors proved the global well-posedness and exponential decay of energy by assuming the weights of the delay are small enough. For more results, concerning Timoshenko system with delay, one can refer to the previous studies [3–9] and so on.

Comparing our result with the work of Feng [8], for laminated Timoshenko beams with time delays and boundary feedbacks, he proved the global well-posedness and exponential decay of energy by assuming the weights of the delay are small enough.

The main objectives of the present work are to establish the global well-posedness and exponential stability of problem (1.2)–(1.4).

Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to problem (1.2)–(1.4) for linear damping and delay terms. To obtain global solutions to problem (1.2)–(1.4), we use the argument combining the semigroup theory (see [10] and [5]) with the energy estimate method. To prove decay estimates, we use a multiplier method.

2. Well-posedness of the problem

In this section, we prove the global existence and the uniqueness of the solution of system (1.2)–(1.4). For this purpose, we adopt the technique of [10] (see also [13]) to prove that the operator $\mathcal{A}$ defined in (2.3) generates a contraction semigroup on the Hilbert space $\mathcal{H}$ given by (2.4).

Let us introduce the following new variables:

$$\begin{aligned} z_1(x,\rho,t) &= \varphi_t(x,t-\tau\rho), \quad x \in (0,L),\ \rho \in (0,1), t > 0,\\ z_2(x,\rho,t) &= \psi_t(x,t-\tau\rho), \quad x \in (0,L),\ \rho \in (0,1), t > 0,\\ z_3(x,\rho,t) &= \omega_t(x,t-\tau\rho), \quad x \in (0,L),\ \rho \in (0,1), t > 0. \end{aligned}$$

Then, it is easy to check that

$$\tau z_{it}(x,\rho,t) + z_{i\rho}(x,\rho,t) = 0 \quad \text{in } (0,L)\times(0,1)\times(0,+\infty) \quad \text{for } i = 1,2,3.$$

Therefore, our problem (1.2)–(1.4) is equivalent to

$$\begin{cases}\rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi+l\omega)_x(x,t) - K_0 l(\omega_x - l\varphi)(x,t) + a_1 z_1(x,1,t) = 0,\\ \tau_1 z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) = 0,\\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi+l\omega)(x,t) + a_2 z_2(x,1,t) = 0,\\ \tau_2 z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) = 0,\\ \rho_1\omega_{tt}(x,t) - K_0(\omega_x - l\varphi)_x(x,t) + Kl(\varphi_x+\psi+l\omega)(x,t) + a_3 z_3(x,1,t) = 0,\\ \tau_3 z_{3t}(x,\rho,t) + z_{3\rho}(x,\rho,t) = 0.\end{cases} \tag{2.1}$$

Now, we present a short discussion of the well-posedness and semigroup formulation of the initial boundary value problem (2.1), (1.3), and (1.4). For this purpose, let $U = (\varphi,\varphi_t,z_1,\psi,\psi_t,z_2,\omega,\omega_t,z_3)^T$. Then U satisfies the problem

$$\begin{cases}U' = \mathcal{A}U,\\ U(0) = (\varphi_0,\varphi_1,f_1(.,-.\tau),\psi_0,\psi_1,f_2(.,-.\tau),\omega_0,\omega_1,f_3(.,-.\tau))^T,\end{cases} \tag{2.2}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix}\varphi\\ u\\ z_1\\ \psi\\ v\\ z_2\\ \omega\\ \tilde\omega\\ z_3\end{pmatrix} = \begin{pmatrix} u\\ \dfrac{K}{\rho_1}(\varphi_x+\psi+l\omega)_x + \dfrac{lK_0}{\rho_1}(\omega_x - l\varphi) - \dfrac{a_2}{\rho_1}z_1(.,1)\\ -\dfrac{1}{\tau}z_{1\rho}\\ v\\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{K}{\rho_2}(\varphi_x+\psi+l\omega) - \dfrac{a_2}{\rho_2}z_2(.,1)\\ -\dfrac{1}{\tau}z_{2\rho}\\ \tilde\omega\\ \dfrac{K_0}{\rho_1}(\omega_x - l\varphi)_x - \dfrac{lK}{\rho_1}(\varphi_x+\psi+l\omega) - \dfrac{a_3}{\rho_1}z_3(.,1)\\ -\dfrac{1}{\tau}z_{3\rho}\end{pmatrix} \tag{2.3}$$

with domain

$$D(\mathcal{A}) = \left\{\begin{array}{l}(\varphi,u,z_1,\psi,v,z_2,\omega,\tilde\omega,z_3)^T \in H,\\ u = z_1(.,0), v = z_2(.,0), \tilde\omega = z_3(.,0), \text{ in } (0,L),\\ K(\varphi_x+\psi+l\omega)(L) = -\alpha u(L),\ b\psi_x(L) = -\mu v(L),\\ K_0(\omega_x - l\varphi)(L) = -\gamma\tilde\omega(L)\end{array}\right\},$$

where

$$H = (H^2(0,L)\cap H^1_*(0,L)\times L^2(0,1,H^1(0,L)))^3$$

and

$$H^1_*(0,L) = \{f\in H^1(0,L) : f(0) = 0\}.$$

Now, the energy space $\mathcal{H}$ is defined as follows:

$$\mathcal{H} := (H^1_*(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1)))^3. \tag{2.4}$$

For $U=(\varphi,u,z_1,\psi,v,z_2,\omega,\tilde{\omega},z_3)^T,\overline{U}=(\overline{\varphi},\overline{u},\overline{z}_1,\overline{\psi},\overline{v},\overline{z}_2,\overline{\omega},\overline{\tilde{\omega}},\overline{z}_3)^T$ and for positive constants ξ_i, we define the inner product in $\mathcal{H}$ as follows:

$$\begin{aligned}\langle U,\overline{U}\rangle_{\mathcal{H}}= & \int_0^L\Big(\rho_1 u\overline{u}+\rho_2 v\overline{v}+\rho_1\tilde{\omega}\overline{\tilde{\omega}}+b\psi_x\overline{\psi}_x+K(\varphi_x+\psi+l\omega)(\overline{\varphi}_x+\overline{\psi}+l\overline{\omega})\\ & +K_0(\omega_x-l\varphi)(\overline{\omega}_x-l\overline{\varphi})+\sum_{i=1}^{3}\xi_i\int_0^1 z_i(x,\rho)\overline{z}_i(x,\rho)\,d\rho\Big)\,dx.\end{aligned}$$

The existence and uniqueness result reads as follows.

Theorem 2.1. *For any $U_0\in\mathcal{H}$, there exists a unique solution $U(x,t)\in C([0,+\infty),\mathcal{H})$ of problem (2.2). Moreover, if $U_0\in D(\mathcal{A})$, then*

$$U\in C([0,+\infty);D(\mathcal{A}))\cap C^1([0,+\infty);\mathcal{H}).$$

Proof. In order to prove the result stated in Theorem 2.1, we will use the semigroup approach. That is, we will show that the operator $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{H}$. In this step, we concern ourselves to prove that the operator $\mathcal{A}$ is dissipative. Indeed, for $U=(\varphi,u,z_1,\psi,v,z_2,\omega,\tilde{\omega},z_3)^T$, we have

$$\begin{aligned}\langle\mathcal{A}U,U\rangle_{\mathcal{H}}= & -\alpha u^2(L)-\mu v^2(L)-\gamma\tilde{\omega}^2(L)-a_1\int_0^L z_1(x,1)u\,dx\\ & -a_2\int_0^L z_2(x,1)v\,dx-a_3\int_0^L z_3(x,1)\tilde{\omega}\,dxdx\\ & -\sum_{i=1}^{3}\frac{\xi_i}{\tau}\int_0^L\int_0^1 z_i(x,\rho)z_{i\rho}(x,\rho)\,d\rho\,dx.\end{aligned}\tag{2.5}$$

Looking now at the last two terms of the right-hand side of (2.5), we have

$$\begin{aligned}\sum_{i=1}^{3}\xi_i\int_0^L\int_0^1 z_i(x,\rho)z_{i\rho}(x,\rho)\,d\rho\,dx= & \sum_{i=1}^{3}\xi_i\int_0^L\int_0^1\frac{1}{2}\frac{\partial}{\partial\rho}z_i^2(x,\rho)\,d\rho\,dx\\ = & \sum_{i=1}^{3}\frac{\xi_i}{2}\int_0^L\{z_i^2(x,1)-z_i^2(x,0)\}\,dx.\end{aligned}\tag{2.6}$$

Consequently, (2.6) becomes

$$\begin{aligned}\langle\mathcal{A}U,U\rangle_{\mathcal{H}}= & -\alpha u^2(L)-\mu v^2(L)-\gamma\tilde{\omega}^2(L)-a_1\int_0^L z_1(x,1)u\,dx\\ & -a_2\int_0^L z_2(x,1)v\,dx-a_3\int_0^L z_3(x,1)\tilde{\omega}\,dxdx\\ & -\sum_{i=1}^{3}\frac{\xi_i}{2\tau}\int_0^L\{z_i^2(x,1)-z_i^2(x,0)\}\,dx.\end{aligned}\tag{2.7}$$

By using Young's inequality, we obtain from (2.7), that

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq& -\alpha u^2(L) - \mu v^2(L) - \gamma\tilde{\omega}^2(L) + (-\tfrac{\xi_1}{4\tau})\int_0^L z_1^2(x,1)\,dx\\
&+(\tfrac{a_1^2\tau}{\xi_1}+\tfrac{\xi_1}{2\tau})\textstyle\int_0^L u^2\,dx + (-\tfrac{\xi_2}{4\tau})\displaystyle\int_0^L z_2^2(x,1)\,dx\\
&+(\tfrac{a_2^2\tau}{\xi_2}+\tfrac{\xi_2}{2\tau})\textstyle\int_0^L v^2\,dx + (-\tfrac{\xi_3}{4\tau})\displaystyle\int_0^L z_3^2(x,1)\,dx + (\frac{a_3^2\tau}{\xi_3}+\frac{\xi_3}{2\tau})\int_0^L \tilde{\omega}^2\,dx\\
\leq& \max(\tfrac{1}{\rho_1}(\tfrac{a_1^2\tau}{\xi_1}+\tfrac{\xi_1}{2\tau}), \tfrac{1}{\rho_2}(\tfrac{a_2^2\tau}{\xi_2}+\tfrac{\xi_2}{2\tau}), \tfrac{1}{\rho_1}(\tfrac{a_3^2\tau}{\xi_3}+\tfrac{\xi_3}{2\tau}))\langle U, U\rangle_{\mathcal{H}}\\
=& c_1\langle U, U\rangle_{\mathcal{H}}.\end{aligned}$$

Consequently, the operator $\mathcal{A}-c_1I$ is dissipative. To show that $\mathcal{A}$ is maximal monotone, it is sufficient to show that the operator $\lambda I-\mathcal{A}$ is surjective for fixed $\lambda > 0$. Indeed, given $(h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9)^T \in \mathcal{H}$, we seek $U = (\varphi, u, z_1, \psi, v, z_2, \omega, \tilde{\omega}, z_3)^T \in D(\mathcal{A})$ solution of the following system of equations:

$$\begin{cases}\lambda\varphi - u = h_1,\\
\lambda u - \dfrac{K}{\rho_1}(\varphi_x+\psi+l\omega)_x - \dfrac{lK_0}{\rho_1}(\omega_x - l\varphi) + \dfrac{a_1}{\rho_1}z_1(.,1) = h_2,\\
\lambda z_1 + \dfrac{1}{\tau}z_{1\rho} = h_3,\\
\lambda\psi - v = h_4,\\
\lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x+\psi+l\omega) + \dfrac{a_2}{\rho_2}z_2(.,1) = h_5,\\
\lambda z_2 + \dfrac{1}{\tau}z_{2\rho} = h_6,\\
\lambda\omega - \tilde{\omega} = h_7,\\
\lambda\tilde{\omega} - \dfrac{K_0}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lK}{\rho_1}(\varphi_x+\psi+l\omega) + \dfrac{a_3}{\rho_1}z_3(.,1) = h_8,\\
\lambda z_3 + \dfrac{1}{\tau}z_{3\rho} = h_9.\end{cases} \tag{2.8}$$

Suppose that we have found (φ, ψ, ω) with the appropriate regularity. Then

$$\begin{cases}u = \lambda\varphi - h_1,\\ v = \lambda\psi - h_4,\\ \tilde{\omega} = \lambda\omega - h_7.\end{cases} \tag{2.9}$$

It is clear that $u \in H_*^1(0,L), v \in H_*^1(0,L)$, and $\omega \in H_*^1(0,L)$. Furthermore, by (2.8), we can find $z_i (i = 1, 2, 3)$ as

$$z_1(x,0) = u(x),\ z_2(x,0) = v(x),\ z_3(x,0) = \tilde{\omega}(x) \qquad \text{for } x \in (0,L).$$

Following the same approach as in [10], by using equations for z_i in (2.8), we obtain

$$\begin{aligned}z_1(x,\rho) &= u(x)e^{-\lambda\tau\rho} + \tau_1 e^{-\lambda\tau\rho}\int_0^\rho h_3(x,s)e^{\lambda\tau s}\,ds,\\
z_2(x,\rho) &= v(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho h_6(x,s)e^{\lambda\tau s}\,ds,\\
z_3(x,\rho) &= \tilde{\omega}(x)e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho h_9(x,s)e^{\lambda\tau s}\,ds.\end{aligned}$$

From (2.9), we obtain

$$\begin{cases} z_1(x,\rho) &=& \lambda\varphi(x)e^{-\lambda\tau\rho} - h_1 e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\displaystyle\int_0^\rho h_3(x,s)e^{\lambda\tau s}\,ds,\\ z_2(x,\rho) &=& \lambda\psi(x)e^{-\lambda\tau\rho} - h_4 e^{-\lambda\tau\rho} + \tau_2 e^{-\lambda\tau\rho}\displaystyle\int_0^\rho h_6(x,s)e^{\lambda\tau s}\,ds,\\ z_3(x,\rho) &=& \lambda\omega(x)e^{-\lambda\tau\rho} - h_7 e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\displaystyle\int_0^\rho h_9(x,s)e^{\lambda\tau s}\,ds.\end{cases}$$

By using (2.8) and (2.9), the functions φ, ψ, and ω satisfy the following system:

$$\begin{cases} \lambda^2\varphi - \dfrac{K}{\rho_1}(\varphi_x+\psi+l\omega)_x - \dfrac{lK_0}{\rho_1}(\omega_x - l\varphi) + \dfrac{a_1}{\rho_1}z_1(.,1) = h_2 + \lambda h_1,\\ \lambda^2\psi - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x+\psi+l\omega) + \dfrac{a_2}{\rho_2}z_2(.,1) = h_5 + \lambda h_4,\\ \lambda^2\omega - \dfrac{K_0}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lK}{\rho_1}(\varphi_x+\psi+l\omega) + \dfrac{a_3}{\rho_1}z_3(.,1) = h_8 + \lambda h_7.\end{cases} \tag{2.10}$$

Using the following

$$\begin{cases} z_1(x,1) &=& u(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\displaystyle\int_0^1 h_3(x,s)e^{\lambda\tau s}\,ds,\\ &=& \lambda\varphi e^{-\lambda\tau} + z_1^0(x)\\ z_2(x,1) &=& v(x)\tau e^{-\lambda\tau} + \tau e^{-\lambda\tau}\displaystyle\int_0^1 h_6(x,s)e^{\lambda\tau s}\,ds,\\ &=& \lambda\psi e^{-\lambda\tau} + z_2^0(x)\\ z_3(x,1) &=& \tilde{\omega}(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\displaystyle\int_0^1 h_9(x,s)e^{\lambda\tau s}\,ds\\ &=& \lambda\omega e^{-\lambda\tau} + z_3^0(x).\end{cases}$$

where for $x \in (0,L)$,

$$\begin{cases} z_1^0(x) &=& -h_1(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\displaystyle\int_0^1 h_3(x,s)e^{\lambda\tau_1 s}\,ds,\\ z_2^0(x) &=& -h_4(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\displaystyle\int_0^1 h_6(x,s)e^{\lambda\tau s}\,ds,\\ z_3^0(x) &=& -h_7(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\displaystyle\int_0^1 h_9(x,s)e^{\lambda\tau s}\,ds,\end{cases}$$

the problem (2.10) can be reformulated as

$$\begin{aligned} &\int_0^L \left(\lambda^2\varphi - \frac{K}{\rho_1}(\varphi_x+\psi+l\omega)_x - \frac{lK_0}{\rho_1}(\omega_x - l\varphi) + \frac{a_1}{\rho_1}\lambda\varphi e^{-\lambda\tau}\right).\omega_1\,dx\\ &\quad = \int_0^L \left(h_2 + \lambda h_1 - \frac{a_1}{\rho_1}z_1^0(x)\right).\omega_1\,dx \quad \text{for all } \omega_1 \in H_*^1(0,L),\\ &\int_0^L \left(\lambda^2\psi - \frac{b}{\rho_2}\psi_{xx} + \frac{K}{\rho_2}(\varphi_x+\psi+l\omega) + \frac{a_2}{\rho_2}\lambda\psi e^{-\lambda\tau}\right).\omega_2\,dx\\ &\quad = \int_0^L \left(h_5 + \lambda h_4 - \frac{a_2}{\rho_2}z_2^0(x)\right).\omega_2\,dx \quad \text{for all } \omega_2 \in H_*^1(0,L),\\ &\int_0^L \left(\lambda^2\omega - \frac{K_0}{\rho_1}(\omega_x - l\varphi)_x + \frac{lK}{\rho_1}(\varphi_x+\psi+l\omega) + \frac{a_3}{\rho_1}\lambda\omega e^{-\lambda\tau}\right).\omega_3\,dx\\ &\quad = \int_0^L \left(h_8 + \lambda h_7 - \frac{a_3}{\rho_1}z_3^0(x)\right).\omega_3\,dx \quad \text{for all } \omega_3 \in H_*^1(0,L).\end{aligned} \tag{2.11}$$

Integrating the first equation in (2.11) by parts, we obtain

$$\begin{aligned}
&\int_0^L \left(\lambda^2\varphi - \frac{K}{\rho_1}(\varphi_x+\psi+l\omega)_x - \frac{lK_0}{\rho_1}(\omega_x - l\varphi) + \frac{a_1}{\rho_1}\lambda\varphi e^{-\lambda\tau}\right).\omega_1\,dx \\
&= \int_0^L \lambda^2\varphi\omega_1\,dx + \frac{K}{\rho_1}\int_0^L(\varphi_x+\psi+l\omega)(\omega_1)_x\,dx - \frac{lK_0}{\rho_1}\int_0^L(\omega_x-l\varphi)\omega_1\,dx \\
&\quad + \frac{a_1}{\rho_1}\int_0^L \lambda e^{-\lambda\tau}\varphi\omega_1\,dx - \frac{K}{\rho_1}(\varphi_x+\psi+l\omega)(L)\omega_1(L) \\
&= \int_0^L \lambda^2\varphi\omega_1\,dx + \frac{K}{\rho_1}\int_0^L(\varphi_x+\psi+l\omega)(\omega_1)_x\,dx - \frac{lK_0}{\rho_1}\int_0^L(\omega_x-l\varphi)\omega_1\,dx \\
&\quad + \frac{a_1}{\rho_1}\int_0^L \lambda e^{-\lambda\tau}\varphi\omega_1\,dx + \frac{\alpha}{\rho_1}u(L)\omega_1(L) \\
&= \int_0^L\left(\lambda^2 + \frac{a_1}{\rho_1}\lambda e^{-\lambda\tau}\right)\varphi\omega_1\,dx + \int_0^L \frac{K}{\rho_1}(\varphi_x+\psi+l\omega)(\omega_1)_x\,dx \\
&\quad - \frac{lK_0}{\rho_1}\int_0^L(\omega_x-l\varphi)\omega_1\,dx + \frac{\alpha}{\rho_1}\lambda\varphi(L)\omega_1(L) - \frac{\alpha}{\rho_1}h_1(L)\omega_1(L).
\end{aligned} \tag{2.12}$$

Integrating the second equation in (2.11) by parts, we obtain

$$\begin{aligned}
&\int_0^L \left(\lambda^2\psi - \frac{b}{\rho_2}\psi_{xx} + \frac{K}{\rho_2}(\varphi_x+\psi+l\omega) + \frac{a_2}{\rho_2}\lambda\psi e^{-\lambda\tau}\right).\omega_2\,dx \\
&= \int_0^L\left(\lambda^2+\frac{a_2}{\rho_2}\lambda e^{-\lambda\tau}\right)\psi\omega_2\,dx + \frac{b}{\rho_2}\int_0^L \psi_x(\omega_2)_x\,dx \\
&\quad + \frac{K}{\rho_2}\int_0^L(\varphi_x+\psi+l\omega)\omega_2\,dx - \frac{b}{\rho_2}\psi_x(L)\omega_2(L) \\
&= \int_0^L\left(\lambda^2+\frac{a_2}{\rho_2}\lambda e^{-\lambda\tau}\right)\psi\omega_2\,dx + \frac{b}{\rho_2}\int_0^L \psi_x(\omega_2)_x\,dx \\
&\quad + \frac{K}{\rho_2}\int_0^L(\varphi_x+\psi+l\omega)\omega_2\,dx + \frac{\mu}{\rho_2}v(L)\omega_2(L) \\
&= \int_0^L\left(\lambda^2+\frac{a_2}{\rho_2}\lambda e^{-\lambda\tau}\right)\psi\omega_2\,dx + \frac{b}{\rho_2}\int_0^L \psi_x(\omega_2)_x\,dx + \frac{K}{\rho_2}\int_0^L(\varphi_x+\psi+l\omega)\omega_2\,dx \\
&\quad + \frac{\mu}{\rho_2}\lambda\psi(L)\omega_2(L) - \frac{\mu}{\rho_2}h_4(L)\omega_2(L).
\end{aligned} \tag{2.13}$$

Integrating the third equation in (2.11) by parts, we obtain

$$\begin{aligned}
&\int_0^L \left(\lambda^2\omega - \frac{K_0}{\rho_1}(\omega_x - l\varphi)_x + \frac{lK}{\rho_1}(\varphi_x+\psi+l\omega) + \frac{a_3}{\rho_1}\lambda\omega e^{-\lambda\tau}\right).\omega_3\,dx \\
&= \int_0^L\left(\lambda^2+\frac{a_3}{\rho_1}\lambda e^{-\lambda\tau}\right)\omega\omega_3\,dx + \frac{K_0}{\rho_1}\int_0^L(\omega_x-l\varphi)(\omega_3)_x\,dx \\
&\quad + \frac{lK}{\rho_1}\int_0^L(\varphi_x+\psi+l\omega)\omega_3\,dx - \frac{K_0}{\rho_1}(\omega_x-l\varphi)(L)\omega_3(L) \\
&= \int_0^L\left(\lambda^2+\frac{a_3}{\rho_1}\lambda e^{-\lambda\tau}\right)\omega\omega_3\,dx + \int_0^L\frac{K_0}{\rho_1}(\omega_x-l\varphi)(\omega_3)_x\,dx \\
&\quad + \frac{lK}{\rho_1}\int_0^L(\varphi_x+\psi+l\omega)\omega_3\,dx + \frac{\gamma}{\rho_1}\tilde{\omega}(L)\omega_3(L)
\end{aligned}$$

$$= \int_0^L \left(\lambda^2 + \frac{a_3}{\rho_1}\lambda e^{-\lambda\tau}\right)\omega\omega_3\,dx + \int_0^L \frac{K_0}{\rho_1}(\omega_x - l\varphi)(\omega_3)_x\,dx$$
$$+ \frac{lK}{\rho_1}\int_0^L (\varphi_x + \psi + l\omega)\omega_3\,dx + \frac{\gamma}{\rho_1}\lambda\omega(L)\omega_3(L) - \frac{\gamma}{\rho_1}h_7(L)\omega_3(L). \tag{2.14}$$

Using (2.12), (2.13), and (2.14), the problem (2.11) is equivalent to the problem

$$\phi((\varphi,\psi,\omega),(\omega_1,\omega_2,\omega_3)) = \mathcal{I}(\omega_1,\omega_2,\omega_3), \tag{2.15}$$

where the bilinear form $\phi : [H^1_*(0,L) \times H^1_*(0,L) \times H^1_*(0,L)]^2 \to \mathbb{R}$ and the linear form $\mathcal{I} : H^1_*(0,L) \times H^1_*(0,L) \times H^1_*(0,L) \to \mathbb{R}$ are defined by

$$\begin{aligned}
&\phi((\varphi,\psi,\omega),(\omega_1,\omega_2,\omega_3))\\
&= \int_0^L \left(\lambda^2 + \frac{a_1}{\rho_1}\lambda e^{-\lambda\tau}\right)\varphi\omega_1\,dx + \int_0^L \frac{K}{\rho_1}(\varphi_x + \psi + l\omega)(\omega_1)_x\,dx\\
&\quad -\frac{lK_0}{\rho_1}\int_0^L (\omega_x - l\varphi)\omega_1\,dx + \frac{\alpha}{\rho_1}\lambda\varphi(L)\omega_1(L)\\
&\quad + \int_0^L \left(\lambda^2 + \frac{a_2}{\rho_2}\lambda e^{-\lambda\tau}\right)\psi\omega_2\,dx + \frac{b}{\rho_2}\int_0^L \psi_x(\omega_2)_x\,dx\\
&\quad +\frac{K}{\rho_2}\int_0^L (\varphi_x + \psi + l\omega)\omega_2\,dx + \frac{\mu}{\rho_2}\lambda\psi(L)\omega_2(L)\\
&\quad + \int_0^L \left(\lambda^2 + \frac{a_3}{\rho_1}\lambda e^{-\lambda\tau}\right)\omega\omega_3\,dx + \int_0^L \frac{K_0}{\rho_1}(\omega_x - l\varphi)(\omega_3)_x\,dx\\
&\quad +\frac{lK}{\rho_1}\int_0^L (\varphi_x + \psi + l\omega)\omega_3\,dx + \frac{\gamma}{\rho_1}\lambda\omega(L)\omega_3(L),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}(\omega_1,\omega_2,\omega_3) &= \int_0^L \left(h_2 + \lambda h_1 - \frac{a_1}{\rho_1}z_1^0(x)\right).\omega_1\,dx + \int_0^L \left(h_5 + \lambda h_4 - \frac{a_2}{\rho_2}z_2^0(x)\right).\omega_2\,dx\\
&\quad + \int_0^L \left(h_8 + \lambda h_7 - \frac{a_3}{\rho_1}z_3^0(x)\right)\omega_3\,dx\\
&\quad +\frac{\alpha}{\rho_1}h_1(L)\omega_1(L) + \frac{\mu}{\rho_2}h_4(L)\omega_2(L) + \frac{\gamma}{\rho_1}h_7(L)\omega_3(L).
\end{aligned}$$

It is easy to verify that ϕ is continuous and coercive and that $\mathcal{I}$ is continuous. Hence applying the Lax–Milgram theorem, we deduce that for all $(\omega_1,\omega_2,\omega_3) \in H^1_*(0,L) \times H^1_*(0,L) \times H^1_*(0,L)$, problem (2.15) admits a unique solution $(\varphi,\psi,\omega) \in H^1_*(0,L) \times H^1_*(0,L)\times H^1_*(0,L)$. It follows from (2.12), (2.13), and (2.14) that $(\varphi,\psi,\omega) \in (H^2(0,L)\cap H^1_*(0,L)) \times (H^2(0,L) \cap H^1_*(0,L)) \times (H^2(0,L) \cap H^1_*(0,L))$. Therefore, the operator $\lambda I - \mathcal{A}$. $\mathcal{A}$ is surjective for any $\lambda > 0$. Hence, the well-posedness result follows from the Hille–Yosida theorem. $\square$

3. Asymptotic stability

In this section, we investigate the asymptotic behavior of system (1.2)–(1.4). For any regular solution of (1.2)–(1.4), we define the energy as

$$\begin{aligned}\mathcal{E}(t) = \frac{1}{2}\int_0^L \Big(&\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + \rho_1|\omega_t|^2 + b|\psi_x|^2 + K|\varphi_x+\psi+l\omega|^2 \\ &+K_0|\omega_x - l\varphi|^2\Big)\,dx + \frac{\xi_1}{2}\int_{t-\tau}^t\int_0^L \varphi_t^2(x,s)\,dx\,ds \\ &+\frac{\xi_2}{2}\int_{t-\tau}^t\int_0^L \psi_t^2(x,s)\,dx\,ds + \frac{\xi_3}{2}\int_{t-\tau}^t\int_0^L \omega_t^2(x,s)\,dx\,ds,\end{aligned} \tag{3.1}$$

where ξ_1, ξ_2, and ξ_3 are strictly positive numbers that will be chosen later. The main result of this paper is the following.

Theorem 3.1. *Let (φ,ψ,ω) be a regular solution of (1.2)–(1.4). Assume that $\frac{4K(1+l^2)L^2}{\pi^2} \le b$, that $\frac{4K_0 l^2 L^2}{\pi^2} \le b$, and that there exist small enough positive constants $a_i^\circ, i=1,2,3$ satisfying $0 < a_i \le a_i^\circ$ $i = 1,2,3$. Then,*

$$\mathcal{E}(t) \le Ce^{-\omega t} \qquad \textit{for all } t \ge 0, \tag{3.2}$$

while C and ω are positive constants, independent of the initial data.

The proof of Theorem 3.1 will be done through some Lemmas.

Lemma 3.2. *For any regular solution of (1.2)–(1.4), the following estimate holds:*

$$\begin{aligned}\tfrac{d\mathcal{E}(t)}{dt} \le& -\alpha\varphi_t^2(L,t) + \left(\tfrac{a_1+\xi_1}{2}\right)\textstyle\int_0^L \varphi_t^2(x,t)\,dx + \left(\tfrac{a_1-\xi_1}{2}\right)\int_0^L \varphi_t^2(x,t-\tau)\,dx \\ &-\mu\psi_t^2(L,t) + \left(\tfrac{a_2+\xi_2}{2}\right)\textstyle\int_0^L \psi_t^2(x,t)\,dx + \left(\tfrac{a_2-\xi_2}{2}\right)\int_0^L \psi_t^2(x,t-\tau)\,dx \\ &-\gamma\omega_t^2(L,t) + \left(\tfrac{a_3+\xi_3}{2}\right)\textstyle\int_0^L \omega_t^2(x,t)\,dx + \left(\tfrac{a_3-\xi_3}{2}\right)\int_0^L \omega_t^2(x,t-\tau)\,dx.\end{aligned} \tag{3.3}$$

Proof. Differentiating (3.1), we get

$$\begin{aligned}\tfrac{d\mathcal{E}(t)}{dt} = \int_0^L \Big(&\rho_1\varphi_t\varphi_{tt} + \rho_2\psi_t\psi_{tt} + \rho_1\omega_t\omega_{tt} + b\psi_x\psi_{xt} + K(\varphi_x+\psi+l\omega)(\varphi_x+\psi+l\omega)_t \\ &+K_0(\omega_x - l\varphi)(\omega_x-l\varphi)_t\Big)(t,x)\,dx \\ &+\frac{\xi_1}{2}\int_0^L \varphi_t^2(x,t)\,dx - \frac{\xi_1}{2}\int_0^L \varphi_t^2(x,t-\tau)\,dx \\ &+\frac{\xi_2}{2}\int_0^L \psi_t^2(x,t)\,dx - \frac{\xi_2}{2}\int_0^L \psi_t^2(x,t-\tau)\,dx \\ &+\frac{\xi_3}{2}\int_0^L \omega_t^2(x,t)\,dx - \frac{\xi_3}{2}\int_0^L \omega_t^2(x,t-\tau)\,dx.\end{aligned}$$

Now, using the equations in (1.2) and exploiting the boundary conditions in (1.3), we obtain

$$\frac{d\mathcal{E}(t)}{dt} \le -\alpha\varphi_t^2(L,t) - a_1\int_0^L \varphi_t(x,t-\tau)\varphi_t(x,t)\,dx$$

$$
\begin{aligned}
&- \mu\psi_t^2(L,t) - a_2\int_0^L \psi_t(x,t-\tau)\psi_t(x,t)\,dx\\
&- \gamma\omega_t^2(L,t) - a_3\int_0^L \varphi_t(x,t-\tau)\varphi_t(x,t)\,dx\\
&+ \frac{\xi_1}{2}\int_0^L \varphi_t^2(x,t)\,dx - \frac{\xi_1}{2}\int_0^L \varphi_t^2(x,t-\tau)\,dx\\
&+ \frac{\xi_2}{2}\int_0^L \psi_t^2(x,t)\,dx - \frac{\xi_2}{2}\int_0^L \psi_t^2(x,t-\tau)\,dx\\
&+ \frac{\xi_3}{2}\int_0^L \omega_t^2(x,t)\,dx - \frac{\xi_3}{2}\int_0^L \omega_t^2(x,t-\tau)\,dx. \qquad (3.4)
\end{aligned}
$$

By applying Young's inequality to the first three terms in (3.4), then (3.3) holds true. □

Next, we define the following functional:

$$
\mathcal{F}(t) := \int_0^L \{\rho_1 x\psi_t\psi_x + \rho_2 x\varphi_t\varphi_x + \rho_1 x\omega_t\omega_x\}(x,t)dx. \qquad (3.5)
$$

Then we have the following estimate:

Lemma 3.3. *Let (φ,ψ,ω) be the solution of (1.2)–(1.4). Then, for any ϵ_1,ϵ_2, ϵ_3, δ_1, δ_2, $\delta_3,\beta_1,\beta_2,\beta_3>0$, we have*

$$
\begin{aligned}
\frac{d\mathcal{F}(t)}{dt} \le &-\int_0^L\left\{\frac{\rho_1}{2}\varphi_t^2 + \frac{\rho_2}{2}\psi_t^2 + \frac{\rho_1}{2}\omega_t^2\right\}dx + \left(a_1L\epsilon_1 + \frac{c^*}{4\beta_1} + \delta_1 c_2\right)\int_0^L \varphi_x^2(x,t)\,dx\\
&+ \left(a_2L\epsilon_2 + \frac{c^*}{4\beta_2} + \delta_2 c_2 - \frac{b}{2}\right)\int_0^L \psi_x^2(x,t)\,dx\\
&+ \left(a_3L\epsilon_3 + \frac{c^*}{4\beta_3} + \delta_3 c_2\right)\int_0^L \omega_x^2(x,t)\,dx\\
&+ \left(\beta_2K^2 + \beta_3K^2l^2 - \frac{K}{2}\right)\int_0^L (\varphi_x+\psi+l\omega)^2(x,t)\,dx\\
&+ \left(\beta_1K_0^2l^2 - \frac{K_0}{2}\right)\int_0^L (\omega_x - l\varphi)^2(x,t)\,dx + \left(\frac{\rho_1L}{2} + \frac{\gamma^2L}{2K_0} + \frac{\gamma^2l^2L^2}{4\delta_1}\right)\omega_t^2(L,t)\\
&+ \left(\frac{\rho_2L}{2} + \frac{\mu^2L}{2b}\right)\psi_t^2(L,t) + \left(\frac{\rho_2L}{2} + \frac{\alpha^2L}{2K} + \frac{\alpha^2L^2}{4\delta_2} + \frac{\alpha^2l^2L^2}{4\delta_3}\right)\varphi_t^2(L,t)\\
&+ \frac{a_1L}{4\epsilon_1}\int_0^L \varphi_t^2(x,t-\tau)\,dx + \frac{a_2L}{4\epsilon_2}\int_0^L \psi_t^2(x,t-\tau)\,dx + \frac{a_3L}{4\epsilon_3}\int_0^L \omega_t^2(x,t-\tau)\,dx,
\end{aligned}
\qquad (3.6)
$$

where $c^*=L^2/\pi^2$ is the Poincaré constant.

Proof. Differentiating the functional $\mathcal{F}$ with respect to t and using (1.2), we find

$$
\begin{aligned}
\frac{d\mathcal{F}(t)}{dt} = &\int_0^L \Big\{Kx(\varphi_x+\psi+l\omega)_x\varphi_x + K_0lx(\omega_x - l\varphi)\varphi_x + bx\psi_{xx}\psi_x\\
&- Kx(\varphi_x+\psi+l\omega)\psi_x + K_0x(\omega_x-l\varphi)_x\omega_x - Klx(\varphi_x+\psi+l\omega)\omega_x\Big\}(x,t)\,dx\\
&+ \int_0^L\left\{\rho_1\frac{x}{2}\frac{d\varphi_t^2}{dx} + \rho_2\frac{x}{2}\frac{d\psi_t^2}{dx} + \rho_1\frac{x}{2}\frac{d\omega_t^2}{dx}\right\}(x,t)\,dx
\end{aligned}
$$

$$- a_1 \int_0^L x\varphi_x(x,t)\varphi_t(x,t-\tau)\,dx - a_2 \int_0^L x\psi_x(x,t)\psi_t(x,t-\tau)\,dx$$
$$- a_3 \int_0^L x\omega_x(x,t)\omega_t(x,t-\tau)\,dx. \tag{3.7}$$

Remark that

$$\int_0^L K_0 l x(\omega_x - l\varphi)\varphi_x\,dx = K_0 l L(\omega_x - l\varphi)(L,t)\varphi(L,t) - \int_0^L K_0 l(\omega_x - l\varphi)\varphi\,dx - \int_0^L K_0 l x(\omega_x - l\varphi)_x\varphi\,dx, \tag{3.8}$$

$$-\int_0^L Kx(\varphi_x + \psi + l\omega)\psi_x\,dx = - KL(\varphi_x + \psi + l\omega)(L,t)\psi(L,t) + \int_0^L K(\varphi_x + \psi + l\omega)\psi\,dx + \int_0^L Kx(\varphi_x + \psi + l\omega)_x\psi\,dx, \tag{3.9}$$

$$-\int_0^L Klx(\varphi_x + \psi + l\omega)\omega_x\,dx = - KlL(\varphi_x + \psi + l\omega)(L,t)\omega(L,t) + \int_0^L Kl(\varphi_x + \psi + l\omega)\omega\,dx + \int_0^L Klx(\varphi_x + \psi + l\omega)_x\omega\,dx. \tag{3.10}$$

Inserting (3.8), (3.9), and (3.10) into (3.7), we obtain

$$\begin{aligned}
\frac{d\mathcal{F}(t)}{dt} &= \int_0^L \left\{ \frac{K}{2} x \frac{d(\varphi_x + \psi + l\omega)^2}{dx} + \frac{b}{2} x \frac{d\psi_x^2}{dx} + \frac{K_0}{2} x \frac{d(\omega_x - l\varphi)^2}{dx} \right\} dx \\
&\quad + K_0 l L(\omega_x - l\varphi)(L,t)\varphi(L,t) - \int_0^L K_0 l(\omega_x - l\varphi)\varphi\,dx \\
&\quad - KL(\varphi_x + \psi + l\omega)(L,t)\psi(L,t) + \int_0^L K(\varphi_x + \psi + l\omega)\psi\,dx \\
&\quad - KlL(\varphi_x + \psi + l\omega)(L,t)\omega(L,t) + \int_0^L Kl(\varphi_x + \psi + l\omega)\omega\,dx \\
&\quad + \int_0^L \left\{ \rho_1 \frac{x}{2} \frac{d\varphi_t^2}{dx} + \rho_2 \frac{x}{2} \frac{d\psi_t^2}{dx} + \rho_1 \frac{x}{2} \frac{d\omega_t^2}{dx} \right\}(x,t)\,dx \\
&\quad - a_1 \int_0^L x\varphi_x(x,t)\varphi_t(x,t-\tau)\,dx - a_2 \int_0^L x\psi_x(x,t)\psi_t(x,t-\tau)\,dx \\
&\quad - a_3 \int_0^L x\omega_x(x,t)\omega_t(x,t-\tau)\,dx.
\end{aligned}$$

Using integration by parts, we obtain

$$
\begin{aligned}
\frac{d\mathcal{F}(t)}{dt} = &-\int_0^L \left\{\frac{\rho_1}{2}\varphi_t^2 + \frac{\rho_2}{2}\psi_t^2 + \frac{\rho_1}{2}\omega_t^2\right\} dx \\
&- \int_0^L \left\{\frac{b}{2}\psi_x^2 + \frac{K}{2}(\varphi_x+\psi+l\omega)^2 + \frac{K_0}{2}(\omega_x - l\varphi)^2\right\} dx \\
&+\tfrac{bL}{2}\psi_x^2(L,t) + \tfrac{KL}{2}(\varphi_x+\psi+l\omega)^2(L,t) + \tfrac{K_0L}{2}(\omega_x-l\varphi)^2(L,t) \\
&+\tfrac{\rho_1 L}{2}\varphi_t^2(L,t) + \tfrac{\rho_2 L}{2}\psi_t^2(L,t) + \tfrac{\rho_1 L}{2}\omega_t^2(L,t) \\
&+K_0 l L(\omega_x - l\varphi)(L,t)\varphi(L,t) - \int_0^L K_0 l(\omega_x - l\varphi)\varphi\, dx \\
&-KL(\varphi_x+\psi+l\omega)(L,t)\psi(L,t) + \int_0^L K(\varphi_x+\psi+l\omega)\psi\, dx \\
&-KlL(\varphi_x+\psi+l\omega)(L,t)\omega(L,t) + \int_0^L Kl(\varphi_x+\psi+l\omega)\omega\, dx \\
&-a_1\textstyle\int_0^L x\varphi_x(x,t)\varphi_t(x,t-\tau)\, dx - a_2\int_0^L x\psi_x(x,t)\psi_t(x,t-\tau)\, dx \\
&-a_3\textstyle\int_0^L x\omega_x(x,t)\omega_t(x,t-\tau)\, dx.
\end{aligned}
\tag{3.11}
$$

Consequently, using the boundary conditions (1.3), we write

$$
\frac{bL}{2}\psi_x^2(L,t) = \frac{\mu^2 L}{2b}\psi_t^2(L,t). \tag{3.12}
$$

Similarly, we get

$$
\frac{KL}{2}(\varphi_x+\psi+l\omega)^2(L,t) = \frac{\alpha^2 L}{2K}\varphi_t^2(L,t), \tag{3.13}
$$

$$
\frac{K_0 L}{2}(\omega_x - l\varphi)^2(L,t) = \frac{\gamma^2 L}{2K_0}\omega_t^2(L,t). \tag{3.14}
$$

By the embedding of $W^{1,1}(0,L)$ in $L^\infty(0,L)$, we have

$$
\begin{aligned}
|\varphi(L,t)|^2 &\le c_1 \textstyle\int_0^L (\varphi^2 + \varphi_x^2)\, dx, \\
|\psi(L,t)|^2 &\le c_1 \textstyle\int_0^L (\psi^2 + \psi_x^2)\, dx, \\
|\omega(L,t)|^2 &\le c_1 \textstyle\int_0^L (\omega^2 + \omega_x^2)\, dx.
\end{aligned}
$$

This implies by Poincaré's inequality that

$$
\begin{aligned}
|\varphi(L,t)|^2 &\le c_2 \textstyle\int_0^L \varphi_x^2\, dx, \\
|\psi(L,t)|^2 &\le c_2 \textstyle\int_0^L \psi_x^2\, dx, \\
|\omega(L,t)|^2 &\le c_2 \textstyle\int_0^L \omega_x^2\, dx,
\end{aligned}
\tag{3.15}
$$

where c_1 and c_2 are two positive constants.

Equation (1.3), Young's inequality, and (3.15) imply for all $\delta_1, \delta_2, \delta_3 > 0$ that

$$
\begin{aligned}
K_0 l L(\omega_x - l\varphi)(L,t)\varphi(L,t) &= -\gamma l L\omega_t(L,t)\varphi(L,t) \\
&\le \delta_1 c_2 \textstyle\int_0^L \varphi_x^2\, dx + \frac{\gamma^2 l^2 L^2 \omega_t^2(L,t)}{4\delta_1}, \\
-KL(\varphi_x+\psi+l\omega)(L,t)\psi(L,t) &= \alpha L\varphi_t(L,t)\psi(L,t) \\
&\le \delta_2 c_2 \textstyle\int_0^L \psi_x^2\, dx + \frac{\alpha^2 L^2 \varphi_t^2(L,t)}{4\delta_2}, \\
-KlL(\varphi_x+\psi+l\omega)(L,t)\omega(L,t) &= \alpha l L\varphi_t(L,t)\omega(L,t) \\
&\le \delta_3 c_2 \textstyle\int_0^L \omega_x^2\, dx + \frac{\alpha^2 l^2 L^2 \varphi_t^2(L,t)}{4\delta_3}.
\end{aligned}
\tag{3.16}
$$

Once again, Young's inequality and Poincaré's inequality, for any $\beta_1, \beta_2, \beta_3 > 0$ give

$$\begin{aligned} -\int_0^L K_0 l(\omega_x - l\varphi)\varphi\,dx &\le \beta_1 K_0^2 l^2 \int_0^L (\omega_x - l\varphi)^2\,dx + \frac{c^*}{4\beta_1}\int_0^L \varphi_x^2\,dx,\\ \int_0^L K(\varphi_x+\psi+l\omega)\psi\,dx &\le \beta_2 K^2 \int_0^L (\varphi_x+\psi+l\omega)^2\,dx + \frac{c^*}{4\beta_2}\int_0^L \psi_x^2\,dx,\\ \int_0^L Kl(\varphi_x+\psi+l\omega)\omega\,dx &\le \beta_3 K^2 l^2 \int_0^L (\varphi_x+\psi+l\omega)^2\,dx + \frac{c^*}{4\beta_3}\int_0^L \omega_x^2\,dx, \end{aligned} \tag{3.17}$$

where $c^* = L^2/\pi^2$ is the Poincaré constant.

On the other hand, for all $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, by using Young's inequalit, the last two terms in the right-hand side of (3.11) can be estimated as follows:

$$\begin{aligned} \left| a_1 \int_0^L x\varphi_x(x,t)\varphi_t(x,t-\tau)\,dx \right| \le\ & a_1 L\epsilon_1 \int_0^L \varphi_x^2(x,t)\,dx \\ & + \tfrac{a_1 L}{4\epsilon_1}\int_0^L \varphi_t^2(x,t-\tau)\,dx, \end{aligned} \tag{3.18}$$

$$\begin{aligned} \left| a_2 \int_0^L x\psi_x(x,t)\psi_t(x,t-\tau)\,dx \right| \le\ & a_2 L\epsilon_2 \int_0^L \psi_x^2(x,t)\,dx \\ & + \tfrac{a_2 L}{4\epsilon_2}\int_0^L \psi_t^2(x,t-\tau)\,dx, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \left| a_3 \int_0^L x\omega_x(x,t)\omega_t(x,t-\tau)\,dx \right| \le\ & a_3 L\epsilon_3 \int_0^L \omega_x^2(x,t)\,dx \\ & + \tfrac{a_3 L}{4\epsilon_3}\int_0^L \omega_t^2(x,t-\tau)\,dx. \end{aligned} \tag{3.20}$$

Inserting (3.12)–(3.20) into (3.11), we get (3.6). □

Next, let us introduce

$$\begin{aligned} \mathcal{F}_1(t) =: \int_0^L \int_{t-\tau}^t e^{s-t}\varphi_t^2(x,s)\,ds\,dx,\ \mathcal{F}_2(t) =: \int_0^L \int_{t-\tau}^t e^{s-t}\psi_t^2(x,s)\,ds\,dx,\\ \text{and } \mathcal{F}_3(t) =: \int_0^L \int_{t-\tau}^t e^{s-t}\omega_t^2(x,s)\,ds\,dx. \end{aligned} \tag{3.21}$$

Then, the following estimates hold.

Lemma 3.4. *Let (φ, ψ, ω) be the solution of (1.2)–(1.4). Then*

$$\tfrac{d\mathcal{F}_1(t)}{dt} \le \int_0^L \varphi_t^2(x,t)\,dx - e^{-\tau}\int_0^L \varphi_t^2(x,t-\tau)\,dx - e^{-\tau}\int_0^L\int_{t-\tau}^t \varphi_t^2(x,s)\,ds\,dx, \tag{3.22}$$

$$\tfrac{d\mathcal{F}_2(t)}{dt} \le \int_0^L \psi_t^2(x,t)\,dx - e^{-\tau}\int_0^L \psi_t^2(x,t-\tau)\,dx - e^{-\tau}\int_0^L\int_{t-\tau}^t \psi_t^2(x,s)\,ds\,dx, \tag{3.23}$$

and

$$\tfrac{d\mathcal{F}_3(t)}{dt} \int_0^L \omega_t^2(x,t)\,dx - e^{-\tau}\int_0^L \omega_t^2(x,t-\tau)\,dx - e^{-\tau}\int_0^L\int_{t-\tau}^t \omega_t^2(x,s)\,ds\,dx. \tag{3.24}$$

Proof. Taking the derivative of $\mathcal{F}_1$ with respect to t, we obtain

$$\tfrac{d\mathcal{F}_1(t)}{dt} = \int_0^L \varphi_t^2(x,t)\,dx - e^{-\tau}\int_0^L \varphi_t^2(x,t-\tau)\,dx - \int_0^L\int_{t-\tau}^t e^{s-t}\varphi_t^2(x,s)\,ds\,dx.$$

Then, (3.22) easily holds, and similarly (3.23) and (3.24). □

Proof. Proof of Theorem 3.1. We define the Lyapunov functional $\mathcal{L}(t)$ as follows:

$$\mathcal{L}(t) := \mathcal{E}(t) + N\mathcal{F}(t) + N_1\mathcal{F}_1(t) + N_2\mathcal{F}_2(t) + N_3\mathcal{F}_3(t), \tag{3.25}$$

where N, N_1, N_2, and N_3 are positive real numbers that be chosen later. Now, from (3.3), (3.6), (3.22), (3.23), and (3.24) and using inequalities, we obtain

$$\begin{aligned}\int_0^L \omega_x^2(x,t)\,dx &\leq 2\int_0^L (\omega_x - l\varphi)^2(x,t)\,dx + 2l^2c^*\int_0^L \varphi_x^2(x,t)\,dx,\\ \int_0^L \varphi_x^2(x,t)\,dx &\leq 2\int_0^L (\varphi_x + \psi + l\omega)^2(x,t)\,dx + 4c^*\int_0^L \psi_x^2(x,t)\,dx\\ &\qquad + 4l^2c^*\int_0^L \omega_x^2(x,t)\,dx.\end{aligned}$$

We get

$$\begin{aligned}\frac{d\mathcal{L}(t)}{dt} \leq & A_1\varphi_t^2(L,t) + A_2\psi_t^2(L,t) \qquad (3.26)\\ &+ A_3\omega_t^2(L,t) + \left\{\left(\frac{a_1+\xi_1}{2}\right) + N_1 - \frac{N\rho_1}{2}\right\}\int_0^L \varphi_t^2(x,t)\,dx\\ &+ \left\{\left(\frac{a_2+\xi_2}{2}\right) + N_2 - \frac{N\rho_2}{2}\right\}\int_0^L \psi_t^2(x,t)\,dx\\ &+ \left\{\left(\frac{a_3+\xi_3}{2}\right) + N_3 - \frac{N\rho_1}{2}\right\}\int_0^L \omega_t^2(x,t)\,dx\\ &+ \left\{\left(\frac{a_1-\xi_1}{2}\right) + N\frac{a_1L}{4\epsilon_1} - N_1e^{-\tau}\right\}\int_0^L \varphi_t^2(x,t-\tau)\,dx\\ &+ \left\{\left(\frac{a_2-\xi_2}{2}\right) + N\frac{a_2L}{4\epsilon_2} - N_2e^{-\tau}\right\}\int_0^L \psi_t^2(x,t-\tau)\,dx\\ &+ \left\{\left(\frac{a_3-\xi_3}{2}\right) + N\frac{a_3L}{4\epsilon_3} - N_3e^{-\tau}\right\}\int_0^L \omega_t^2(x,t-\tau)\,dx\\ &+ N\Bigg(+ \frac{4c^*a_1L\epsilon_1}{1-8l^4c^{*2}} + \frac{2c^{*2}}{2\beta_1(1-8l^4c^{*2})} + \frac{4c^*\delta_1c_2}{1-8l^4c^{*2}+a_2L\epsilon_2} + \frac{8l^2c^*a_3L\epsilon_3}{1-8l^4c^{*2}}\\ &+ \frac{2l^2c^{*2}}{\beta_3(1-8l^4c^{*2})} + \frac{8l^2c^*\delta_3c_2}{1-8l^4c^{*2}} + \delta_2c_2 + \frac{c^*}{4\beta_2} - \frac{b}{2}\Bigg)\int_0^L \psi_x^2(x,t)\,dx\\ &+ N\Bigg(\frac{4l^2c^*a_3L\epsilon_3}{1-8l^4c^{*2}} + \frac{l^2c^{*2}}{\beta_3(1-8l^4c^{*2})} + \frac{4l^2c^*\delta_3c_2}{1-8l^4c^{*2}} + \frac{2a_1L\epsilon_1}{1-8l^4c^{*2}} + \frac{c^*}{2\beta_1(1-8l^4c^{*2})}\\ &+ \frac{2\delta_1c_2}{1-8l^4c^{*2}} + \beta_2K^2 + \beta_3K^2l^2 - \frac{K}{2}\Bigg)\int_0^L (\varphi_x+\psi+l\omega)^2(x,t)\,dx\\ &+ N\Bigg(\frac{2a_3L\epsilon_3}{1-8l^4c^{*2}} + \frac{c^*}{2\beta_3(1-8l^4c^{*2})} + \frac{2\delta_3c_2}{1-8l^4c^{*2}} + \frac{8l^2c^*a_1L\epsilon_1}{1-8l^4c^{*2}}\\ &+ \frac{2l^2c^{*2}}{\beta_1(1-8l^4c^{*2})} + \frac{8l^2c^*\delta_1c_2}{1-8l^4c^{*2}} + \beta_1K_0^2l^2 - \frac{K_0}{2}\Bigg)\int_0^L (\omega_x - l\varphi)^2(x,t)\,dx\\ &- N_1e^{-\tau}\int_0^L\int_{t-\tau}^t \varphi_t^2(x,s)\,ds\,dx - N_2e^{-\tau}\int_0^L\int_{t-\tau}^t \psi_t^2(x,s)\,ds\,dx\\ &- N_3e^{-\tau}\int_0^L\int_{t-\tau}^t \omega_t^2(x,s)\,ds\,dx, \qquad (3.27)\end{aligned}$$

where

$$\begin{aligned} A_1 &= -a + N\left(\frac{\rho_2 L}{2} + \frac{\alpha^2 L}{2K} + \frac{\alpha^2 L^2}{4\delta_2} + \frac{\alpha^2 l^2 L^2}{4\delta_3}\right), \\ A_2 &= -\mu + N\left(\frac{\rho_2 L}{2} + \frac{\mu^2 L}{2b}\right), \\ A_3 &= -\gamma + N\left(\frac{\rho_1 L}{2} + \frac{\gamma^2 L}{2K_0} + \frac{\gamma^2 l^2 L^2}{4\delta_1}\right). \end{aligned}$$

At this point, we have to choose our constants in (3.27) very carefully.

First, it is clear that for any $\alpha > 0$, $\mu > 0$, and $\gamma > 0$, and for sufficiently small N, we get $A_i < 0,\ i = 1, 2, 3$.

Second, we may choose β_1, β_2, and β_3 such that

$$\begin{aligned} \beta_1 K_0^2 l^2 - \frac{K_0}{2} &\le -\frac{K_0}{4}, \\ \beta_2 K^2 + \beta_3 K^2 l^2 - \frac{K}{2} &\le -\frac{K}{4}, \\ \frac{c^*}{4\beta_2} - \frac{b}{2} &\le -\frac{b}{4}, \\ \frac{c^*}{4\beta_1} - \frac{b}{2} &\le -\frac{b}{4}, \\ \frac{c^*}{4\beta_3} - \frac{b}{2} &\le -\frac{b}{4} \end{aligned} \tag{3.28}$$

Of course, in order to get (3.28), we have to assume that

$$\begin{aligned} 4K(1+l^2)c^* &= \frac{4K(1+l^2)L^2}{\pi^2} \le b, \\ 4K_0 l^2 c^* &= \frac{4K_0 l^2 L^2}{\pi^2} \le b. \end{aligned}$$

For any fixed $\delta_1,\ \delta_3 > 0$, we pick $\delta_2, \epsilon_1, \epsilon_2 > 0$ and $\epsilon_3 > 0$ so small such that

$$\begin{aligned} &\tfrac{4c^* a_1 L\epsilon_1}{1-8l^4 {c^*}^2} + \tfrac{2{c^*}^2}{2\beta_1(1-8l^4{c^*}^2)} + \tfrac{4c^*\delta_1 c_2}{1-8l^4{c^*}^2 + a_2 L\epsilon_2} + \tfrac{8l^2 c^* a_3 L\epsilon_3}{1-8l^4{c^*}^2} + \tfrac{2l^2{c^*}^2}{\beta_3(1-8l^4{c^*}^2)} + \tfrac{8l^2c^*\delta_3 c_2}{1-8l^4{c^*}^2} + \delta_2 c_2 \le \tfrac{b}{8}, \\ &\tfrac{4l^2c^* a_3 L\epsilon_3}{1-8l^4{c^*}^2} + \tfrac{l^2{c^*}^2}{\beta_3(1-8l^4{c^*}^2)} + \tfrac{4l^2c^*\delta_3 c_2}{1-8l^4{c^*}^2} + \tfrac{2a_1 L\epsilon_1}{1-8l^4{c^*}^2} + \tfrac{c^*}{2\beta_1(1-8l^4{c^*}^2)} + \tfrac{2\delta_1 c_2}{1-8l^4{c^*}^2} \le \tfrac{K}{8}, \\ &\tfrac{2a_3 L\epsilon_3}{1-8l^4{c^*}^2} + \tfrac{c^*}{2\beta_3(1-8l^4{c^*}^2)} + \tfrac{2\delta_3 c_2}{1-8l^4{c^*}^2} + \tfrac{8l^2c^* a_1 L\epsilon_1}{1-8l^4{c^*}^2} + \tfrac{2l^2{c^*}^2}{\beta_1(1-8l^4{c^*}^2)} + \tfrac{8l^2c^*\delta_1 c_2}{1-8l^4{c^*}^2} \le \tfrac{K_0}{8}. \end{aligned}$$

After that, we fix N_1, N_2, and N_3 such that $\frac{N\rho_1}{2} - N_1 > 0$, $\frac{N\rho_2}{2} - N_2 > 0$, and $\frac{N\rho_3}{2} - N_3 > 0$. Now the main goal is to choose the sets of pairs (a_1, ξ_1), (a_2, ξ_2), and (a_3, ξ_3) such that

$$\begin{cases} \frac{a_1+\xi_1}{2} < \frac{N\rho_1}{2} - N_1, \\ a_1(\frac{1}{2} + \frac{NL}{4\epsilon_1}) - \frac{\xi_1}{2} \le N_1 e^{-\tau}, \end{cases}$$

$$\begin{cases} \frac{a_2+\xi_2}{2} < \frac{N\rho_2}{2} - N_2, \\ a_2(\frac{1}{2} + \frac{NL}{4\epsilon_2}) - \frac{\xi_2}{2} \le N_2 e^{-\tau}, \end{cases}$$

and

$$\begin{cases} \frac{a_3+\xi_3}{2} < \frac{N\rho_3}{2} - N_3, \\ a_3(\frac{1}{2} + \frac{NL}{4\epsilon_3}) - \frac{\xi_3}{2} \le N_3 e^{-\tau}. \end{cases}$$

Obviously, for $a_i,\ i = 1, 2, 3$, small enough satisfying

$$a_i < a_i^\circ = \min\left\{\frac{N_i e^{-\tau} + (N\rho_i/2 - N_i)}{1 + NL/(4\epsilon_i)}, (N\rho_i - 2N_i)\right\},\ i = 1, 2, 3,$$

there exists $\xi_i,\ i=1,2,3$ such that

$$a_i\left(1+\frac{NL}{2\epsilon_i}\right)-2N_ie^{-\tau}\le\xi_i<(N\rho_i-2N_i)-a_i,\ i=1,2,3.$$

From this, we infer $A_i<0$ $(i=1,2,3)$ if $(a,\mu,\gamma)\longrightarrow(0,0,0)$ or if $(a,\mu,\gamma)\longrightarrow(\infty,\infty,\infty)$, then $(N,N_i)\longrightarrow(0,0)$ $(i=1,2,3)$ and consequently a_i^0 goes to zero.

Then, from the above, we deduce that there exists a positive constant $\eta_1>0$ such that (3.27) becomes

$$\begin{aligned}\tfrac{d\mathcal{L}(t)}{dt}\le&-\eta_1\textstyle\int_0^L(\psi_t^2+\psi_x^2+\varphi_t^2+\omega_t^2+(\varphi_x+\psi+l\omega)^2+(\omega_x-l\varphi)^2)\,dx\\&-\eta_1\textstyle\int_0^L\displaystyle\int_{t-\tau}^t(\varphi_t^2(x,s)+\psi_t^2(x,s)+\omega_t^2(x,s))\,ds\,dx\quad\text{for all } t\ge0,\end{aligned}$$

which implies from (3.1), that there exists also η_2, such that

$$\frac{d\mathcal{L}(t)}{dt}\le-\eta_2\mathcal{E}(t)\quad\text{for all } t\ge0.\tag{3.29}$$

On the other hand, from (3.1), (3.5), (3.21), and (3.25), and for sufficiently small N, we deduce that there exist two positive constants B_1 and B_2 depending on N,N_1,N_2,N_3, and L such that

$$B_1\mathcal{E}(t)\le\mathcal{L}(t)\le B_2\mathcal{E}(t)\quad\text{for all } t\ge0.\tag{3.30}$$

Now, by combining (3.29) and (3.30), there exists $\Lambda>0$, such that

$$\frac{d\mathcal{L}(t)}{dt}\le-\Lambda\mathcal{L}(t)\quad\text{for all } t\ge0.\tag{3.31}$$

Consequently, integrating (3.31) and using once again (3.30), we obtain (3.2). □

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