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EMBEDDING TOPOLOGICAL SPACES IN A TYPE OF GENERALIZED TOPOLOGICAL SPACES

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Abstract. A stack on a nonempty set X is a collection of nonempty subsets of X that is closed under the operation of superset. Let (X, τ) be an arbitrary topological space with a stack $\mathcal{S}$, and let $X^* = X \cup \{p\}$ for $p \notin X$. In the present paper, using the stack $\mathcal{S}$ and the topological closure operator associated to the space (X, τ), we define an envelope operator on X^* to construct a generalized topology $\mu_{\mathcal{S}}$ on X^*. We then show that the space $(X^*, \mu_{\mathcal{S}})$ is the generalized extension of the space (X, τ). We also provide conditions under which $(X^*, \mu_{\mathcal{S}})$ becomes a generalized Hausdorff space.

1. Introduction and preliminaries

The concept of generalized open sets, which led to the formation of a new topological structure called generalized topology, first was appeared in Levin's 1963 attempt in the paper "Semi-open Sets and Semi-continuity in Topological Spaces" when he tried to generalize a topology by replacing open sets with semi-open sets; see [6]. From that date until 1997 a lots of work were done in the field of generalization of open sets as well as topology, for example, some of these efforts appeared in the form of δ-open sets [12], θ-open sets [12], preopen sets [8], α-open sets [9], feebly open sets [7] and β-open sets [1]. Investigation of generalized open sets was continued until Á. Császár [3] in 1997 stated a universal definition of these concepts in the form of γ-open set. This object later inspired Császár provides a basic framework for defining the concept of generalized topology.

Now, let us recall some of notions related to a generalized topological space from [3, 4, 5].

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Let X be an arbitrary set and let $\mathcal{P}(X)$ be the set of all subsets of X, equipping X with a generalized topology means to choose a collection $\mu \subseteq \mathcal{P}(X)$ that is closed under arbitrary union, and contains the empty set. The pair (X, μ) is called a generalized topological space. Those subsets of X, which are members of μ, are called μ-open (sub)set in the generalized space X. A subset $F \subseteq X$ is called μ-closed in (X, μ) if its complement $X \setminus F$ is a μ-open set. For $A \subseteq X$, the union of all μ-open subsets of A is μ-open and denoted by $int_\mu A$. Similarly, the intersection of all μ-closed supersets of A is μ-closed and denoted by $cl_\mu A$. Also, for $A \subseteq X$, an operator $f : \mathcal{P}(X) \to \mathcal{P}(X)$ is extensive if and only if $A \subseteq f(A)$, idempotent if and only if $f(f(A)) = f(A)$ and monotonous if and only if $A \subseteq B \subseteq X$ implies $f(A) \subseteq f(B)$. An operator that is extensive, monotonous, and idempotent is called an envelope operator.

Below, we present some concepts related to the generalized topology that are used throughout the paper.

Definition 1.1 ([2])**.** A generalized space (X, μ) is said to be generalized Hausdorff or simply μ-Hausdorff if for any two distinct points $x, y \in X$, there exist two disjoint μ-open sets U and V containing x and y, respectively.

Definition 1.2 ([13])**.** Suppose that μ is a generalized topology on X and that A is a nonempty subset of X. The generalized subspace topology of X on A is the generalized topology $\mu_A = \{A \cap U : U \in \mu\}$ on A. The pair (A, μ_A) is called a generalized subspace of the generalized space (X, μ).

Definition 1.3. Let (X, μ) is a generalized topological space and let A be a nonempty subset of X. We call A a generalized dense subset of X, if $cl_\mu A = X$.

Definition 1.4. A generalized topological space Y is a generalized extension of a (generalized) topological space X if Y contains X as a generalized dense subspace and also we call a generalized extension Y, a one-point generalized extension of X if $Y \setminus X$ is a singleton.

Since stacks have an important role in this paper, we provide an explanation in the form of an example after its definition.

Definition 1.5 ([11])**.** A nonnull collection $\mathcal{S}$ of nonempty subsets of a set X is a stack on X if every superset of a member of $\mathcal{S}$ is also a member of $\mathcal{S}$.

For a topological space (X, τ), the following collections are apparently examples of stacks on X: The collection of all uncountable subsets of X, the collection of all subsets of X whose closure has nonempty interior, for $p \in X$, the collection $\{A \subseteq X; p \in A\}$, for $A \subseteq X$ the collection $\{B \subseteq X : B \cap A \neq \emptyset\}$, and the collection of all dense subsets of X. In fact, except the collection of dense subsets of X other examples are grills (a stack is called a grill on X if $A \cup B \in \mathcal{S}$ implies $A \in \mathcal{S}$ or $B \in \mathcal{S}$; see [11]) on X.

Throughout the paper, for a topological space (X, τ) and $A \subseteq X$, the interior and closure of A in the space (X, τ) is denoted by $int_\tau A$ (in short, $intA$) and $cl_\tau A$ (in short, clA). Also, the X-complement of A means $X \setminus A$.

2. Main results

In this section, we are going to exhibit an interesting way of construction of a one-point generalized extension of a topological space. For this purpose we first consider a topological space (X, τ) with a stack $\mathcal{S}$ on it, and then with the help of the topological closure operator associated to the space (X, τ) and the stack $\mathcal{S}$, we define an envelope operator on the set $X^* = X \cup \{p\}$ (for $p \notin X$). Hence according to [5], we will have a generalized topology, say, $\mu_{\mathcal{S}}$ on X^* that was induced by the envelope operator. The space $(X^*, \mu_{\mathcal{S}})$ is the generalized extension of the space (X, τ), for which we show that the generalized space $(X^*, \mu_{\mathcal{S}})$ contains the space (X, τ) as a generalized dense subspace. We then introduce conditions under which the space $(X^*, \mu_{\mathcal{S}})$ becomes a generalized Hausdorff space. Finally, we close our paper with examples of one-point generalized extension of some topological spaces.

Now let us introduce and describe the structure of the envelope operator in question.

Let (X, τ) be a topological space and let $\mathcal{S}$ be a stack on X. For $p \notin X$, put $X^* = X \cup \{p\}$ and define the operator $f_{\mathcal{S}} : \mathcal{P}(X^*) \to \mathcal{P}(X^*)$, by

$$f_{\mathcal{S}}(A) = \begin{cases} clA & if\ clA \notin \mathcal{S},\ for\ A \subseteq X, \\ clA \cup \{p\} & if\ clA \in \mathcal{S},\ for\ A \subseteq X, \\ cl(A \setminus \{p\}) \cup \{p\} & if\ p \in A. \end{cases}$$

In the first step, we have the following proposition regarding the envelope operator $f_{\mathcal{S}}$.

Proposition 2.1. *For a topological space (X, τ) and a stack $\mathcal{S}$ on X, the operator $f_{\mathcal{S}}$ has the following property:*

(1) *For $A, B \subseteq X^*$ that $p \in A, B$, we have $f_{\mathcal{S}}(A \cup B) = cl((A \cup B) \setminus \{p\}) \cup \{p\} = cl(A \setminus \{p\}) \cup cl(B \setminus \{p\}) \cup \{p\} = f_{\mathcal{S}}(A) \cup f_{\mathcal{S}}(B)$.*
(2) *For $A \subseteq X$ and $p \in B$, if $clA \notin \mathcal{S}$, then $f_{\mathcal{S}}(A \cup B) = cl((A \cup B) \setminus \{p\}) \cup \{p\} = cl(A \cup (B \setminus \{p\})) \cup \{p\} = clA \cup cl(B \setminus \{p\}) \cup \{p\} = f_{\mathcal{S}}(A) \cup f_{\mathcal{S}}(B)$, and also, if $clA \in \mathcal{S}$, then $f_{\mathcal{S}}(A \cup B) = cl((A \cup B) \setminus \{p\}) \cup \{p\} = clA \cup \{p\} \cup cl(B \setminus \{p\}) \cup \{p\} = f_{\mathcal{S}}(A) \cup f_{\mathcal{S}}(B)$.*
(3) *For $A, B \subseteq X$, if $cl(A \cup B) \notin \mathcal{S}$, then since $\mathcal{S}$ is a stack, thus $clA, clB \notin \mathcal{S}$, hence $f_{\mathcal{S}}(A \cup B) = cl(A \cup B) = clA \cup clB = f_{\mathcal{S}}(A) \cup f_{\mathcal{S}}(B)$, but if $cl(A \cup B) \in \mathcal{S}$, then as shown in the next example, $f_{\mathcal{S}}$ cannot preserve finite unions, that is, in this case $f_{\mathcal{S}}(A \cup B) \neq f_{\mathcal{S}}(A) \cup f_{\mathcal{S}}(B)$.*

Example 2.2. Take (X, τ) the real line with the standard topology, and $\mathcal{S} = \{A \subseteq X : clA = X\}$. For $A = (-\infty, 0]$ and $B = [0, \infty)$, we have $cl(A \cup B) \in \mathcal{S}$ but $clA \notin \mathcal{S}$ and $clB \notin \mathcal{S}$. Thus $f_{\mathcal{S}}(A \cup B) = \mathcal{R}^* = \mathcal{R} \cup \{p\}$, $f_{\mathcal{S}}(A) = (-\infty, 0]$ and $f_{\mathcal{S}}(B) = [0, \infty)$. Therefore $\mathcal{R}^* = f_{\mathcal{S}}(A \cup B) \neq f_{\mathcal{S}}(A) \cup f_{\mathcal{S}}(B) = \mathcal{R}$.

As we saw, Example 2.2 shows that the operator $f_{\mathcal{S}}$ cannot preserve the finitely additive property, which means that this operator cannot satisfy the Kuratowski closure axioms and so, its induced topological structure cannot be topology.

Next theorem states that for any stack $\mathcal{S}$ on a topological space (X, τ), the operator $f_{\mathcal{S}}$ is an envelope operator on X^*.

Theorem 2.3. *Let (X, τ) be a topological space and let $\mathcal{S}$ be a stack on X. For $p \notin X$, put $X^* = X \cup \{p\}$. Then the operator $f_{\mathcal{S}} : \mathcal{P}(X^*) \to \mathcal{P}(X^*)$, defined by*

$$f_{\mathcal{S}}(A) = \begin{cases} clA & if\ clA \notin \mathcal{S},\ for\ A \subseteq X, \\ clA \cup \{p\} & if\ clA \in \mathcal{S},\ for\ A \subseteq X, \\ cl(A \setminus \{p\}) \cup \{p\} & if\ p \in A, \end{cases}$$

is an extensive, monotonous, and idempotent operator, inducing a generalized topology say $\mu_{\mathcal{S}}$ on X^*, such that $(X^*, \mu_{\mathcal{S}})$ contains X as a generalized dense subspace. Moreover, every τ-open set is $\mu_{\mathcal{S}}$-open, that is, $\tau \subseteq \mu_{\mathcal{S}}$.

Proof. If $A \subseteq X^*$, then $A \subseteq f_{\mathcal{S}}(A)$ is evident, so $f_{\mathcal{S}}$ is extensive.
Now, we verify that for any $A \subseteq B \subseteq X^*$, $f_{\mathcal{S}}(A) \subseteq f_{\mathcal{S}}(B)$, proving that $f_{\mathcal{S}}$ is monotone.

Case (1): $A \subseteq B \subseteq X$.

If $clA \in \mathcal{S}$, then since $\mathcal{S}$ is a stack on X, thus $clB \in \mathcal{S}$ and therefore $f_{\mathcal{S}}(A) = clA \cup \{p\} \subseteq clB \cup \{p\} = f_{\mathcal{S}}(B)$.

If $clB \in \mathcal{S}$ but $clA \notin \mathcal{S}$, then $f_{\mathcal{S}}(A) = clA \subseteq clB \cup \{p\} = f_{\mathcal{S}}(B)$.

If $clB, clA \notin \mathcal{S}$, then $f_{\mathcal{S}}(A) = clA \subseteq clB = f_{\mathcal{S}}(B)$.

Case (2): $A \subseteq X$ and $p \in B \supseteq A$. Then $f_{\mathcal{S}}(B) = cl(B \setminus \{p\}) \cup \{p\}$ and if $clA \notin \mathcal{S}$ (resp., $clA \in \mathcal{S}$), then $f_{\mathcal{S}}(A) = clA$ (resp., $f_{\mathcal{S}}(A) = clA \cup \{p\}$), in both clearly $f_{\mathcal{S}}(A) \subseteq f_{\mathcal{S}}(B)$.

Case (3): $p \in A \subseteq B$. Then $f_{\mathcal{S}}(A) = cl(A \setminus \{p\}) \cup \{p\} \subseteq cl(B \setminus \{p\}) \cup \{p\} = f_{\mathcal{S}}(B)$.

We next show that $f_{\mathcal{S}}$ is an idempotent operator, that is, $f_{\mathcal{S}}(f_{\mathcal{S}}(A)) = f_{\mathcal{S}}(A)$, for any $A \subseteq X^*$.

Case (1): $A \subseteq X$. If $clA \notin \mathcal{S}$, then $f_{\mathcal{S}}(f_{\mathcal{S}}(A)) = f_{\mathcal{S}}(clA) = clA = f_{\mathcal{S}}(A)$, while if $clA \in \mathcal{S}$, then $f_{\mathcal{S}}(f_{\mathcal{S}}(A)) = f_{\mathcal{S}}(clA \cup \{p\})$ (by Proposition 2.1(2))= $f_{\mathcal{S}}(clA) \cup f_{\mathcal{S}}(\{p\}) = clA \cup \{p\} = f_{\mathcal{S}}(A)$.

Case (2): $p \in A$. If $cl(A \setminus \{p\}) \in \mathcal{S}$, then $f_{\mathcal{S}}(f_{\mathcal{S}}(A)) = f_{\mathcal{S}}[cl(A \setminus \{p\}) \cup \{p\}]$ (by Proposition 2.1(2))= $f_{\mathcal{S}}[cl(A \setminus \{p\})] \cup f_{\mathcal{S}}(\{p\}) = cl(A \setminus \{p\}) \cup \{p\} = f_{\mathcal{S}}(A)$.

If $cl(A \setminus \{p\}) \notin \mathcal{S}$, then $f_{\mathcal{S}}(f_{\mathcal{S}}(A)) = f_{\mathcal{S}}[cl(A \setminus \{p\}) \cup \{p\}]$ (by Proposition 2.1(2))= $f_{\mathcal{S}}[cl(A \setminus \{p\})] \cup f_{\mathcal{S}}(\{p\}) = cl(A \setminus \{p\}) \cup \{p\} = f_{\mathcal{S}}(A)$.
Hence, we show that $f_{\mathcal{S}}$ is an envelope operator on X^*, so from [5], $f_{\mathcal{S}}$ induces a generalized topology $\mu_{\mathcal{S}}$ on X^* such that $cl_{\mu_{\mathcal{S}}} A = f_{\mathcal{S}}(A)$ for any $A \subseteq X^*$.

We now verify that every τ-open set is $\mu_{\mathcal{S}}$-open. Let $U (\subseteq X)$ be τ-open. Then $f_{\mathcal{S}}(X^* \setminus U) = cl[(X^* \setminus U) \setminus \{p\}] \cup \{p\} = cl(X \setminus U) \cup \{p\} = (X \setminus U) \cup \{p\} = X^* \setminus U$, so that U is $\mu_{\mathcal{S}}$-open.
In the following, we show that X is a generalized subspace of X^*. As every τ-open set U of X is $\mu_{\mathcal{S}}$-open set of X^*, by putting $V = U \in \tau \subseteq \mu_{\mathcal{S}}$, we can write $U = X \cap V$, which shows that for every τ-open set $U (\subseteq X)$, there exists a $\mu_{\mathcal{S}}$-open set V such that $U = V \cap X$.

Also we show that the intersection of every $\mu_{\mathcal{S}}$-open set of X^* with X is a τ-open set of X. Given any $\mu_{\mathcal{S}}$-open set U of X^*, then

$$f_{\mathcal{S}}(X^* \setminus U) = X^* \setminus U. \tag{2.1}$$

Now, if $p \notin U$, then $cl[(X^* \setminus U) \setminus \{p\}] \cup \{p\} = X^* \setminus U$ (by (2.1)), thus $cl(X \setminus U) \cup \{p\} = X^* \setminus U$, and hence $cl(X \setminus U) = X \setminus U$. So, $X \setminus U$ is τ-closed and thus $U(= U \cap X)$ is τ-open.
Now, if $p \in U$, then $cl(X^* \setminus U) = X^* \setminus U$ (using (2.1) and since $p \notin (X^* \setminus U)$). Hence $cl[(X \cup \{p\}) \cap (X^* \setminus U)] = (X \cup \{p\}) \cap (X^* \setminus U))$ which implies $cl[X \cap (X \setminus U)] = X \cap (X \setminus U))$, so $cl(X \setminus (U \cap X)) = X \setminus (U \cap X))$ therefore $U \cap X$ is τ-open.

Finally, as every stack on X contains X, thus $cl_{\mu_{\mathcal{S}}} X = X^*$, proving that X is generalized dense in $X^*_{\mathcal{S}}$. □

Herein, we need the following definition.

Definition 2.4 ([13])**.** A generalized space (X, μ) is said to be generalized T_1 or simply μ-T_1 if for any two distinct points $x, y \in X$ there exist two μ-open sets U_x and U_y such that $U_x \cap \{x, y\} = \{x\}$ and $U_y \cap \{x, y\} = \{y\}$, or equivalently every singleton is μ-closed in X.

The following theorem proposes conditions under which the space $(X^*, \mu_{\mathcal{S}})$ becomes a generalized T_1-space.

Theorem 2.5. *Let $\mathcal{S}$ be a stack on a T_1-space (X, τ) such that for every $x \in X$, $\{x\} \notin \mathcal{S}$. Adjoin to X a new object $p \notin X$. Then there exists a generalized topology on $X^* = X \cup \{p\}$ satisfying the following properties:*

(1) *X^* is generalized T_1.*
(2) *X is generalized dense in X^*.*

Proof. Let us consider the space $(X^*, \mu_{\mathcal{S}})$ as constructed in Theorem 2.3. Now for any $x \in X$, $f_{\mathcal{S}}(x) = \{x\}$ as $cl(\{x\}) = \{x\} \notin \mathcal{S}$, and $f_{\mathcal{S}}(p) = cl(\{p\} \setminus \{p\}) \cup \{p\} = \{p\}$ and this proves (1). Again, since $clX = X \in \mathcal{S}$, we have $f_{\mathcal{S}}(X) = clX \cup \{p\} = X \cup \{p\} = X^*$, proving (2). □

In the next theorem, we provide conditions under which the space $(X^*, \mu_{\mathcal{S}})$ becomes a generalized Hausdorff space.

Theorem 2.6. *Let $\mathcal{S}$ be a stack on a Hausdorff space (X, τ) such that for every $x \in X$, $\{x\} \notin \mathcal{S}$. If for every point $x \in X$, there is an open neighborhood U of x such that $clU \notin \mathcal{S}$, then one can construct a one-point generalized extension $X^* = X \cup \{p\}$ (where $p \notin X$) satisfying the following properties:*

(1) *X^* is generalized Hausdorff.*
(2) *X is generalized dense in X^*.*

Proof. We consider again the one-point generalized extension space $(X^*, \mu_{\mathcal{S}})$ in Theorem 2.3. Let x and y be two distinct points of X^*. If both of them lie in X, since (X, τ) is Hausdorff, there exist disjoint τ-open sets U and V in X contains them, respectively, and from Theorem 2.3, U and V are $\mu_{\mathcal{S}}$-open in X^*.

On the other hand, if $x \in X$ and $y = p$, then by the hypothesis, there is a τ-open neighborhood U of x such that $clU \notin \mathcal{S}$. Let $V = X^* \setminus U$. Now, since $clU \notin \mathcal{S}$, we have $f_{\mathcal{S}}(U) = U$. Thus V is a $\mu_{\mathcal{S}}$-open neighborhood of y in X^*. Consequently, U and V are the required disjoint $\mu_{\mathcal{S}}$-open neighborhoods of x and y, respectively, in X^*. Hence X^* is $\mu_{\mathcal{S}}$-Hausdorff, this proves part (1).

For (2), again $clX \in \mathcal{S}$ implies $cl_{\mu_{\mathcal{S}}}X = X^*$ (Repeat the proof of Theorem 2.5(2)). □

In the following examples, as the final endeavor, we will determine one-point generalized extensions of some topological spaces.

Example 2.7. Take (X, τ) to be a discrete topological space, and let $\mathcal{S}$ be a stack on it. Let p be a point such that $p \notin X$. Put $Y = X \cup \{p\}$. From the argument of Theorem 2.3, we verify that the generalized topology $\mu_{\mathcal{S}}$ on Y is $\mu_{\mathcal{S}} = \tau \cup \{A \subseteq Y : p \in A,\ cl_\tau(Y \setminus A) \notin \mathcal{S}\}$.

To determine $\mu_{\mathcal{S}}$ we notice that for all $A \subseteq Y$; ($A \in \mu_{\mathcal{S}}$ if and only if $A = int_{\mu_{\mathcal{S}}}A = Y \setminus cl_{\mu_{\mathcal{S}}}(Y \setminus A)$).

Let A be a subset of Y.

Case (1): $A \subseteq X$. Now, since $p \in Y \setminus A$, thus Theorem 2.3 yields $cl_{\mu_{\mathcal{S}}}(Y \setminus A) = cl_\tau((Y \setminus A) \setminus \{p\}) \cup \{p\} = Y \setminus A$ and this concludes that $int_{\mu_{\mathcal{S}}}A = A$. This shows that $\tau \subseteq \mu_{\mathcal{S}}$ (which is once proved in Theorem 2.3).

Case (2): Now, let $p \in A$ (thus $Y \setminus A \subseteq X$) and consider two cases.

First, $cl_\tau(Y \setminus A) \in \mathcal{S}$. By Theorem 2.3, we have $cl_{\mu_{\mathcal{S}}}(Y \setminus A) = cl_\tau(Y \setminus A) \cup \{p\} = (Y \setminus A) \cup \{p\}$ and this concludes that $int_{\mu_{\mathcal{S}}}A = A \setminus \{p\} \neq A$. Hence in this case, A cannot be $\mu_{\mathcal{S}}$-open in Y.

In the second case, let us assume that $cl_\tau(Y \setminus A) \notin \mathcal{S}$, therefore by using Theorem 2.3, we have $cl_{\mu_{\mathcal{S}}}(Y \setminus A) = cl_\tau(Y \setminus A) = (Y \setminus A)$ and this concludes that $int_{\mu_{\mathcal{S}}}A = A$. So A belongs to $\mu_{\mathcal{S}}$ in this case.

Consequently, $\mu_{\mathcal{S}} = \tau \cup \{A \subseteq Y : p \in A, cl_\tau(Y \setminus A) \notin \mathcal{S}\}$.

Example 2.8. We recall that a nowhere dense subset of a topological space is a set whose closure has empty interior. Now, let (X, τ) be a topological space. It is not difficult to check that the set $\mathcal{S}^* = \{A \subseteq X : int_\tau A \neq \emptyset\}$ is a stack on X. If in the previous example, we put $\mathcal{S} = \mathcal{S}^* = \{A \subseteq X : int_\tau A \neq \emptyset\}$, then

$$\begin{aligned}\mu_{\mathcal{S}} &= \tau \cup \{A \subseteq Y : p \in A, cl_\tau(Y \setminus A) \notin \mathcal{S}^*\}\\ &= \tau \cup \{A \subseteq Y : p \in A,\ int_\tau cl_\tau(Y \setminus A) = int_\tau cl_\tau(X \setminus (A \cap X)) = \emptyset\}.\end{aligned}$$

It says that $A \subseteq Y$ is $\mu_{\mathcal{S}}$-open set in $(Y, \mu_{\mathcal{S}})$ if and only if A is τ-open set in (X, τ) or A is a set containing p with this property that the X-complement of whose intersection with X is nowhere dense subset of (X, τ).

Definition 2.9. A strong generalized topological space (Y, μ) is a generalized topological space that $Y \in \mu$.

Every topological space is a strong generalized topological space but the reverse is not true in general.

Example 2.10. Note, by putting $\mathcal{S} = \{A \subseteq X : cl_\tau A = X\}$ as a stack on (X, τ) in Example 2.7, we have

$$\begin{aligned}\mu_{\mathcal{S}} &= \tau \cup \{A \subseteq Y : p \in A,\ cl_\tau(Y \setminus A) \notin \mathcal{S}\}\\ &= \tau \cup \{A \subseteq Y : p \in A,\ cl_\tau(Y \setminus A) = cl_\tau(X \setminus (A \setminus \{p\}) \neq X\}\\ &= \tau \cup \{A \subseteq Y : p \in A, int_\tau(A \setminus \{p\}) \neq \emptyset\}.\end{aligned}$$

Clearly, in all the above examples, the space $(Y, \mu_{\mathcal{S}})$ is a strong generalized topological space.

Remark 2.11. As an important note, we mention that the harvest of replacing the role of stacks by grills in our argument, is not a generalized topology, in fact what we make is a topology on X^*; see [10].

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