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HERMITE-HADAMARD TYPE INEQUALITIES FOR MAPPINGS WHOSE DERIVATIVES ARE $s-$CONVEX IN THE SECOND SENSE VIA FRACTIONAL INTEGRALS

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Abstract. In this paper we establish Hermite-Hadamard type inequalities for mappings whose derivatives are $s-$convex in the second sense and concave.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known that the Hermite-Hadamard inequality for convex function. Both inequalities hold in the reserved direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found; see, for example see ([1]- [21]).

Definition 1.1. ([18]) A function $f : [0, \infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

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for all $x, y \in [0,\infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$. This class of $s-$convex functions is usually denoted by K_s^2.

In ([15]) Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$convex functions in the second sense:

Theorem 1.2. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an $s-$convex function in the second sense ,where $s \in (0,1)$ and let $a, b \in [0,\infty)$, $a < b$. If $f' \in L^1([a,b])$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1} \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2)

The following results are proved by M.I.Bhatti et al. (see [8]).

Theorem 1.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $|f''|$ is convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a,b]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \qquad (1.3)\\ \leq\ & \frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}\left[\frac{|f''(a)|+|f''(b)|}{2}\right] \\ \leq\ & \frac{(b-a)^2}{\alpha+1}\beta(2,\alpha+1)\left[\frac{|f''(a)|+|f''(b)|}{2}\right] \end{aligned}$$

where β is Euler Beta function.

Theorem 1.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $p \in \mathbb{R}, p > 1$ such that $|f''|^{\frac{p}{p-1}}$ is convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \qquad (1.4)\\ \leq\ & \frac{(b-a)^2}{\alpha+1}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

where β is Euler Beta function.

Theorem 1.5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $q \geq 1$ such that $|f''|^q$ is convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \qquad (1.5)\\ \leq\ & \frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)}\left[\begin{array}{c} \left(\frac{2\alpha+4}{3\alpha+9}|f''(a)|^q + \frac{\alpha+5}{3\alpha+9}|f''(b)|^q\right)^{\frac{1}{q}} \\ +\left(\frac{\alpha+5}{3\alpha+9}|f''(a)|^q + \frac{2\alpha+4}{3\alpha+9}|f''(b)|^q\right)^{\frac{1}{q}} \end{array}\right]. \end{aligned}$$

Theorem 1.6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $p \in \mathbb{R}, p > 1$ with $q = \frac{p}{p-1}$ such that $|f''|^q$ is concave function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \\ \leq & \frac{(b-a)^2}{\alpha+1} \beta^{\frac{1}{p}} (p+1, \alpha p+1) \left| f''\left(\frac{a+b}{2} \right) \right| \end{aligned} \tag{1.6}$$

where β is Euler Beta function.

We will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further this paper.

Definition 1.7. Let $f \in L[a,b]$. The Reimann-Liouville integrals $J_{a^+}^\alpha f(x)$ and $J_{b^-}^\alpha f(x)$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$ is the Gamma function and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$ the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities, see [3]-[25].

In this paper, we establish fractional integral ineqalities of Hermite-Hadamard type for mappings whose derivatives are $s-$convex and concave.

2. Main results

In order to prove our main theorems we need the following lemma (see [8]).

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°,the interior of I. Assume that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a,b]$, then the following identity for fractional integral with $\alpha > 0$ holds:*

$$\begin{aligned} & \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \\ = & \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 t(1-t^\alpha) \left[f''(ta+(1-t)b) + f''((1-t)a+tb) \right] dt \end{aligned} \tag{2.1}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$.

Theorem 2.2. *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a twice differentiable function on I° and let $a, b \in I^{\circ}$ with $a < b$ and $f'' \in L[a, b]$. If $|f''|$ is $s-$convex in the second sense on I for some fixed $s \in (0, 1]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \right| \qquad (2.2) \\ \leq\; & \frac{(b-a)^2}{2(\alpha+1)} \left[\frac{\alpha}{(s+2)(\alpha+s+2)} + \beta(2, s+1) - \beta(\alpha+2, s+1) \right] \\ & \times \left[|f''(a)| + |f''(b)| \right] \end{aligned}$$

where β is Euler Beta function.

Proof. From Lemma 2.1 since $|f''|$ is $s-$convex in the second sense on I, we have

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \right| \\ \leq\; & \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 |t(1-t^{\alpha})| \left[|f''(ta + (1-t)b)| + |f''((1-t)a + tb)| \right] dt \\ \leq\; & \frac{(b-a)^2}{2(\alpha+1)} \left[\int_0^1 t(1-t^{\alpha}) \left[t^s |f''(a)| + (1-t)^s |f''(b)| \right] dt \right. \\ & \left. + \int_0^1 t(1-t^{\alpha}) \left[(1-t)^s |f''(a)| + t^s |f''(b)| \right] dt \right] \\ =\; & \frac{(b-a)^2}{2(\alpha+1)} \left[\int_0^1 t^{s+1}(1-t^{\alpha})\, dt + \int_0^1 t(1-t^{\alpha})(1-t)^s dt \right] \left[|f''(a)| + |f''(b)| \right] \\ =\; & \frac{(b-a)^2}{2(\alpha+1)} \left[\frac{\alpha}{(s+2)(\alpha+s+2)} + \beta(2, s+1) - \beta(\alpha+2, s+1) \right] \\ & \times \left[|f''(a)| + |f''(b)| \right] \end{aligned}$$

where we used the fact that

$$\int_0^1 t^{s+1}(1-t^{\alpha})\, dt = \frac{\alpha}{(s+2)(\alpha+s+2)}$$

and

$$\int_0^1 t(1-t^{\alpha})(1-t)^s dt = \beta(2, s+1) - \beta(\alpha+2, s+1)$$

which completes the proof. □

Remark 2.3. In Theorem 2.2 if we choose $s = 1$ then (2.2) reduces the inequality (1.3) of Theorem 1.3.

Theorem 2.4. *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a twice differentiable function on I°. Suppose that $a, b \in I^{\circ}$ with $a < b$ and $f'' \in L[a, b]$. If $|f''|^q$ is $s-$convex in the second sense on I for some fixed $s \in (0, 1]$, $p, q > 1$, then the following inequality for fractional integrals with $\alpha \in (0, 1]$ holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right| \leq \frac{(b-a)^2}{\alpha+1}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\left[\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right]^{\frac{1}{q}} \tag{2.3}$$

where β is Euler Beta function and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1, using the well known Hölder inequality and $|f''|^q$ is $s-$convex in the second sense on I, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 |t(1-t^{\alpha})|\left[|f''(ta+(1-t)b)|+|f''((1-t)a+tb)|\right]dt \\
\leq\ & \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 t^p(1-t^{\alpha})^p\,dt\right)^{\frac{1}{p}} \\
& \times\left[\left(\int_0^1 |f''(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}+\left(\int_0^1 |f''((1-t)a+tb)|^q\,dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 t^p(1-t^{\alpha})^p\,dt\right)^{\frac{1}{p}} \\
& \times\left[\begin{array}{c}\left(\int_0^1 (t^s|f''(a)|^q+(1-t)^s|f''(b)|^q)dt\right)^{\frac{1}{q}} \\ +\left(\int_0^1 ((1-t)^s|f''(a)|^q+t^s|f''(b)|^q)\,dt\right)^{\frac{1}{q}}\end{array}\right] \\
=\ & \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 t^p(1-t^{\alpha})^p\,dt\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(|f''(a)|^q\frac{1}{s+1}+|f''(b)|^q\frac{1}{s+1}\right)^{\frac{1}{q}} \\ +\left(|f''(a)|^q\frac{1}{s+1}+|f''(b)|^q\frac{1}{s+1}\right)^{\frac{1}{q}}\end{array}\right] \\
\leq\ & \frac{(b-a)^2}{\alpha+1}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\left[\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right]^{\frac{1}{q}}
\end{aligned}$$

where we used the fact that

$$\int_0^1 t^s dt=\int_0^1 (1-t)^s dt=\frac{1}{s+1}$$

and

$$\int_0^1 t^p(1-t^{\alpha})^p\,dt\leq\int_0^1 t^p(1-t)^{\alpha p}dt=\beta(p+1,\alpha p+1)$$

which completes the proof. ◻

Remark 2.5. In Theorem 2.4 if we choose $s=1$ then (2.3) reduces the inequality (1.4) of Theorem 1.4.

Theorem 2.6. *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a twice differentiable function on I°. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$. If $|f''|^q$ is $s-$convex in the second sense on I for some fixed $s \in (0, 1]$ and $q \geq 1$ then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \qquad (2.4) \\ \leq\ & \frac{\alpha (b-a)^2}{4(\alpha+1)(\alpha+2)} \\ & \times \left[\left(|f''(a)|^q \tfrac{2\alpha+4}{(s+2)(\alpha+s+2)} + |f''(b)|^q \tfrac{[\beta(2,s+1)-\beta(\alpha+2,s+1)](2\alpha+4)}{\alpha} \right)^{\frac{1}{q}} \right. \\ & \left. + \left(|f''(a)|^q \tfrac{[\beta(2,s+1)-\beta(\alpha+2,s+1)](2\alpha+4)}{\alpha} + |f''(b)|^q \tfrac{2\alpha+4}{(s+2)(\alpha+s+2)} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. From Lemma 2.1, using power mean inequality and $|f''|^q$ is $s-$convex in the second sense on I we have

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \\ \leq\ & \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 |t(1-t^\alpha)| \left[|f''(ta+(1-t)b)| + |f''((1-t)a+tb)| \right] dt \\ \leq\ & \frac{(b-a)^2}{2(\alpha+1)} \left(\int_0^1 t(1-t^\alpha) dt \right)^{1-\frac{1}{q}} \left[\left(\int_0^1 t(1-t^\alpha) |f''(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_0^1 t(1-t^\alpha) |f''((1-t)a+tb)|^q dt \right)^{\frac{1}{q}} \right] \\ \leq\ & \frac{(b-a)^2}{2(\alpha+1)} \left(\int_0^1 t(1-t^\alpha) dt \right)^{1-\frac{1}{q}} \\ & \times \left[\left(\int_0^1 \left[t^{s+1}(1-t^\alpha) |f''(a)|^q + t(1-t^\alpha)(1-t)^s |f''(b)|^q \right] dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_0^1 t(1-t^\alpha)(1-t)^s |f''(a)|^q + t^{s+1}(1-t^\alpha) |f''(b)|^q dt \right)^{\frac{1}{q}} \right] \\ =\ & \frac{(b-a)^2}{2(\alpha+1)} \left(\frac{\alpha}{2(\alpha+2)} \right)^{1-\frac{1}{q}} \\ & \times \left[\left(|f''(a)|^q \tfrac{\alpha}{(s+2)(\alpha+s+2)} + |f''(b)|^q \left[\beta(2,s+1) - \beta(\alpha+2,s+1) \right] \right)^{\frac{1}{q}} \right. \\ & \left. + \left(|f''(a)|^q \left[\beta(2,s+1) - \beta(\alpha+2,s+1) \right] + |f''(b)|^q \tfrac{\alpha}{(s+2)(\alpha+s+2)} \right)^{\frac{1}{q}} \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha (b-a)^2}{4(\alpha+1)(\alpha+2)} \\
&\quad \times \left[\left(|f''(a)|^q \tfrac{(2\alpha+4)}{(s+2)(\alpha+s+2)} + |f''(b)|^q \tfrac{[\beta(2,s+1)-\beta(\alpha+2,s+1)](2\alpha+4)}{\alpha} \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(|f''(a)|^q \tfrac{[\beta(2,s+1)-\beta(\alpha+2,s+1)](2\alpha+4)}{\alpha} + |f''(b)|^q \tfrac{2\alpha+4}{(s+2)(\alpha+s+2)} \right)^{\frac{1}{q}} \right]
\end{aligned}
$$

where we used the fact that

$$\int_0^1 t^{s+1} (1-t^\alpha)\, dt = \frac{\alpha}{(s+2)(\alpha+s+2)}$$

and

$$\int_0^1 t(1-t^\alpha)(1-t)^s\, dt = \beta(2, s+1) - \beta(\alpha+2, s+1)$$

which completes the proof. □

Remark 2.7. In Theorem 2.6 if we choose $s = 1$ then (2.4) reduces the inequality (1.5) of Theorem 1.5.

The following result holds for $s-$concavity.

Theorem 2.8. *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a twice differentiable function on I°. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$. If $|f''|^q$ is $s-$concave in the second sense on I for some fixed $s \in (0, 1]$ and $p, q > 1$, then the following inequality for fractional integrals with $\alpha \in (0, 1]$ holds:*

$$
\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \qquad (2.5)\\
&\leq \frac{(b-a)^2}{\alpha+1} \beta^{\frac{1}{p}}(p+1, \alpha p+1)\, 2^{\frac{s-1}{q}} \left| f''\left(\frac{a+b}{2}\right) \right|
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and β is Euler Beta function.

Proof. From Lemma 2.1 and using the Hölder inequality we have

$$
\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \right| \qquad (2.6)\\
&\leq \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 |t(1-t^\alpha)| \left[|f''(ta+(1-t)b)| + |f''((1-t)a+tb)| \right] dt \\
&\leq \frac{(b-a)^2}{2(\alpha+1)} \left(\int_0^1 t^p (1-t^\alpha)^p\, dt \right)^{\frac{1}{p}} \\
&\quad \times \left[\left(\int_0^1 |f''(ta+(1-t)b)|^q\, dt \right)^{\frac{1}{q}} + \left(\int_0^1 |f''((1-t)a+tb)|^q\, dt \right)^{\frac{1}{q}} \right]
\end{aligned}
$$

Since $|f''|^q$ is $s-$concave using inequality (1.2) we get (see [2])

$$\int_0^1 |f''(ta+(1-t)b)|^q\, dt \leq 2^{s-1} \left| f''\left(\frac{a+b}{2}\right) \right|^q \qquad (2.7)$$

and

$$\int_0^1 |f''((1-t)a+tb)|^q \, dt \leq 2^{s-1} \left| f''\left(\frac{b+a}{2}\right)\right|^q \tag{2.8}$$

Using (2.7) and (2.8) in (2.6), we have

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \\ \leq\ & \frac{(b-a)^2}{\alpha+1}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\, 2^{\frac{s-1}{q}} \left| f''\left(\frac{a+b}{2}\right)\right| \end{aligned}$$

which completes the proof. □

Remark 2.9. In Theorem 2.8 if we choose $s = 1$ then (2.5) reduces inequality (1.6) of Theorem 1.6.

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