Khayyam J. Math. 6 (2020), no. 2, 236-242
DOI: 10.22034/kjm.2020.109820

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ANDERSON'S THEOREM FOR SOME CLASS OF OPERATORS

MEHDI NAIMI[1] AND MOHAMMED BENHARRAT[1*]

Communicated by A.M. Peralta

Abstract. Anderson's theorem states that if the numerical range of an $n\times n$ matrix is contained in the closed unit disk and intersects with the unit circle at more than n points, then the numerical range coincides with the closed unit disk. In an infinite-dimensional setting, an analogue of this result for a compact operator was established by Gau and Wu and for operator being the sum of a normal and compact operator by Birbonshi et al. We consider here three classes of operators: Operators being the sum of compact and operator with numerical radius strictly less than 1, operators with essentially numerical range coinciding with the convex hull of its essential spectrum, and quasicompact operators.

1. Introduction

Let H be an infinite-dimensional complex Hilbert space and let A be a bounded linear operator on H. The numerical range and the numerical radius of A are defined by

$$W(A) := \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$$

and

$$w(A) := \sup\{|z| : z \in W(A)\},$$

respectively. It is well known that $W(A)$ is a convex set in the complex plane $\mathbb{C}$, and its closure contains the spectrum of A. Also,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|,$$

and hence the numerical radius defines an equivalent norm in the Banach algebra $B(H)$, the set of all bounded operators in H. For proofs of these facts and further

Date: Received: 10 August 2019; Revised: 30 January 2020; Accepted: 17 February 2020.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47A10; 47A56.
Key words and phrases. Numerical range, essentially numerical range, essentially normal operator, hyponormal operator, quasicompact operator.

background on numerical range, we refer the reader to the book of Gustafson and Rao [8]. One of the interesting and important problems of operator theory is the determination of the numerical range of the concrete operator and calculation of its numerical radius. In this paper, we will concentrate on the investigations concerning the Anderson's theorem for operators in the infinite-dimensional case. Anderson's theorem states that if A is an $n \times n$ matrix whose numerical range $W(A)$ is contained in the closed unit disc $\overline{\mathbb{D}}$ ($\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$) with their boundaries $\partial W(A)$ and $\partial \mathbb{D}$ intersecting at more than n points, then $W(A)$ and $\overline{\mathbb{D}}$ coincide (see [10, p. 507]). For an interesting survey about this theorem, see the third section of [7]. Gau and Wu [6] extended this result to compact operators in the following theorem.

Theorem 1.1 ([6, Theorem 1])**.** *Let A be a compact operator on a Hilbert space H. If $W(A) \subseteq \overline{\mathbb{D}}$ and the intersection $\overline{W(A)} \cap \partial \mathbb{D}$ is infinite, then $W(A) = \overline{\mathbb{D}}$.*

Recently the above result was generalized by Birbonshi et al. to operators being the sum of a normal and compact operator.

Theorem 1.2 ([1, Theorem 4])**.** *Let $A = N + K$, where N is normal and K is a compact operator on a Hilbert space H. If $W(A) \subseteq \overline{\mathbb{D}}$ and the intersection $\overline{W(A)} \cap \partial \mathbb{D}$ is infinite while the essential spectrum of A, $\sigma_{ess}(A)$ is contained in $\mathbb{D}$, then $W(A) = \overline{\mathbb{D}}$.*

Our goal in this paper is to generalize the Anderson's theorem to a wider class of operators in the infinite-dimensional setting. More precisely, we consider three classes of operators:

- Operators being the sum of compact and operator with numerical radius strictly less than one.
- Operators with essentially numerical range coinciding with convex hull of its essential spectrum.
- Quasicompact operators.

2. Main results

Recall that an operator $A \in B(H)$ is called Fredholm if the dimension of the kernel $N(A)$, (noted $\alpha(A)$), and the codimension of its range $R(A)$, (noted $\beta(A)$), are finite. In that case, the Fredholm index is defined by

$$ind(A) = \alpha(A) - \beta(A).$$

An operator is called Weyl if it is Fredholm with index zero. The set of complex numbers λ such that the operator $A - \lambda I$ is not Fredholm is denoted by $\sigma_{ess}(A)$ and called the essential spectrum of A. In what follows, we denote by $\partial \mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ the unidimensional torus.

Let us start with the following result.

Theorem 2.1. *Let $A \in B(H)$ be such that $A = C + K$, where C is an operator with $w(C) < 1$ and K is a compact operator on a Hilbert space H. If $W(A) \subseteq \overline{\mathbb{D}}$ and γ is a closed arc of $\partial \mathbb{D}$ such that the intersection $\overline{W(A)} \cap \gamma$ is infinite, then $\gamma \subset W(A)$.*

Proof. Let d_A be the support function of the compact convex set $\overline{W(A)}$:

$$d_A(\theta) = \max \overline{W(\Re(e^{-i\theta}A)}, \qquad \theta \in \mathbb{R}, \tag{2.1}$$

where $\Re(A) = (A + A^*)/2$ is the real part of A. The condition $W(A) \subseteq \overline{\mathbb{D}}$ is equivalent to $d_A(\theta) \leq 1$ for all $\theta \in \mathbb{R}$, we have also that $d_A(\theta) = 1$ if and only if $e^{i\theta} \in \overline{W(A)}$. By the condition imposed on $\gamma \cap \overline{W(A)}$, we can take a sequence $e^{i\theta_n}$, $n \geq 1$, $\theta_n \in [0, 2\pi)$ of distinct points in

$$\Gamma = \{e^{i\theta} \in \gamma : d_A(\theta) = 1\}.$$

We may assume that $e^{i\theta_n}$ converges to $e^{i\theta_0}$ for some $\theta_0 \in [0, 2\pi]$; thus $d_A(\theta_0) = 1$, which is contained in the convex hull of the spectrum of the operator $\Re(e^{-i\theta_0}A)$.

It is clear that $\Re(\overline{W(e^{-i\theta}C))} = W(\Re e^{-i\theta}C)$ for every $\theta \in \mathbb{R}$. On the other hand, $\sigma_{ess}(\Re(e^{-i\theta}C)) \subseteq \overline{W(\Re(e^{-i\theta}C)}$ for all $\theta \in \mathbb{R}$. The assumption $w(C) < 1$ clearly implies that $\sigma_{ess}(\Re(e^{-i\theta}C))$ is contained in the open real segment $]-1, 1[$. Note that the essential spectrum is invariant under additive compact perturbations, so that $\sigma_{ess}(\Re(e^{-i\theta}A)) = \sigma_{ess}(\Re(e^{-i\theta}C))$. We infer that $1 \notin \sigma_{ess}(\Re(e^{-i\theta}A))$ is an isolated eigenvalue of $\Re(e^{-i\theta}A)$.

According now to [9, Theorem 3.3], for $e^{i\theta} \in \Gamma$, there exist a neighborhood U_θ of $e^{i\theta}$ and two open analytic arcs $e^{i\theta} \in \gamma_j(\theta), j = 1, 2$ such that

$$\partial W(A) \cap U_\theta \subset \gamma_1(\theta) \cup \gamma_2(\theta) \subset W(A).$$

If $e^{i\theta} \in \Gamma'$, where is Γ' is the set of the limit points of Γ, without loss of generality, we may assume that $\gamma_1(\theta)$ contains infinitely many points of the unit circle, and thus $\gamma_1(\theta)$ lies in $\partial\mathbb{D}$. Therefore, $\gamma_1(\theta) \subset W(A) \cap \Gamma$. As the whole arc $\gamma_1(\theta)$ is a subset of Γ, we infer that $e^{i\theta}$ is an interior point of Γ'. Then Γ' is not only closed but also open in γ, so $\Gamma' = \gamma$ and $\Gamma = \gamma$, then $\gamma \subset W(A)$. □

Corollary 2.2. *Let $A \in B(H)$ be such that $A = C + K$, where C is an operator with $w(C) < 1$ and K is a compact operator on a Hilbert space H. If $W(A) \subseteq \overline{\mathbb{D}}$ and the intersection $\overline{W(A)} \cap \partial\mathbb{D}$ is infinite, then $W(A) = \overline{\mathbb{D}}$.*

Proof. It follows immediately from Theorem 2.1 that if we take $\gamma = \partial\mathbb{D}$, then $\partial\mathbb{D} \subset W(A)$. Since the numerical range is always convex, then $\overline{\mathbb{D}} \subset W(A)$, and hence $\overline{\mathbb{D}} = W(A)$. □

The fact that $K(H)$, the set of all compact operators on H, is a closed two-sided ideal in $B(H)$, enables us to define the Calkin algebra over H as the quotient algebra $C(H) = B(H)/K(H)$. We shall use π to denote the natural homomorphism of $B(H)$ onto $C(H)$; $\pi(A) = A + K(H)$, $A \in B(H)$.

The essential numerical range $W_e(A)$ is (by the definition) the numerical range of the operator $\pi(A)$ in $C(H)$; see [5]. Equivalently,

$$W_e(A) = \cap \overline{W(A + K)},$$

where the intersection runs over the compact operators K. It follows that $W_e(A)$ is a compact convex set containing $\sigma_{ess}(A)$, and invariant under compact perturbations. Also, the essential numerical range obeys the projection property. For instance, for an operator $A \in B(H)$, we have $\Re(W_e(A)) = W_e(\Re(A))$. Let $co(M)$ denote the convex hull of a subset M of $\mathbb{C}$. We have the following result.

Theorem 2.3. *Let $A \in B(H)$ be such that $W_e(A) = co(\sigma_{ess}(A))$. If $W(A) \subseteq \overline{\mathbb{D}}$ and γ is a closed arc of $\partial\mathbb{D}$ such that the intersection $\overline{W(A)} \cap \gamma$ is infinite while $\sigma_{ess}(A) \cap \gamma = \emptyset$, then $\gamma \subset W(A)$.*

Proof. By the condition imposed on $\gamma \cap \overline{W(A)}$, we may infer that $\Gamma = \{e^{i\theta} \in \gamma : d_A(\theta) = 1\}$ has a nonvoid set of the limit points Γ'. Since $co(\sigma_{ess}(A)) = W_e(A)$ and $co(\sigma_{ess}(e^{-i\theta}A)) = e^{-i\theta}co(\sigma_{ess}(A))$, then $co(\sigma_{ess}(e^{-i\theta}A)) = W_e(e^{-i\theta}A)$ for any $\theta \in [0, 2\pi)$. We have also

$$\Re(co(\sigma_{ess}(e^{-i\theta}A)) = \Re(W_e(e^{-i\theta}A)) = W_e(\Re(e^{-i\theta}A)) = co(\sigma_{ess}(\Re(e^{-i\theta}A))).$$

Now, let $e^{i\theta} \in \Gamma$. As $\sigma_{ess}(A) \cap \gamma = \emptyset$ and $\sigma_{ess}(A) \subseteq \overline{\mathbb{D}}$, we conclude that $\gamma \cap co(\sigma_{ess}(A)) = \emptyset$. Observe that $1 \in \sigma(\Re e^{-i\theta}A)$, but not in $\sigma_{ess}(e^{-i\theta}A)$, then $1 \notin co(\sigma_{ess}(\Re(e^{-i\theta}A)))$. Thus, 1 is an isolated eigenvalue of $\Re(e^{-i\theta}A)$ of finite multiplicity. Moreover, it is apparent that $\Gamma \subset W(A)$. Using similar arguments as in the proof of Theorem 2.1, we can now derive that $\gamma \subset W(A)$. □

Corollary 2.4. *Let $A \in B(H)$ be such that $W_e(A) = co(\sigma_{ess}(A))$. If $W(A) \subseteq \overline{\mathbb{D}}$ and the intersection $\partial\mathbb{D} \cap \overline{W(A)}$ is infinite while $\sigma_{ess}(A) \subset \mathbb{D}$, then $W(A) = \overline{\mathbb{D}}$.*

Remark 2.5.
(1) An example of operators satisfying $W_e(A) = co(\sigma_{ess}(A))$, are essentially normal operators, that is, operators such that $A^*A - AA^*$ is compact.
(2) It is clear that if A is normal plus compact, then A is essentially normal, so our theorem generalizes [1, Theorem 4.]. However, its the proof is inspired from [1].
(3) Not all essentially normal operators are normal plus compact. For example, the unilateral shift S defined on l^2_+ by $Se_n = e_{n+1}$, where $\{e_n, n \geq 0\}$ is an orthonormal basis, satisfies

$$S^*S - SS^* = I - SS^* = e_0e_0^*.$$

Hence $S^*S - SS^*$ is of finite rank, so compact. Thus the unilateral shift S is essentially normal. Furthermore, it has a nonzero Fredholm index,

$$ind(S) = \alpha(S) - \beta(S) = -1.$$

But, it well known that every normal Fredholm operator is Weyl. Also, the index is invariant under compact perturbations, so the same persists for normal plus compact operators. Consequently, S cannot be a normal plus compact.

An operator $A \in B(H)$ is said to be hyponormal if $A^*A - AA^* \geq 0$. Thus the notion of hyponormality can be viewed as a generalization of normality. It is well known that a hyponormal operator can have a nonclosed numerical range, for example the numerical range of the unilateral shift S on l^2_+ is the open unit disk.

Corollary 2.6. *Let $A = T + K$, where T is hyponormal and K is a nonzero compact operator on a Hilbert space H. If $W(A) \subseteq \overline{\mathbb{D}}$ and γ is a closed arc of $\partial\mathbb{D}$ such that the intersection $\overline{W(A)} \cap \gamma$ is infinite while $\sigma_{ess}(A) \cap \gamma = \emptyset$, then $\gamma \subset W(A)$.*

Proof. Since $\sigma_{ess}(A) \subset W_e(A)$, by [11, Lemma 2.9], we can deduce that $W_e(A) = co(\sigma_{ess}(A))$, then it follows from Theorem 2.3 that $\gamma \subset W(A)$. □

Remark 2.7. The compact operator K in the last corollary must be nonzero. Indeed, if $K = 0$ and $\gamma \subset W(T)$, then $\gamma \subset W(T) \subset co(\sigma(T))$ for a hyponormal operator T. Therefore $\gamma \subset \partial\sigma(T)$, by [11, Lemma 2.1], we infer that $\gamma \subset \sigma_{ess}(T)$ and this contradicts the hypothesis.

Remark 2.8. The condition that $\sigma_{ess}(A) \cap \gamma = \emptyset$ is essential. We can easily construct an operator A satisfying the conditions in the above corollary, for example, $A = K \oplus \alpha S$, where

$$K = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$

S is the unilateral shift and $0 < \alpha < 1$. It is obvious that $W(A) = W(K)$, then $W(A)$ is the closed unit disk. Recall that here $\sigma_{ess}(\alpha S) = \alpha\partial\mathbb{D} \subset \mathbb{D}$. However, this is not the case if $A = \alpha K \oplus S$, here $\sigma_{ess}(A) = \sigma_{ess}(S) = \partial\mathbb{D} \not\subset \mathbb{D}$ and $W(A) = W(S) = \mathbb{D}$.

Definition 2.9 ([2, Defintion I.1])**.** An operator $A \in B(H)$ is called quasicompact if there exist a positive integer m and compact operator $K \neq 0$ on H such that $\|A^m - K\| < 1$.

We say that A is a Riesz operator if $\lambda I - A$ is Fredholm for all nonzero complex numbers λ. Certainly, we have the implications: Compact $\Rightarrow$ Riesz $\Rightarrow$ quasicompact; see [4]. The following theorem gives an important and useful characterization of quasicompact operators.

Theorem 2.10 ([2, Theorem I.6])**.** *If $A \in B(H)$ is quasicompact, then for all complex number λ such that $|\lambda| \geq 1$, $(\lambda I - A)$ is a Weyl operator that is a Fredholm operator of index zero.*

As a consequence of this theorem, A is quasicompact if (and only if) $\sigma(A)\cap\partial\mathbb{D}$ contains only finitely many poles of the resolvent $(\lambda I - A)^{-1}$, and if A is a power-bounded operator the residue of $(\lambda I - A)^{-1}$ at each peripheral pole is of finite rank. However, the concept of quasicompact operator plays a crucial role and seems to be more appropriate, because it evokes not only the special configuration of the spectrum of A, but also the fact that the spectral values of greatest modulus are eigenvalues associated with finite-dimensional generalized eigenspaces.

Now, Anderson's theorem for this class read as follows.

Theorem 2.11. *Let A be a quasicompact operator on a Hilbert space H. If $W(A) \subset \overline{\mathbb{D}}$ and $\overline{W(A^m)}$ intersects $\partial\mathbb{D}$ at infinitely many points with m is the positive integer given in Definition 2.9, then $W(A^m) = \overline{\mathbb{D}}$ and $\partial\mathbb{D} \subset W(A)^m$, where $W(A)^m = \{z^m : z \in W(A)\}$.*

To achieve our goal, we need the following basic result, which is essentially due to Choi and Li [3].

Lemma 2.12. *Suppose $A \in B(H)$ and that $\phi = e^{2\pi i/k}$ for a positive integer $k > 1$. Let $\tilde{A} = A \oplus \phi A \oplus \cdots \oplus \phi^{k-1}A$. Then*

$$W(\tilde{A})^k = \left\{co[\cup_{j=1}^{k}\phi^j W(A)]\right\}^k \tag{2.2}$$

is a convex set satisfying the following inclusion:

$$W(A^k) = W(\tilde{A}^k) \subseteq W(\tilde{A})^k. \tag{2.3}$$

Proof of Theorem 2.11. By the power inequality, $w(A^m) \leq w(A)^m$ for any positive integer m, $W(A^m) \subset \overline{\mathbb{D}}$. Moreover A being quasicompact means that $A^m = C + K$, where C is an operator such that $\|C\| < 1$ and K is a compact operator. By Corollary 2.2, we conclude that $W(A^m) = \overline{\mathbb{D}}$. Finally taking into account Lemma 2.12, we get from (2.2) and (2.3) that

$$W(A^m) \subseteq \left\{co[\cup_{j=1}^m \phi^j W(A)]\right\}^m;$$

then

$$W(A^m) \subseteq \left\{co[\cup_{j=1}^m \phi^j W(A)]\right\}^m \subseteq \overline{\mathbb{D}}.$$

Moreover, while $W(A^m) = \overline{\mathbb{D}}$, then

$$\left\{co[\cup_{j=1}^m \phi^j W(A)]\right\}^m = \overline{\mathbb{D}}.$$

This implies

$$\partial\mathbb{D} \cap \left\{co[\cup_{j=1}^m \phi^j W(A)]\right\}^m = \partial\mathbb{D}.$$

Then

$$\partial\mathbb{D} \cap \left\{\cup_{j=1}^m \phi^j W(A)\right\}^m = \partial\mathbb{D}$$

and, as $(\phi^j)^m = 1$,

$$\partial\mathbb{D} \cap (W(A)^m) = \partial\mathbb{D}.$$

□

Remark 2.13.

(1) Since a compact operator is quasicompact, then this theorem may be viewed as a generalization of Theorem 1.1. Also this theorem remains valid if we take the operator A in the one of the following classes.

(2) An operator $A \in B(H)$ is said to be power compact if there exists a positive integer N such that A^N is compact. Also, $A \in B(H)$ is called polynomially compact if there exists a nonzero complex polynomial $p \in \mathbb{C}[x]$ such that the operator $p(A)$ is compact. Certainly we have

compact $\Longrightarrow$ power compact $\Longrightarrow$ polynomially compact $\Longrightarrow$ quasicompact

These implications are strict.

(3) An operator $A \in B(H)$ is said to be quasinilpotent if $\|A^n\|^{1/n} \underset{n\to\infty}{\longrightarrow} 0$ and is said to be nilpotent if $A^n = 0$ for some n. So a quasinilpotent is also quasicompact.

As in [1] and [6], the results of the paper remain valid with $\partial\mathbb{D}$ and $\mathbb{D}$ replaced by an arbitrary elliptical disk and its boundary, respectively. In order to see that, it suffices to consider a suitable affine transformation

$$aA + bA^* + cI$$

of A in place of A itself.

Acknowledgement. The authors are grateful to anonymous reviewers for valuable remarks and comments, which significantly contributed to the quality of the paper. This work was supported by the Laboratory of Fundamental and

Applicable Mathematics of Oran (LMFAO) and the Algerian research project: PRFU, no. C00L03ES310120180002.

References

1. R. Birbonshi, I. M. Spitkovsky and P.D. Srivastava, *A note on Anderson's theorem in the infinite-dimensional,* J. Math. Anal. Appl. **461** (2018), no.1, 349–353.
2. A. Brunel and D. Revuz, *Quelques applications probabilistes de la quasi-compacité*, Ann. Inst. H. Poincaré Sect. B (N.S.) **10** (1974) 301–337.
3. M-D. Choi and C-K. Li, *Numerical Ranges of the Powers of an Operator*, J. Math. Anal. Appl. **365** (2010) 458–466.
4. J.F. Feinstein and H. Kamowitz, *Quasicompact and Riesz endomorphisms of Banach algebras*, J. Funct. Anal. **225** (2005), no. 2, 427–438.
5. P.A. Fillmore, J.G. Stampfli and J.P. Williams, *On the essential numerical range, the essential spectrum, and a problem of Halmos*, Acta Sci. Math. (Szeged) **33** (1972) 179–192.
6. H.-L. Gau and P.Y. Wu, *Anderson's theorem for compact operators*, Proc. Amer. Math. Soc. **134** (11) (2006) 3159–3162.
7. H.-L. Gau and P.Y. Wu, *Excursions in numerical ranges*, Bull. Inst. Math. Acad. Sin. (N.S.) **9** (2014), no. 3, 351–370.
8. K.E. Gustafson and K.M. Rao, *Numerical Range. The Field of Vlues of Linear Operators and Matrices*, Springer-Verlag, New York, 1997.
9. F.J. Narcowich, *Analytic properties of the boundary of the numerical range*, Indiana Univ. Math. J. **29** (1980), no. 1, 67–77.
10. M. Radjabalipour and H. Radjavi, *On the geometry of numerical ranges*, Pacific J. Math. **61** (1975) 507–511.
11. S. Zhu, *Approximate unitary equivalence of normaloid type operators*, Banach J. Math. Anal. **9** (2015), no. 3, 173–193.

[1] Department of Systems Engineering, National Polytechnic School of Oran-Maurice Audin (Ex. ENSET of Oran), BP 1523 Oran-El M'naouar, 31000 Oran, Algeria.

Email address: mehdi.naimi@univ-mosta.dz

Email address: mohammed.benharrat@enp-oran.dz, mohammed.benharrat@gmail.com