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GENERALIZED PERIPHERALLY MULTIPLICATIVE MAPS BETWEEN REAL LIPSCHITZ ALGEBRAS WITH INVOLUTION

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Abstract. Let (X, d) and (Y, ρ) be compact metric spaces, let τ and η be Lipschitz involutions on X and Y, respectively, let $\mathcal{A} = \mathrm{Lip}(X, d, \tau)$, and let $\mathcal{B} = \mathrm{Lip}(Y, \rho, \eta)$, where $\mathrm{Lip}(X, d, \tau) = \{f \in \mathrm{Lip}(X, d) : f \circ \tau = \bar{f}\}$. For each $f \in \mathcal{A}$, $\sigma_{\pi,\mathcal{A}}(f)$ denotes the peripheral spectrum of f. We prove that if $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ are surjective mappings that satisfy $\sigma_{\pi,\mathcal{B}}(T_1(f)T_2(g)) = \sigma_{\pi,\mathcal{A}}(S_1(f)S_2(g))$ for all $f, g \in \mathcal{A}$, then there are $\kappa_1, \kappa_2 \in \mathrm{Lip}(Y, \rho, \eta)$ with $\kappa_1\kappa_2 = 1_Y$ and a Lipschitz homeomorphism φ from (Y, ρ) to (X, d) with $\tau \circ \varphi = \varphi \circ \eta$ on Y such that $T_j(f) = \kappa_j \cdot (S_j(f) \circ \varphi)$ for all $f \in \mathcal{A}$ and $j = 1, 2$. Moreover, we show that the same result holds for surjective mappings $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ that satisfy $\sigma_{\pi,\mathcal{B}}(T_1(f)T_2(g)) \cap \sigma_{\pi,\mathcal{A}}(S_1(f)S_2(g)) \neq \emptyset$ for all $f, g \in \mathcal{A}$.

1. Introduction and preliminaries

The symbol $\mathbb{K}$ denotes a field that can be $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{A}$ be an algebra over $\mathbb{K}$ with unit $e_{\mathcal{A}}$ and $f \in \mathcal{A}$. The *spectrum* of f is denoted by $\sigma_{\mathcal{A}}(f)$ and defined by

$$\sigma_{\mathcal{A}}(f) = \{\lambda \in \mathbb{C} : f - \lambda e_{\mathcal{A}} \text{ is not invertible in } \mathcal{A}\},$$

if $\mathbb{K} = \mathbb{C}$ and

$$\sigma_{\mathcal{A}}(f) = \{a + ib \in \mathbb{C} : (f - ae_{\mathcal{A}})^2 + b^2 e_{\mathcal{A}} \text{ is not invertible in } \mathcal{A}\},$$

if $\mathbb{K} = \mathbb{R}$. It is known that $\sigma_{\mathcal{A}}(f)$ is a nonempty compact subset of $\mathbb{C}$ wherever, $\mathcal{A}$ is a Banach algebra with unit over $\mathbb{K}$ (see [9, Theorem 1.2.8] for $\mathbb{K} = \mathbb{C}$

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and [11, Corollary 1.1.8 and Theorem 1.1.19] for $\mathbb{K} = \mathbb{R}$). Let $\mathcal{A}$ be a Banach algebra with unit $e_{\mathcal{A}}$ over $\mathbb{K}$ and let $f \in \mathcal{A}$. The *peripheral spectrum* of f is denoted by $\sigma_{\pi,\mathcal{A}}(f)$ and defined by

$$\sigma_{\pi,\mathcal{A}}(f) = \{\lambda \in \sigma_{\mathcal{A}}(f) : |\lambda| = \max\{|w| : w \in \sigma_{\mathcal{A}}(f)\}\}.$$

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with unit over $\mathbb{K}$.

(i) A map $T : \mathcal{A} \to \mathcal{B}$ is called *multiplicatively spectrum preserving*, if $\sigma_{\mathcal{B}}(T(f)T(g)) = \sigma_{\mathcal{A}}(fg)$ for all $f, g \in \mathcal{A}$.

(ii) A map $T : \mathcal{A} \to \mathcal{B}$ is called *peripherally multiplicative spectrum preserving*, if $\sigma_{\pi,\mathcal{B}}(T(f)T(g)) = \sigma_{\pi,\mathcal{A}}(fg)$ for all $f, g \in \mathcal{A}$.

(iii) A map $T : \mathcal{A} \to \mathcal{B}$ is called *weakly peripherally multiplicative spectrum preserving*, if $\sigma_{\pi,\mathcal{B}}(T(f)T(g)) \cap \sigma_{\pi,\mathcal{A}}(fg) \neq \emptyset$ for all $f, g \in \mathcal{A}$.

(iv) Four mappings $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ are called *jointly peripherally multiplicative spectrum preserving*, if

$$\sigma_{\pi,\mathcal{B}}(T_1(f)T_2(g)) = \sigma_{\pi,\mathcal{A}}(S_1(f)S_2(g))$$

for all $f, g \in \mathcal{A}$.

(v) Four mappings $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ are called *jointly weakly peripherally multiplicative spectrum preserving*, if

$$\sigma_{\pi,\mathcal{B}}(T_1(f)T_2(g)) \cap \sigma_{\pi,\mathcal{A}}(S_1(f)S_2(g)) \neq \emptyset$$

for all $f, g \in \mathcal{A}$.

Definition 1.2. Let $(\mathcal{A}, \|\cdot\|)$ and $(\mathcal{B}, \|\cdot\|)$ be normed algebras over $\mathbb{K}$.

(i) A map $T : \mathcal{A} \to \mathcal{B}$ is called *norm multiplicative*, if $\|T(f)T(g)\| = \|fg\|$ for all $f, g \in \mathcal{A}$.

(ii) Four mappings $S_1, S_2 : \mathcal{A} \to \mathcal{B}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ are called *jointly norm multiplicative*, if $\|T_1(f)T_2(g)\| = \|S_1(f)S_2(g)\|$ for all $f, g \in \mathcal{A}$.

Let X be a compact Hausdorff space. We denote by $C_{\mathbb{K}}(X)$ the set of all $\mathbb{K}$-valued continuous functions on X. It is known that $C_{\mathbb{K}}(X)$ is a unital commutative Banach algebra over $\mathbb{K}$ with unit 1_X, the constant function on X with value 1, under the uniform norm $\|\cdot\|_X$ defined by

$$\|f\|_X = \sup\{|f(x)| : x \in X\} \qquad (f \in C_{\mathbb{K}}(X)).$$

We write $C(X)$ instead of $C_{\mathbb{C}}(X)$. A *complex function algebra* on X is a complex subalgebra A of $C(X)$, which separates the points of X and containing 1_X. A *complex Banach function algebra* on X is a complex function algebra A on X such that it is a unital Banach algebra under an algebra norm $\|\cdot\|$. If the algebra norm on A is the uniform norm $\|\cdot\|_X$, then A is called a *complex uniform function algebra* on X. Let $f \in C(X)$. The *range* of f is denoted by $\mathrm{Ran}_X(f)$ and defined by $\mathrm{Ran}_X(f) = \{f(x) : x \in X\}$. The set of all $x \in X$ for which $\|f\|_X = |f(x)|$ is called the *maximizing set* of f and denoted by $M(f)$. The *peripheral range* of f is denoted by $\mathrm{Ran}_{\pi,X}(f)$ and defined by $\mathrm{Ran}_{\pi,X}(f) = \{f(x) : x \in X, \|f\|_X = |f(x)|\}$. In fact, $\mathrm{Ran}_{\pi,X}(f)$ is the image of $M(f)$ by f. It is known [16, Lemma 1] that $\mathrm{Ran}_{\pi,X}(f) = \sigma_{\pi,A}(f)$ for all $f \in A$, where X is a compact Hausdorff space and A is a complex uniform function algebra on X.

Definition 1.3. Let X and Y be compact Hausdorff spaces and let $\mathcal{A}$ and $\mathcal{B}$ be subalgebras of $C(X)$ and $C(Y)$ over $\mathbb{K}$, respectively.

(i) A map $T : \mathcal{A} \to \mathcal{B}$ is called *multiplicatively range preserving*, if $\mathrm{Ran}_Y(T(f)T(g)) = \mathrm{Ran}_X(fg)$ for all $f, g \in \mathcal{A}$.

(ii) A map $T : \mathcal{A} \to \mathcal{B}$ is called *peripherally multiplicative range preserving*, if $\mathrm{Ran}_{\pi,Y}(T(f)T(g)) = \mathrm{Ran}_{\pi,X}(fg)$ for all $f, g \in \mathcal{A}$.

(iii) A map $T : \mathcal{A} \to \mathcal{B}$ is called *weakly peripherally multiplicative range preserving*, if $\mathrm{Ran}_{\pi,Y}(T(f)T(g)) \cap \mathrm{Ran}_{\pi,X}(fg) \neq \emptyset$ for all $f, g \in \mathcal{A}$.

(iv) Four mappings $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ are called *jointly peripherally multiplicative range preserving*, if

$$\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) = \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g))$$

for all $f, g \in \mathcal{A}$.

(v) Four mappings $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ are called *jointly weakly peripherally multiplicative range preserving*, if

$$\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) \cap \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)) \neq \emptyset$$

for all $f, g \in \mathcal{A}$.

The study on the multiplicatively spectrum preserving maps was initiated by Molnár in [18]. For a first countable compact Hausdorff space X, he showed [18, Theorem 5] that if $T : C(X) \to C(X)$ is a surjective multiplicatively spectrum preserving map, then there exist a homeomorphism $\varphi : X \to X$ and a continuous function $\kappa : X \to \{-1, 1\}$ such that $T(f) = \kappa \cdot (f \circ \varphi)$, for all $f \in C(X)$, that is, T is a weighted composition operator from $C(X)$ onto itself. Lambert, Luttman, and Tonev [12, Theorem 1] characterized the surjective norm multiplicative maps between complex uniform function algebras and showed that these mappings are essentially weighted composition operator in modulus. Lee [14] generalized their result to complex function algebras.

Let $\mathcal{A}$ be a unital commutative Banach algebra over $\mathbb{K}$. A *character* of $\mathcal{A}$ is a nonzero homomorphism $\varphi : \mathcal{A} \to \mathbb{C}$, where $\mathbb{C}$ is regarded as an algebra over $\mathbb{K}$. We denote by $\Delta(\mathcal{A})$ the set of all characters of $\mathcal{A}$. Let A be a complex Banach function algebra on a compact Hausdorff space X. For each $x \in X$, the map $\delta_{A,x} : A \to \mathbb{C}$ defined by $\delta_{A,x}(f) = f(x)$ $(f \in A)$, is an element of $\Delta(A)$, which is called the *evaluation character* on A at x. A is called *natural* if for every $\varphi \in \Delta(A)$ we have $\varphi = \delta_{A,x}$ for some $x \in X$.

Rao and Roy [19] obtained a generalization of Molnár's theorem for natural uniform function algebras. Hatori, Miura, and Takagi in [5] generalized the result of Rao and Roy by replacing the multiplicatively spectrum preserving condition by a weaker multiplicatively range preserving condition. Luttman and Tonev [16] replaced the range by the peripheral range or, equivalently, by the peripheral spectrum. Lambert, Luttman, and Tonev [12] and Lee and Luttman [15] characterized weakly peripherally multiplicative spectrum preserving between complex uniform function algebras and showed that these mappings are essentially weighted composition operators.

There has been a recent surge of work done on characterizing the mappings between Banach algebras that preserve certain spectral properties; see [4] for a survey. These problems are known as *spectral preserver problems*.

Let X be a topological space. A self-map $\tau : X \to X$ is called a *topological involution* on X if τ is continuous and $\tau(\tau(x)) = x$ for all $x \in X$. Clearly, such τ is a homeomorphism from X onto X.

Let X be a compact Hausdorff space and let τ be a topological involution on X. The map $\tau^* : C(X) \to C(X)$ defined by $\tau^*(f) = \bar{f} \circ \tau$ is an algebra involution on $C(X)$, which is called the *algebra involution induced by* τ on $C(X)$. We now define

$$C(X, \tau) := \{f \in C(X) : \tau^*(f) = f\}.$$

Then $C(X, \tau)$ is a unital self-adjoint uniformly closed real subalgebra of $C(X)$ that separates the points of X. Moreover, $i1_X \notin C(X, \tau)$ and $C(X) = C(X, \tau) \oplus iC(X, \tau)$. Note that $C(X, \tau) = C_{\mathbb{R}}(X)$ if and only if τ is the identity map on X. A *real function algebra* on (X, τ) is a real subalgebra $\mathcal{A}$ of $C(X, \tau)$ that separates the points of X and containing 1_X. A *real Banach function algebra* on (X, τ) is a real function algebra $\mathcal{A}$ on (X, τ) such that it is a unital Banach algebra under an algebra norm $\|\cdot\|$. If the algebra norm on $\mathcal{A}$ is the uniform norm $\|\cdot\|_X$, then $\mathcal{A}$ is called a *real uniform function algebra* on (X, τ). For each $x \in X$, the map $\delta_{\mathcal{A},x} : \mathcal{A} \to \mathbb{C}$ defined by $\delta_{\mathcal{A},x}(f) = f(x)$ $(f \in \mathcal{A})$, is an element of $\Delta(\mathcal{A})$, which is called the *evaluation character* on $\mathcal{A}$ at x. Also $\mathcal{A}$ is called *natural* if for every $\varphi \in \Delta(\mathcal{A})$ we have $\varphi = \delta_{\mathcal{A},x}$ for some $x \in X$. This algebra was defined explicitly by Kulkarni and Limaye [10]. For further general facts about $C(X, \tau)$ and real uniform function algebras, we refer the reader to [11].

Let X be a compact Hausdorff space and let $\mathcal{A}$ be a nonempty subset of $C(X)$. A function $f \in \mathcal{A}$ is said to be a *peaking function* in $\mathcal{A}$ if $\mathrm{Ran}_{\pi,X}(f) = \{1\}$. We denote by $\mathcal{P}(\mathcal{A})$ the set of all peaking functions in $\mathcal{A}$. For $x \in X$, the set of all $f \in \mathcal{P}(\mathcal{A})$ for which $f(x) = 1$ is denoted by $\mathcal{P}_x(\mathcal{A})$. A nonempty subset F of X is called a *peak set* of $\mathcal{A}$ if there exists a function $f \in \mathcal{P}(\mathcal{A})$ such that $F = \{x \in X : f(x) = 1\}$. For a nonempty subset E of X, we denote by $\mathcal{F}_E(\mathcal{A})$ the set of all $f \in \mathcal{A}$ for which $\|f\|_X = |f(x)| = 1$ for all $x \in E$.

Let X be a compact Hausdorff space, let $\tau : X \to X$ be a topological involution on X, and let $\mathcal{A}$ be a nonempty subset of $C(X, \tau)$. An (i)-*peaking function* in $\mathcal{A}$ is a function $f \in \mathcal{A}$ with $\mathrm{Ran}_{\pi,X}(f) = \{i, -i\}$. A nonempty subset F of X is called an (i)-*peak set* of $\mathcal{A}$ if there exists an (i)-*peaking function* $f \in \mathcal{A}$ such that $F = \{x \in X : f(x) = i\}$. Note that F is a compact set and $F \cap \tau(F) = \emptyset$. For $x \in X$, we denote by $i\mathcal{P}_x(\mathcal{A})$ the set of all $f \in \mathcal{P}(\mathcal{A})$ for which $f(x) = i$. These types of functions were studied extensively in [17].

Spectral preserver problems have yet to be investigated for real Banach function algebras, or for real Banach algebras. Lee [13] studied the jointly uniform norm multiplicative maps, jointly peripherally multiplicative spectrum preserving maps, and jointly weakly peripherally multiplicative spectrum preserving maps between real uniform function algebras, using the peaking functions, (i)-peaking functions, and the version of Bishop's lemma for real uniform function algebras. Moreover he showed that such mappings are essentially weighted composition operators on the Choquet boundaries of mentioned algebras. To see the version of Bishop's lemma for complex uniform function algebras we refer [3, Theorem 2.4.1].

Let (X, d) and (Y, ρ) be metric spaces. A map $\varphi : X \to Y$ is called a *Lipschitz mapping* from (X, d) to (Y, ρ) if there exists a constant C such that $\rho(\varphi(x), \varphi(y)) \leq Cd(x, y)$ for all $x, y \in X$. Let (X, d) be a metric space. A function $f : X \to \mathbb{K}$ is called a $\mathbb{K}$-*valued Lipschitz function* on (X, d) if f is a Lipschitz mapping from (X, d) to the Euclidean metric space $\mathbb{K}$. We denote by $p_{(X,d)}(f)$ the *Lipschitz constant* of $f : X \to \mathbb{K}$, that is,

$$p_{(X,d)}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y\right\}.$$

Let (X, d) be a compact metric spaces. We denote by $\mathrm{Lip}_{\mathbb{K}}(X, d)$ the set of all $\mathbb{K}$-valued Lipschitz functions on (X, d). Then $\mathrm{Lip}_{\mathbb{K}}(X, d)$ is a subalgebra of $C_{\mathbb{K}}(X)$ over $\mathbb{K}$ containing 1_X that separates the points of X. Moreover, $\mathrm{Lip}_{\mathbb{K}}(X, d)$ is a commutative unital Banach algebra over $\mathbb{K}$ with the Lipschitz algebra norm

$$\|f\|_{\mathrm{Lip}(X,d)} = \|f\|_X + p_{(X,d)}(f) \quad (f \in \mathrm{Lip}_{\mathbb{K}}(X, d)).$$

We write $\mathrm{Lip}(X, d)$ instead of $\mathrm{Lip}_{\mathbb{C}}(X, d)$. It is known that $\mathrm{Lip}(X, d)$ with the norm $\|\cdot\|_{\mathrm{Lip}(X,d)}$ is a natural complex Banach function algebra on X. These algebras were first introduced by Sherbert [20, 21].

Jiméneze-Vargas and Villegas-Vallecillos [8] studied and characterized peripherally multiplicative spectrum preserving maps between Lipschitz algebras on compact metric spaces. Weakly peripherally multiplicative range preserving maps between Lipschitz algebras on pointed compact metric spaces were characterized by Jiméneze-Vargas, Luttman, and Villegas-Vallecillos [7]. Jiméneze-Vargas et al. [6] characterized jointly weakly peripherally multiplicative range preserving maps between Lipschitz algebras on pointed compact metric spaces.

Let (X, d) be a metric space. A self-map $\tau : X \to X$ is called a *Lipschitz involution* on (X, d) if $\tau(\tau(x)) = x$ for all $x \in X$ and τ is a Lipschitz mapping on (X, d). Note that if τ is a Lipschitz involution on (X, d), then τ is a topological involution on (X, d). Let (X, d) be a compact metric space and let τ be a Lipschitz involution on (X, d). Then $\tau^*(\mathrm{Lip}(X, d)) = \mathrm{Lip}(X, d)$. We now define

$$\mathrm{Lip}(X, d, \tau) := \{f \in \mathrm{Lip}(X, d) : \tau^*(f) = f\}.$$

In fact, $\mathrm{Lip}(X, d, \tau) = \mathrm{Lip}(X, d) \cap C(X, \tau)$. The following result is a modification of [1, Theorem 2.7].

Theorem 1.4. *Let (X, d) be a compact metric space and let τ be a Lipschitz involution on (X, d). Suppose that $\mathcal{A} = \mathrm{Lip}(X, d, \tau)$ and $A = \mathrm{Lip}(X, d)$. Then the following statements hold:*

(i) *$\mathcal{A}$ is self-adjoint real subalgebra of A, $1_X \in \mathcal{A}$, and $i1_X \notin \mathcal{A}$.*
(ii) *$A = \mathcal{A} \oplus i\mathcal{A}$ and $\mathcal{A}$ separates the points of X.*
(iv) *$\mathcal{A}$ is closed in $(A, \|\cdot\|_{\mathrm{Lip}(X,d)})$.*
(v) *$(\mathcal{A}, \|\cdot\|_{\mathrm{Lip}(X,d)})$ is a real natural Banach function algebra on (X, d, τ).*
(vi) *$\mathcal{A} = \mathrm{Lip}_{\mathbb{R}}(X, d)$, if and only if τ is the identity map on X.*

The algebras $\mathrm{Lip}(X, d, \tau)$ are called *real Lipschitz algebras with involution* and were first studied in [1]. We know from [2] that $\mathrm{Ch}(\mathrm{Lip}(X, d, \tau)) = X$, where $\mathrm{Ch}(A, X, \tau)$ is the Choquet boundary of real function algebra A with respect to (X, τ).

In Section 2, we first give a version of Bishop's lemma for real Lipschitz algebras with involution. Next, we obtain some results, which are useful in Sections 3-5. In Section 3, we characterize jointly uniform norm multiplicative maps between real Lipschitz algebras with involution and show that these mappings are essentially composition operators in modulus. In Section 4, we characterize jointly peripherally multiplicative spectrum preserving maps between real Lipschitz algebras with involution and show that these mappings are essentially weighted composition operators. In Section 5, we first show that jointly weakly peripherally multiplicative spectrum preserving maps between real Lipschitz algebras with involution are jointly peripherally multiplicative spectrum preserving. Finally, we show that jointly weakly peripherally multiplicative spectrum preserving maps are essentially weighted composition operators. In this work, we follow almost the same method of Lee in [13].

2. Bishop's lemma and some results in real Lipschitz algebras

The key result to the work on peripheral multiplicativity and its generalizations is Bishop's lemma. In this section, we first give a version of Bishop's lemma for $\mathrm{Lip}(X, d, \tau)$, where (X, d) is a compact metric space and τ is a Lipschitz involution on X. For this purpose, we need the following lemmas.

Lemma 2.1. *Let (X, d) be a compact metric space and let $\tau : X \to X$ be a Lipschitz involution on (X, d). Then for each $x \in X$, there is a function $f \in \mathrm{Lip}(X, d, \tau)$ with $f(x) = f(\tau(x)) = 1$ and $0 \le f(y) < 1$ for all $y \in X \setminus \{x, \tau(x)\}$.*

Proof. For each $z \in X$ and every $\delta > 0$, define the function $h_{z,\delta} : X \to \mathbb{C}$ by

$$h_{z,\delta}(y) = \max\{0, 1 - \frac{d(z, y)}{\delta}\} \qquad (y \in X).$$

It is easy to see that $h_{z,\delta} \in \mathrm{Lip}(X, d)$, $h_{z,\delta}(z) = 1$ and $0 \le h_{z,\delta}(y) < 1$ for all $y \in X \setminus \{z\}$.

To prove the result, we first assume that $x \in X$ with $\tau(x) = x$. Define the function $f : X \to \mathbb{C}$ by $f = h_{x,1}\tau^*(h_{x,1})$, where τ^* is the algebra involution on $C(X)$ induced by τ. Then $f \in \mathrm{Lip}(X, d, \tau)$, $f(x) = f(\tau(x)) = 1$ and $0 \le f(y) < 1$ for all $y \in X \setminus \{x, \tau(x)\}$.

We now assume that $x \in X$ with $\tau(x) \neq x$. Take $0 < \delta < d(x, \tau(x))/2$. Then $h_{x,\delta}(\tau(x)) = h_{\tau(x),\delta}(x) = 0$. Define the function $g : X \to \mathbb{C}$ by $g = h_{x,\delta} + h_{\tau(x),\delta}$. Then $g \in \mathrm{Lip}(X, d)$ and $g(x) = g(\tau(x)) = 1$. Let $y \in X \setminus \{x, \tau(x)\}$. Clearly, $0 < d(y, x) < \delta$ implies that $d(y, \tau(x)) \ge \delta$ and $0 < d(y, \tau(x)) < \delta$ implies that $d(y, x) \ge \delta$ since $0 < \delta < \frac{d(x,\tau(x))}{2}$. If $d(y, x) \ge \delta$ and $d(y, \tau(x)) \ge \delta$, then

$$g(y) = h_{x,\delta}(y) + h_{\tau(x),\delta}(y) = 0.$$

If $0 < d(y, x) < \delta$ and $d(y, \tau(x)) \ge \delta$, then

$$g(y) = h_{x,\delta}(y) + h_{\tau(x),\delta}(y) = 1 - d(y, x)/\delta < 1.$$

If $d(y,x) \geq \delta$ and $0 < d(y,\tau(x)) < \delta$, then

$$g(y) = h_{x,\delta}(y) + h_{\tau(x),\delta}(y) = 1 - \frac{d(y,\tau(x))}{\delta} < 1.$$

Therefore, $0 \leq g(y) < 1$ for all $y \in X \setminus \{x,\tau(x)\}$ and so $\|g\|_X = 1$. Define the function $f : X \to \mathbb{C}$ by $f = g\tau^*(g)$. Then $f \in \mathrm{Lip}(X,d,\tau)$, $f(x) = f(\tau(x)) = 1$, and $0 \leq f(y) < 1$ for all $y \in X \setminus \{x,\tau(x)\}$. □

As an immediate consequence of Lemma 2.1, we obtain the following result.

Corollary 2.2. *Let (X,d) be a compact metric space and let $\tau : X \to X$ be a Lipschitz involution on (X,d). If $x \in X$, then $\mathcal{P}_x(\mathrm{Lip}(X,d,\tau))$ is nonempty and $\{x,\tau(x)\}$ is a peak set for $\mathrm{Lip}(X,d,\tau)$ with respect to X.*

Lemma 2.3. *Let (X,d) be a compact metric space and let $\tau : X \to X$ be a Lipschitz involution on (X,d). Then for each $x \in X$ with $\tau(x) \neq x$, there is a function $f \in \mathrm{Lip}(X,d,\tau)$ with $f(x) = i$, $f(\tau(x)) = -i$ and $|f(y)| < 1$ for all $y \in X \setminus \{x,\tau(x)\}$.*

Proof. Let $x \in X$ with $\tau(x) \neq x$. Take $0 < \delta < d(x,\tau(x))/2$. Define the function $g_{x,\delta} : X \to \mathbb{C}$ by

$$g_{x,\delta}(y) = i \max\{0, 1 - \frac{d(x,y)}{\delta}\} \qquad (y \in X).$$

Then $g_{x,\delta} \in \mathrm{Lip}(X,d)$, $g_{x,\delta}(x) = i$, $g_{x,\delta}(\tau(x)) = 0$, and $|g_{x,\delta}(y)| < 1$ for all $y \in X \setminus \{x\}$. Define the function $f : X \to \mathbb{C}$ by $f = g_{x,\delta} + \tau^*(g_{x,\delta})$. Then $f \in \mathrm{Lip}(X,d,\tau)$, $f(x) = i$ and $f(\tau(x)) = -i$. Let $y \in X \setminus \{x,\tau(x)\}$. Then there exist $\alpha,\beta \in [0,1)$ such that $g_{x,\delta}(y) = i\alpha$ and $g_{x,\delta}(\tau(y)) = i\beta$. Therefore, $f(y) = i(\alpha - \beta)$ and so $|f(y)| < 1$. □

As an immediate consequence of Lemma 2.3, we obtain the following result.

Corollary 2.4. *Let (X,d) be a compact metric space and let $\tau : X \to X$ be a Lipschitz involution on (X,d). If $x \in X$ with $\tau(x) \neq x$, then $i\mathcal{P}_x(\mathrm{Lip}(X,d,\tau))$ is nonempty and $\{x,\tau(x)\}$ is an (i)-peak set for $\mathrm{Lip}(X,d,\tau)$ with respect to X.*

We are now in a position to state and prove the peak and (i)-peak versions of Bishop's lemma for $\mathrm{Lip}(X,d,\tau)$.

Theorem 2.5. *Let (X,d) be a compact metric space, let $\tau : X \to X$ be a Lipschitz involution on (X,d), let $f \in \mathrm{Lip}(X,d,\tau)$, and let $x \in X$.*

(i) (Peak version of Bishop's lemma for $\mathrm{Lip}(X,d,\tau)$). *If $f(x) \neq 0$, then there is a function $h \in \mathcal{P}_x(\mathrm{Lip}(X,d,\tau))$ such that $M(fh) = M(h) = \{x,\tau(x)\}$ and $\mathrm{Ran}_{\pi,X}(fh) = \{f(x), f(\tau(x))\}$.*

(ii) *If $f(x) = 0$, then for each $\varepsilon > 0$ there is a function $h \in \mathcal{P}_x(\mathrm{Lip}(X,d,\tau))$ such that $\|fh\|_X < \varepsilon$.*

Proof. (i) Let $f(x) \neq 0$. Take $f_1 = \frac{1}{|f(x)|} f$. Then $f_1 \in \mathrm{Lip}(X, d, \tau)$ and $|f_1(x)| = 1$. Define the function $g : X \to \mathbb{C}$ by

$$g(y) = \begin{cases} 1, & y \in X, \quad |f_1(y)| < 1, \\ 2 - |f_1(y)|, & y \in X, \quad 1 \leq |f_1(y)| \leq 2, \\ 0, & y \in X, \quad |f_1(y)| > 2. \end{cases} \tag{2.1}$$

Then $g \in \mathrm{Lip}(X, d, \tau)$, $g(x) = g(\tau(x)) = 1$, and $0 \leq g(y) \leq 1$ for all $y \in X$. By Lemma 2.1, there exists a function $k \in \mathrm{Lip}(X, d, \tau)$ with $k(x) = k(\tau(x)) = 1$ and $0 \leq k(y) < 1$ for all $y \in X \setminus \{x, \tau(x)\}$. Define the function $h : X \to \mathbb{C}$ by $h = gk$. Then $h \in \mathrm{Lip}(X, d, \tau)$, $h(\tau(x)) = h(x) = 1$, and $|h(y)| = |g(y)k(y)| < 1$ for all $y \in X \setminus \{x, \tau(x)\}$. Hence, $M(h) = \{x, \tau(x)\}$ and $h \in \mathcal{P}_x(\mathrm{Lip}(X, d, \tau))$. Let $y \in X \setminus \{x, \tau(x)\}$. If $|f_1(y)| < 1$, then

$$|f_1(y)h(y)| = |f_1(y)g(y)k(y)| = |f_1(y)k(y)| < 1.$$

If $1 \leq |f_1(y)| \leq 2$, then

$$|f_1(y)h(y)| = |f_1(y)g(y)k(y)| = |f_1(y)|(2 - |f_1(y)|)|k(y)| < 1.$$

If $|f_1(y)| > 2$, then

$$|f_1(y)h(y)| = |f_1(y)g(y)k(y)| = 0 < 1.$$

Hence, $|f_1(y)h(y)| < |f_1(x)|$ for all $y \in X \setminus \{x, \tau(x)\}$. On the other hand, $(f_1h)(x) = f_1(x)h(x) = f_1(x)$ and $(f_1h)(\tau(x)) = f_1(\tau(x))h(\tau(x)) = f_1(\tau(x))$. Hence, $\|f_1h\|_X = |f_1(x)| = |f_1(\tau(x))|$, $M(f_1h) = \{x, \tau(x)\}$, and $\mathrm{Ran}_{\pi,X}(f_1h) = \{f_1(x), f_1(\tau(x))\}$. Since $fh = |f(x)|f_1h$ and $|f(x)| > 0$, we conclude that $M(fh) = M(f_1h) = \{x, \tau(x)\}$ and

$$\begin{aligned} \mathrm{Ran}_{\pi,X}(fh) &= \{|f(x)|\lambda : \lambda \in \mathrm{Ran}_{\pi,X}(f_1h)\} \\ &= \{|f(x)|f_1(x), |f(x)|f_1(\tau(x))\} \\ &= \{f(x), f(\tau(x))\}. \end{aligned}$$

(ii) Let $f(x) = 0$. If $f(y) = 0$ for all $y \in X$, then (ii) holds for all $h \in \mathcal{P}_x(\mathrm{Lip}(X, d, \tau))$. Note that, by Corollary 2.2, $\mathcal{P}_x(\mathrm{Lip}(X, d, \tau)) \neq \emptyset$. We now assume that $f(x_0) \neq 0$ for some $x_0 \in X$. Let $\varepsilon > 0$ be given. Set $F = \{y \in X : |f(y)| \geq \min\{\varepsilon, \frac{|f(x_0)|}{2}\}\}$. Then F is a τ-invariant compact subset of X, $x_0 \in F$, and $x \notin F$. By Lemma 2.1, there exists a function $g \in \mathrm{Lip}(X, d, \tau)$ with $g(x) = g(\tau(x)) = 1$ and $|g(y)| < 1$ for all $y \in X \setminus \{x, \tau(x)\}$. Clearly, $\|g\|_F < 1$. Hence, there exists $m \in \mathbb{N}$ such that $(\|g\|_F)^m < \frac{\varepsilon}{\|f\|_X}$. Assume that $h = g^m$. It is easy to see that $h \in \mathcal{P}_x(\mathrm{Lip}(X, d, \tau))$. Since

$$|f(y)h(y)| \leq \|f\|_X |h(y)| < \|f\|_X \frac{\varepsilon}{\|f\|_X} = \varepsilon,$$

for all $y \in F$ and

$$|f(y)h(y)| = |f(y)||h(y)| \leq |f(y)| < \min\{\varepsilon, \frac{|f(x_0)|}{2}\} \leq \varepsilon,$$

for all $y \in X \setminus F$, we deduce that $\|fh\|_X < \varepsilon$. □

Theorem 2.6. $\big((i)$-peak version of Bishop's lemma for $\mathrm{Lip}(X,d,\tau)\big)$. *Let (X,d) be a compact metric space, let $\tau : X \to X$ be a Lipschitz involution on (X,d), let $f \in \mathrm{Lip}(X,d,\tau)$, and let $x \in X$ with $\tau(x) \neq x$. If $f(x) \neq 0$, then there is a function $h \in i\mathcal{P}_x(\mathrm{Lip}(X,d,\tau))$ such that $M(fh) = M(h) = \{x,\tau(x)\}$ and $\mathrm{Ran}_{\pi,X}(fh) = \{if(x), -if(\tau(x))\}$.*

Proof. Let $f(x) \neq 0$. Take $f_1 = \frac{1}{|f(x)|}f$. Then $f_1 \in \mathrm{Lip}(X,d,\tau)$ and $|f_1(x)| = 1$. We consider the function $g : X \to \mathbb{C}$ defined by (2.1). By Lemma 2.3, there exists a function $k \in \mathrm{Lip}(X,d,\tau)$ with $k(x) = i$, $k(\tau(x)) = -i$ and $|k(y)| < 1$ for all $y \in X \setminus \{x,\tau(x)\}$. Define the function $h : X \to \mathbb{C}$ by $h = gk$. Then $h \in \mathrm{Lip}(X,d,\tau)$, $h(x) = g(x)k(x) = i$, $h(\tau(x)) = g(\tau(x))k(\tau(x)) = -i$, and $|h(y)| = |g(y)k(y)| < 1$ for all $y \in X \setminus \{x,\tau(x)\}$. Hence, $h \in i\mathcal{P}_x(\mathrm{Lip}(X,d,\tau))$. Let $y \in X \setminus \{x,\tau(x)\}$. If $|f_1(y)| < 1$, then

$$|f_1(y)h(y)| = |f_1(y)g(y)k(y)| = |f_1(y)k(y)| < 1.$$

If $1 \leq |f_1(y)| \leq 2$, then

$$|f_1(y)h(y)| = |f_1(y)g(y)k(y)| = |f_1(y)|(2 - |f_1(y)|)|k(y)| < 1.$$

If $|f_1(y)| > 2$, then

$$|f_1(y)h(y)| = |f_1(y)g(y)k(y)| = 0 < 1.$$

Therefore, $|f_1(y)h(y)| < |f_1(x)|$ for all $y \in X \setminus \{x,\tau(x)\}$. Hence, $M(f_1h) = \{x,\tau(x)\}$. This implies that $M(fh) = \{x,\tau(x)\}$ since $fh = |f(x)|f_1h$ and $|f(x)| > 0$. On the other hand, $(f_1h)(x) = f_1(x)h(x) = if_1(x)$ and $(f_1h)(\tau(x)) = f_1(\tau(x))h(\tau(x)) = -if_1(\tau(x))$. Hence,

$$\|f_1h\|_X = |f_1(x)h(x)| = |f_1(\tau(x))h(\tau(x))|.$$

Therefore, $\mathrm{Ran}_{\pi,X}(f_1h) = \{if_1(x), -if_1(\tau(x))\}$. Since $fh = |f(x)|f_1h$ and $|f(x)| > 0$, we have

$$\begin{aligned}\mathrm{Ran}_{\pi,X}(fh) &= \{|f(x)|\lambda : \lambda \in \mathrm{Ran}_{\pi,X}(f_1h)\}\\ &= \{i|f(x)|f_1(x), -i|f(x)|f_1(\tau(x))\}\\ &= \{if(x), -if(\tau(x))\}.\end{aligned}$$

□

We now give some lemmas in real Lipschitz algebras $\mathrm{Lip}(X,d,\tau)$, which are useful in the next sections.

Lemma 2.7. *Let (X,d) be a compact metric space, let $\tau : X \to X$ be a Lipschitz involution on (X,d), and let $f, g \in \mathrm{Lip}(X,d,\tau)$. Then the following statements hold:*

- (i) *If $\|fh\|_X \leq \|gh\|_X$ for all $h \in \mathcal{P}(\mathrm{Lip}(X,d,\tau))$, then $|f(x)| \leq |g(x)|$ for all $x \in X$.*
- (ii) *If $\|fh\|_X = \|gh\|_X$ for all $h \in \mathcal{P}(\mathrm{Lip}(X,d,\tau))$, then $|f(x)| = |g(x)|$ for all $x \in X$.*

Proof. (i) For each $x \in X$ and every $\delta > 0$ the function $h_{x,\delta} : X \to \mathbb{C}$ defined by

$$h_{x,\delta}(y) = \max\{0, 1 - \frac{d(x,y)}{\delta}\} \qquad (y \in X),$$

belongs to $\mathrm{Lip}(X,d)$ and satisfies $h_{x,\delta}(x) = 1$ and $0 \leq h_{x,\delta}(y) < 1$ for all $y \in X \setminus \{x\}$. Assume that $\|fh\|_X \leq \|gh\|_X$ for all $h \in \mathcal{P}(\mathrm{Lip}(X,d,\tau))$, but $|g(x_0)| < |f(x_0)|$ for some $x_0 \in X$. Choose a positive number ε with $|g(x_0)| < \varepsilon < |f(x_0)|$.

We first consider the case $\tau(x_0) = x_0$. The continuity of g at x_0 implies that there exists $\delta > 0$ such that $|g(y)| < \varepsilon$ for all $y \in X$ with $d(y, x_0) < \delta$. Define the function $k : X \to \mathbb{C}$ by $k = h_{x_0,\delta}\tau^*(h_{x_0,\delta})$. Then $k \in \mathrm{Lip}(X,d,\tau)$, $k(x_0) = k(\tau(x_0)) = 1$ and $0 \leq k(y) < 1$ for all $y \in X \setminus \{x_0, \tau(x_0)\}$ and so $k \in \mathcal{P}(\mathrm{Lip}(X,d,\tau))$. If $y \in X$ with $d(y,x_0) \geq \delta$, then $k(y) = 0$ and so $|g(y)k(y)| = 0 < \varepsilon$. If $y \in X$ with $d(y,x_0) < \delta$, then $|g(y)k(y)| < \varepsilon\|k\|_X \leq \varepsilon$, since $|g(y)| < \varepsilon$ and $\|k\|_X = 1$. Therefore, $|gk(y)| < \varepsilon$ for all $y \in Y$ and $\|fk\|_X \geq |f(x_0)k(x_0)| = |f(x_0)| > \varepsilon$ and so $\|gk\|_X \leq \varepsilon < \|fk\|_X$, which is a contradiction.

We now consider the case $\tau(x_0) \neq x_0$. Then $|g(\tau(x_0))| = |\bar{g}(x_0)| = |g(x_0)| < \varepsilon$. By the continuity of g at x_0 and $\tau(x_0)$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $|g(y)| < \varepsilon$ for all $y \in X$ with $d(y,x_0) < \delta_1$ and $|g(z)| < \varepsilon$ for all $z \in X$ with $d(z, \tau(x_0)) < \delta_2$. Choose a positive number δ with $0 < \delta < \min\{\delta_1, \delta_2, d(x_0, \tau(x_0))/2\}$. Define the function $f_1 : X \to \mathbb{C}$ by $f_1 = h_{x_0,\delta} + h_{\tau(x_0),\delta}$. Then $f_1 \in \mathrm{Lip}(X,d)$, $f_1(x_0) = f_1(\tau(x_0)) = 1$ and $\|f_1\|_X = 1$. Define the function $k : X \to \mathbb{C}$ by $k = f_1\tau^*(f_1)$. Then $k \in \mathrm{Lip}(X,d,\tau)$, $k(x_0) = k(\tau(x_0)) = 1$ and $|k(y)| < 1$ for all $y \in X \setminus \{x_0, \tau(x_0)\}$. Hence, $k \in \mathcal{P}(\mathrm{Lip}(X,d,\tau))$. For $y \in X$, one can see that $0 < d(y,x_0) < \delta$ implies that $d(y,\tau(x_0)) \geq \delta$ and $0 < d(y,\tau(x_0)) < \delta$ implies that $d(y,x_0) \geq \delta$, since $0 < \delta < \frac{d(x_0,\tau(x_0))}{2}$. If $y \in X$ with $d(y,x_0) \geq \delta$ and $d(y,\tau(x_0)) \geq \delta$, then $k(y) = 0$ and so

$$|g(y)k(y)| = 0 < \varepsilon.$$

If $y \in X$ with $d(y,x_0) < \delta$ and $d(y,\tau(x_0)) \geq \delta$, then

$$|g(y)k(y)| < \varepsilon\|k\|_X = \varepsilon,$$

since $\delta \leq \delta_1$ and so $|g(y)| < \varepsilon$. If $y \in X$ with $d(y,x_0) \geq \delta$ and $d(y,\tau(x_0)) < \delta$, then

$$|g(y)k(y)| < \varepsilon\|k\|_X = \varepsilon,$$

since $\delta \leq \delta_2$ and so $|g(y)| < \varepsilon$. Therefore, $|g(y)k(y)| < \varepsilon$ for all $y \in X$ and $\|fk\|_X \geq |f(x_0)k(x_0)| = |f(x_0)| > \varepsilon$ and so $\|gk\|_X \leq \varepsilon < \|fk\|_X$, which is a contradiction.

(ii) This follows from (i). □

Lemma 2.8. *Let (X,d) be a compact metric space, let $\tau : X \to X$ be a Lipschitz involution on (X,d), and let $x, y \in X$. Then $x_\tau = y_\tau$ if and only if $\mathcal{F}_{x_\tau}(\mathrm{Lip}(X,d,\tau)) \subseteq \mathcal{F}_{y_\tau}(\mathrm{Lip}(X,d,\tau))$, where $z_\tau = \{z, \tau(z)\}$ for $z \in X$.*

Proof. The necessity part is clear. Assume that $x_\tau \neq y_\tau$. Then $y \in X \setminus x_\tau$. By Lemma 2.1, there exists a function $f \in \mathrm{Lip}(X,d,\tau)$ such that $f(x) = f(\tau(x)) = 1$, $0 \leq f(z) < 1$ for all $z \in X \setminus x_\tau$. Therefore, $f \in \mathcal{F}_{x_\tau}(\mathrm{Lip}(X,d,\tau))$ and $0 \leq f(y) = f(\tau(y)) < 1$. Thus, $f \in \mathcal{F}_{x_\tau}(\mathrm{Lip}(X,d,\tau)) \setminus \mathcal{F}_{y_\tau}(\mathrm{Lip}(X,d,\tau))$. Hence, $\mathcal{F}_{x_\tau}(\mathrm{Lip}(X,d,\tau)) \nsubseteq \mathcal{F}_{y_\tau}(\mathrm{Lip}(X,d,\tau))$ and so the sufficiency part holds. □

Lemma 2.9. *Let X be a compact Hausdorff space and let A be a natural complex Banach function algebra on X. Then*

$$\mathrm{Ran}_{\pi,X}(f) = \sigma_{\pi,A}(f),$$

for each $f \in A$.

Proof. To prove the result, it is sufficient to note that $f(X) = \sigma_A(f)$ for all $f \in A$, which is a known result. However, for the proof, let $f \in A$. Since A is a natural complex Banach function algebra on the compact Hausdorff space X, we have

$$\Delta(A) = \{\delta_{A,x} : x \in X\}. \tag{2.2}$$

Since A is a commutative unital complex Banach algebra, we have

$$\sigma_A(f) = \hat{f}(\Delta(A)), \tag{2.3}$$

by [9, Theorem 2.2.5]. Apply (2.2) and (2.3), we have

$$\begin{aligned} f(X) &= \{f(x) : x \in X\} = \{\delta_{A,x}(f) : x \in X\} = \{\hat{f}(\delta_{A,x}) : x \in X\} \\ &= \{\hat{f}(\phi) : \phi \in \Delta(A)\} = \hat{f}(\Delta(A)) = \sigma_A(f). \end{aligned}$$

□

Lemma 2.10. *Let (X, d) be a compact metric space, let $\tau : X \to X$ be a Lipschitz involution on (X, d), let $\mathcal{A} = \mathrm{Lip}(X, d, \tau)$, and let $A = \mathrm{Lip}(X, d)$. Then*

(i) *$\sigma_{\mathcal{A}}(f) = \sigma_A(f)$ for all $f \in \mathcal{A}$.*
(ii) *$\mathrm{Ran}_{\pi,X}(f) = \sigma_{\pi,\mathcal{A}}(f)$ for all $f \in \mathcal{A}$.*

Proof. (i) Let $f \in \mathcal{A}$ and let $\lambda \in \sigma_{\mathcal{A}}(f)$. Then there exist real numbers α and β such that $\lambda = \alpha + i\beta$ and $(f - \alpha 1_X)^2 + \beta^2 1_X$ is not invertible in $\mathcal{A}$. Hence, there exists $x \in X$ such that $((f - \alpha 1_X)^2 + \beta^2 1_X)(x) = 0$. Thus, $(f(x) - \alpha)^2 + \beta^2 = 0$, which implies that $f(x) = \alpha$ and $\beta = 0$. Hence, $(f - \alpha 1_X)(x) = 0$ and so $\lambda = \alpha \in \sigma_A(f)$.

Conversely, let $\lambda \in \sigma_A(f)$. Then $f - \lambda 1_X$ is not invertible in A. Hence, there exist $x \in X$ and real numbers α and β such that $\lambda = \alpha + i\beta$ and $f(x) = \lambda$. Thus, $f(x) - \alpha = i\beta$ and so $((f - \alpha 1_X)^2 + \beta^2 1_X)(x) = 0$. Consequently, $(f - \alpha 1_X)^2 + \beta^2 1_X$ is not invertible in $\mathcal{A}$. Therefore, $\lambda = \alpha + i\beta \in \sigma_{\mathcal{A}}(f)$.

(ii) Let $f \in \mathcal{A}$. By (i), we have

$$\sigma_{\mathcal{A}}(f) = \sigma_A(f). \tag{2.4}$$

Since, A is a natural complex Banach function algebra on X, by Lemma 2.9, we have

$$\mathrm{Ran}_{\pi,X}(f) = \sigma_{\pi,A}(f). \tag{2.5}$$

From (2.4) and (2.5), we deduce that $\mathrm{Ran}_{\pi,X}(f) = \sigma_{\pi,\mathcal{A}}(f)$. □

Lemma 2.11. *Let (X, d) and (Y, ρ) be compact metric spaces and let $\tau : X \to X$ and $\eta : Y \to Y$ be Lipschitz involutions on (X, d) and (Y, ρ), respectively. If the map $\varphi : Y \to X$ satisfies $f \circ \varphi \in \mathrm{Lip}(Y, \rho, \eta)$ for all $f \in \mathrm{Lip}(X, d, \tau)$, then φ is a Lipschitz mapping from (Y, ρ) to (X, d).*

Proof. Let $h \in \mathrm{Lip}(X,d)$. By Theorem 1.4(ii), there exist $f, g \in \mathrm{Lip}(X,d,\tau)$ such that $h = f + ig$. Hence, $h \circ \varphi = (f \circ \varphi) + i(g \circ \varphi)$ and so $h \circ \varphi \in \mathrm{Lip}(Y,\rho)$. Define the map $T : \mathrm{Lip}(X,d) \to \mathrm{Lip}(Y,\rho)$ by

$$T(h) = h \circ \varphi \qquad (h \in \mathrm{Lip}(X,d)).$$

Then T is an algebra homomorphism from $\mathrm{Lip}(X,d)$ to $\mathrm{Lip}(Y,\rho)$ with $T(1_X) = 1_Y$. By [20, Theorem 5.1], there exists a Lipschitz mapping θ from (Y,ρ) to (X,d) such that

$$T(h) = h \circ \theta.$$

Therefore, $h \circ \varphi = h \circ \theta$ for all $h \in \mathrm{Lip}(X,d)$. This implies that $\varphi = \theta$ since $\mathrm{Lip}(X,d)$ separates the point of X. Hence, φ is a Lipschitz mapping from (Y,ρ) to (X,d). □

3. Jointly uniform norm multiplicative maps

In this section, we give a description of surjective jointly uniform norm multiplicative maps between real Lipschitz algebras with involution. Throughout this section, we assume that (X,d) and (Y,ρ) are compact metric spaces, that $\tau : X \to X$ and $\eta : Y \to Y$ are Lipschitz involutions on (X,d) and (Y,ρ), respectively, that $x_\tau = \{x, \tau(x)\}$ for $x \in X$, that $X_\tau = \{x_\tau : x \in X\}$, that $y_\eta = \{y, \eta(y)\}$ for $y \in Y$, that $Y_\eta = \{y_\eta : y \in Y\}$, that $\mathcal{A} = \mathrm{Lip}(X,d,\tau)$, and that $\mathcal{B} = \mathrm{Lip}(Y,\rho,\eta)$.

Lemma 3.1. *Let $f, g \in \mathcal{A}$, let $x \in X$, let $y \in Y$, and let four mappings $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ be jointly uniform norm multiplicative satisfying $|T_1(f)(y)T_2(g)(y)|$
$= |S_1(f)(x)S_2(g)(x)|$. Then $x \in M(S_1(f)S_2(g))$ if and only if $y \in M(T_1(f)T_2(g))$.*

Proof. The proof is straightforward. □

Theorem 3.2. *Let four mappings $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ be surjective jointly uniform norm multiplicative. Then there is a bijection mapping $\Psi : X_\tau \to Y_\eta$ such that*

$$|T_1(f)(y)T_2(g)(y)| = |S_1(f)(x)S_2(g)(x)|,$$

for all $f, g \in \mathcal{A}$, $x \in X$ and $y \in \Psi(x_\tau)$. Moreover, if $f, g \in \mathcal{A}$, $x \in X$ and $y \in \Psi(x_\tau)$, then $x \in M(S_1(f)S_2(g))$ if and only if $y \in M(T_1(f)T_2(g))$.

Proof. We divide the proof into several parts. By the hypotheses, we have

$$\|T_1(f)T_2(g)\|_Y = \|S_1(f)S_2(g)\|_X \tag{3.1}$$

for all $f, g \in \mathcal{A}$.

Part 1. *Let $f, g \in \mathcal{A}$ and $j \in \{1,2\}$. Then $|S_j(f)(x)| \leq |S_j(g)(x)|$ for all $x \in X$ if and only if $|T_j(f)(y)| \leq |T_j(g)(y)|$ for all $y \in Y$.*

Suppose that $|S_1(f)(x)| \leq |S_1(g)(x)|$ for all $x \in X$. Then

$$\|S_1(f)\|_X \leq \|S_1(g)\|_X. \tag{3.2}$$

Let $k \in \mathcal{P}(\mathcal{B})$. The surjectivity of T_2 implies that $k = T_2(h)$ for some $h \in \mathcal{A}$. Since

$$|S_1(f)(x)S_2(h)(x)| \le |S_1(g)(x)S_2(h)(x)|,$$

for all $x \in X$, we deduce that

$$\|S_1(f)S_2(h)\|_X \le \|S_1(g)S_2(h)\|_X. \tag{3.3}$$

By $k = T_2(h)$, (3.1), and (3.3), we have

$$\begin{aligned}\|T_1(f)k\|_Y &= \|T_1(f)T_2(h)\|_Y = \|S_1(f)S_2(h)\|_X \le \|S_1(g)S_2(h)\|_X \\ &= \|T_1(g)T_2(h)\|_Y = \|T_1(g)k\|_Y. \end{aligned}\tag{3.4}$$

Since (3.4) holds for all $k \in \mathcal{P}(\mathcal{B})$, by Lemma 2.7, we deduce that $|T_1(f)(y)| \le |T_1(g)(y)|$ for all $y \in Y$.

Similarly, we can show that if $|S_2(f)(x)| \le |S_2(g)(x)|$ for all $x \in X$, then $|T_2(f)(y)| \le |T_2(g)(y)|$ for all $y \in Y$.

Conversely, suppose that $|T_2(f)(y)| \le |T_2(g)(y)|$ for all $y \in Y$. Then

$$\|T_2(f)\|_Y \le \|T_2(g)\|_Y. \tag{3.5}$$

Let $k \in \mathcal{P}(\mathcal{A})$. The surjectivity of S_1 implies that $k = S_1(h)$ for some $h \in \mathcal{A}$. Since

$$|T_1(h)(y)T_2(f)(y)| \le |T_1(h)(y)T_2(g)(y)|,$$

for all $y \in Y$, we deduce that

$$\|T_1(h)T_2(f)\|_Y \le \|T_1(h)T_2(g)\|_Y. \tag{3.6}$$

By $k = S_1(h)$, (3.1), and (3.6), we have

$$\begin{aligned}\|kS_2(f)\|_X &= \|S_1(h)S_2(f)\|_X = \|T_1(h)T_2(f)\|_Y \le \|T_1(h)T_2(g)\|_Y \\ &= \|S_1(h)S_2(g)\|_X = \|kS_2(g)\|_X. \end{aligned}\tag{3.7}$$

Since (3.7) holds for all $k \in \mathcal{P}(\mathcal{A})$, by Lemma 2.7, we deduce that $|S_2(f)(x)| \le |S_2(g)(x)|$ for all $x \in X$.

Similarly, we can show that if $|T_1(f)(y)| \le |T_1(g)(y)|$ for all $y \in Y$, then $|S_1(f)(x)| \le |S_1(g)(x)|$ for all $x \in X$.

Part 2. *Let $x \in X$ and let $\mathscr{A}_j(x) = S_j^{-1}(\mathcal{F}_{x_\tau}(\mathcal{A}))$ for $j \in \{1, 2\}$. Then $S_1(h)S_2(k) \in \mathcal{F}_{x_\tau}(\mathcal{A})$ and $\|T_1(h)T_2(k)\|_Y = \|S_1(h)S_2(k)\|_X = 1$, for all $h \in \mathscr{A}_1(x)$ and $k \in \mathscr{A}_2(x)$.*

Let $h \in \mathscr{A}_1(x)$ and $k \in \mathscr{A}_2(x)$. Then $S_1(h), S_2(k) \in \mathcal{F}_{x_\tau}(\mathcal{A})$. Hence,

$$\|S_1(h)\|_X = |S_1(h)(x)| = |S_1(h)(\tau(x))| = 1$$

and

$$\|S_2(k)\|_X = |S_2(k)(x)| = |S_2(k)(\tau(x))| = 1.$$

Therefore,

$$1 = |(S_1(h)S_2(k))(x)| \le \|S_1(h)\|_X\|S_2(k)\|_X = 1 = |(S_1(h)S_2(k))(x)|,$$

and so

$$1 = \|S_1(h)S_2(k)\|_X = |(S_1(h)S_2(k))(x)| = |(S_1(h)S_2(k))(\tau(x))|.$$

Thus, $S_1(h)S_2(k) \in F_{x_\tau}(\mathcal{A})$. Moreover, by (3.1), we have $\|T_1(h)T_2(k)\|_Y = 1$.

Part 3. *Let $x \in X$ and let*

$$A_{x_\tau} = \bigcap_{h\in\mathscr{A}_1(x),k\in\mathscr{A}_2(x)} M(T_1(h)T_2(k)).$$

Then A_{x_τ} is a nonempty compact subset of Y.

Clearly, $M(T_1(h)T_2(k))$ is a nonempty compact subset of Y for all $h \in \mathscr{A}_1(x)$ and $k \in \mathscr{A}_2(x)$. Therefore, A_{x_τ} is a compact subset of Y. Let $n \in \mathbb{N}$, $h_1, \dots, h_n \in \mathscr{A}_1(x)$ and $k_1, \dots, k_n \in \mathscr{A}_2(x)$. Then $S_1(h_1)\dots S_1(h_n)$ and $S_2(k_1)\dots S_2(k_n) \in F_{x_\tau}(\mathcal{A})$. By the surjectivity of S_1 and S_2, there exist $h, k \in \mathcal{A}$ such that

$$S_1(h_1)\dots S_1(h_n) = S_1(h) \quad \text{and} \quad S_2(k_1)\dots S_2(k_n) = S_2(k). \tag{3.8}$$

By the argument above, $h \in S_1^{-1}(F_{x_\tau}(\mathcal{A})) = \mathscr{A}_1(x)$ and $k \in S_2^{-1}(F_{x_\tau}(\mathcal{A})) = \mathscr{A}_2(x)$. Since $1 = \|S_1(h)S_2(k)\|_X = \|T_1(h)T_2(k)\|_Y$, there exists $y_0 \in Y$ such that $1 = \|T_1(h)T_2(k)\|_Y = |T_1(h)(y_0)T_2(k)(y_0)|$.

Let $i \in \{1, \dots, n\}$. Then $|S_1(h)(x)| \le |S_1(h_i)(x)|$ and $|S_2(k)(x)| \le |S_2(k_i)(x)|$ for all $x \in X$. By Part 1, we have $|T_1(h)(y)| \le |T_1(h_i)(y)|$ and $|T_2(k)(y)| \le |T_2(k_i)(y)|$ for all $y \in Y$. Thus,

$$\begin{aligned} 1 &= \|T_1(h)T_2(k)\|_Y = |T_1(h)(y_0)||T_2(k)(y_0)| \le |T_1(h_i)(y_0)||T_2(k_i)(y_0)| \\ &\le \|T_1(h_i)T_2(k_i)\|_Y = \|S_1(h_i)S_2(k_i)\|_X = 1. \end{aligned}$$

Hence,

$$\|T_1(h_i)T_2(k_i)\|_Y = |T_1(h_i)(y_0)T_2(k_i)(y_0)| = 1. \tag{3.9}$$

Since (3.9), holds for all $i \in \{1, \dots, n\}$, we deduce that $y_0 \in \cap_{i=1}^n M(T_1(h_i)T_2(k_i))$ and so $\cap_{i=1}^n M(T_1(h_i)T_2(k_i)) \neq \emptyset$. By the finite intersection property,

$$\bigcap_{h\in\mathscr{A}_1(x),k\in\mathscr{A}_2(x)} M(T_1(h)T_2(k)) \neq \emptyset,$$

that is, $A_{x_\tau} \neq \emptyset$.

Part 4. *Let $x \in X$, let $y \in A_{x_\tau}$, and let $f, g \in \mathcal{A}$. Then $T_1(f)T_2(g) \in F_{y_\eta}(\mathcal{B})$ if and only if $S_1(f)S_2(g) \in F_{x_\tau}(\mathcal{A})$.*

Let $T_1(f)T_2(g) \in F_{y_\eta}(\mathcal{B})$. Then

$$1 = \|T_1(f)T_2(g)\|_Y = |T_1(f)(y)T_2(g)(y)| = |T_1(f)(\eta(y))T_2(\eta(y))|.$$

By (3.1), we have $\|S_1(f)S_2(g)\|_X = 1$. This implies that $\|S_1(f)\|_X > 0$ and $\|S_2(g)\|_X > 0$. We claim that $S_1(f)(x)S_2(g)(x) \neq 0$. Assume that $S_1(f)(x)S_2(g)(x) = 0$. Without loss of generality, we can assume that $S_1(f)(x) = 0$. By Theorem 2.5(ii), there exists $h \in \mathcal{P}_x(\mathcal{A})$ such that

$$\|S_1(f)h\|_X < \frac{1}{\|S_2(g)\|_X}.$$

The surjectivity of S_1 and S_2 implies that there exist $k_1, k_2 \in \mathcal{A}$ such that $S_1(k_1) = S_2(k_2) = h$. Since $h \in F_{x_\tau}(\mathcal{A})$, we deduce that $k_1 \in \mathscr{A}_1(x)$ and $k_2 \in \mathscr{A}_2(x)$. Hence,

$y \in M(T_1(k_1)T_2(k_2))$ since $y \in A_{x_\tau}$. This implies that

$$|(T_1(k_1)T_2(k_2))(y)| = \|T_1(k_1)T_2(k_2)\|_Y = \|S_1(k_1)S_2(k_2)\|_X = (\|h\|_X)^2 = 1,$$

and so $T_1(k_1)T_2(k_2) \in F_{y_\eta}(\mathcal{B})$. Hence,

$$\begin{aligned}1 &= |(T_1(f)T_2(g))(y)||(T_1(k_1)T_2(k_2))(y)| \le \|T_1(f)T_2(g)T_1(k_1)T_2(k_2)\|_Y \\ &\le \|T_1(f)T_2(k_2)\|_Y\|T_1(k_1)T_2(g)\|_Y = \|S_1(f)S_2(k_2)\|_X\|S_1(k_1)S_2(g)\|_X \\ &= \|S_1(f)h\|_X\|hS_2(g)\|_X \le \|S_1(f)h\|_X\|h\|_X\|S_2(g)\|_X \\ &< \frac{1}{\|S_2(g)\|_X}\|S_2(g)\|_X = 1,\end{aligned}$$

since $\|h\|_X = 1$ and $\|S_2(g)\|_X > 0$, which is a contradiction. Hence, our claim is justified. By Theorem 2.5(i), there exist $h_1, h_2 \in \mathcal{P}_x(\mathcal{A})$ such that

$$\mathrm{Ran}_{\pi,X}(S_1(f)h_2) = \{S_1(f)(x), S_1(f)(\tau(x))\}$$

and

$$\mathrm{Ran}_{\pi,X}(S_2(g)h_1) = \{S_2(g)(x), S_2(g)(\tau(x))\}.$$

Hence, $\|S_1(f)h_2\|_X = |S_1(f)(x)|$ and $\|S_2(g)h_1\|_X = |S_2(g)(x)|$. By the surjectivity of S_1 and S_2, there exist $k_1, k_2 \in \mathcal{A}$ such that $S_1(k_1) = h_1$ and $S_2(k_2) = h_2$. Hence, $k_1 \in \mathscr{A}_1(x)$ and $k_2 \in \mathscr{A}_2(x)$. Thus, $y \in M(T_1(k_1)T_2(k_2))$ since $y \in A_{x_\tau}$. Therefore,

$$\begin{aligned}\|S_1(f)S_2(g)\|_X &\ge |S_1(f)(x)||S_2(g)(x)| = \|S_1(f)h_2\|_X\|h_1S_2(g)\|_X \\ &= \|S_1(f)S_2(k_2)\|_X\|S_1(k_1)S_2(g)\|_X \\ &= \|T_1(f)T_2(k_2)\|_Y\|T_1(k_1)T_2(g)\|_Y \\ &\ge \|T_1(f)T_2(k_2)T_1(k_1)T_2(g)\|_Y \\ &\ge |T_1(f)(y)T_2(g)(y)||T_1(k_1)(y)T_2(k_2)(y)| \\ &= 1 \\ &= \|S_1(f)S_2(g)\|_X.\end{aligned}$$

Hence, $|S_1(f)(x)S_2(g)(x)| = 1$. Therefore, $S_1(f)S_2(g) \in F_{x_\tau}(\mathcal{A})$. Similarly, we can show that $S_1(f)S_2(g) \in F_{x_\tau}(\mathcal{A})$ implies that $T_1(f)T_2(g) \in F_{y_\eta}(\mathcal{B})$.

Part 5. *Let $x \in X$ and let $y \in A_{x_\tau}$. Then $A_{x_\tau} = y_\eta = \{y, \eta(y)\}$.*

Since $y \in A_{x_\tau}$, we have $y \in M(T_1(h)T_2(k))$ for all $h \in \mathscr{A}_1(x)$ and $k \in \mathscr{A}_2(x)$. It follows that $\eta(y) \in M(T_1(h)T_2(k))$ for all $h \in \mathscr{A}_1(x)$ and $k \in \mathscr{A}_2(x)$ since $M(k)$ is η-invariant for all $k \in \mathcal{B}$. Hence,

$$y_\eta = \{y, \eta(y)\} \subseteq A_{x_\tau}. \tag{3.10}$$

Suppose that $w \in A_{x_\tau} \setminus \{y, \eta(y)\}$. By Lemma 2.1, there exists a function $k \in \mathcal{P}_y(\mathcal{B})$ such that $0 \le k(w) < 1$. The surjectivity of T_1 and T_2 implies that there exist $h_1, h_2 \in \mathcal{A}$ such that $T_1(h_1) = T_2(h_2) = k$. Since $k \in F_{y_\eta}(\mathcal{B})$, we deduce that $k^2 \in F_{y_\eta}(\mathcal{B})$ and so $T_1(h_1)T_2(h_2) \in F_{y_\eta}(\mathcal{B})$. By Part 4, $S_1(h_1)S_2(h_2) \in F_{x_\tau}(\mathcal{A})$.

Hence, by Part 4, $k^2 = T_1(h_1)T_2(h_2) \in F_{w_\eta}(\mathcal{B})$ since $w \in A_{x_\tau}$. This implies that $|k^2(w)| = 1$ and so $|k(w)| = 1$, which is a contradiction to $k(w) < 1$. Therefore,

$$A_{x_\tau} \subseteq \{y, \eta(y)\} = y_\eta. \tag{3.11}$$

By (3.10) and (3.11), we have $A_{x_\tau} = y_\eta = \{y, \eta(y)\}$.

Part 5 allows us to define a map $\Psi : X_\tau \to Y_\eta$ by

$$\Psi(x_\tau) = A_{x_\tau} = y_\eta \qquad (x \in X,\ y \in A_{x_\tau}). \tag{3.12}$$

Part 6. *The mapping $\Psi : X_\tau \to Y_\eta$ defined by (3.12) is a bijection.*

Let $x_\tau, z_\tau \in X_\tau$, and suppose that $\Psi(x_\tau) = \Psi(z_\tau)$. Let $h \in F_{x_\tau}(\mathcal{A})$ be given. The surjectivity of S_1 and S_2 implies that there exist $k_1, k_2 \in \mathcal{A}$ such that $S_1(k_1) = S_2(k_2) = h$. Since $h^2 \in F_{x_\tau}(\mathcal{A})$, we deduce that $S_1(k_1)S_2(k_2) \in F_{x_\tau}(\mathcal{A})$. By the definition of Ψ and Part 4, $T_1(k_1)T_2(k_2) \in F_{\Psi(x_\tau)}(\mathcal{B})$ and so $T_1(k_1)T_2(k_2) \in F_{\Psi(z_\tau)}(\mathcal{B})$. Applying Part 4 and the definition of Ψ, we deduce that $h^2 = S_1(k_1)S_2(k_2) \in F_{z_\tau}(\mathcal{A})$. This implies that $h \in F_{z_\tau}(\mathcal{A})$. Hence, $F_{x_\tau}(\mathcal{A})$ is a subset of $F_{z_\tau}(\mathcal{A})$. Therefore, $x_\tau = z_\tau$ by Lemma 2.8, and so Ψ is injective.

Now let $y \in Y$. Define $\mathscr{B}_1(y) = T_1^{-1}(\mathcal{F}_{y_\eta}(\mathcal{B}))$, $\mathscr{B}_2(y) = T_2^{-1}(\mathcal{F}_{y_\eta}(\mathcal{B}))$, and

$$B_{y_\eta} = \bigcap_{h \in \mathscr{B}_1(y), k \in \mathscr{B}_2(y)} M(S_1(h)S_2(k)). \tag{3.13}$$

Then B_{y_η} is a nonempty τ-invariant subset of X, where its proof is analogous to the proof of Part 3. Assume that $x \in B_{y_\eta}$. Let $k \in \mathcal{F}_{y_\eta}(\mathcal{B})$ be given. The surjectivity of T_1 and T_2 implies that there exist $h_1, h_2 \in \mathcal{A}$ such that $k = T_1(h_1) = T_2(h_2)$. Clearly, $h_1 \in \mathscr{B}_1(y)$ and $h_2 \in \mathscr{B}_2(y)$. Hence, by (3.13) and $x \in B_{y_\eta}$, we deduce that $x \in M(S_1(h_1)S_2(h_2))$. This implies that $S_1(h_1)S_2(h_2) \in \mathcal{F}_{x_\tau}(\mathcal{A})$ since $M(f)$ is a τ-invariant subset of X for all $f \in \mathcal{A}$. By the definition of Ψ and Part 4, we deduce that $k^2 = T_1(h_1)T_2(h_2) \in \mathcal{F}_{\Psi(x_\tau)}(\mathcal{B})$. This implies that $k \in \mathcal{F}_{\Psi(x_\tau)}(\mathcal{B})$. Hence, $\mathcal{F}_{y_\eta}(\mathcal{B})$ is a subset of $\mathcal{F}_{\Psi(x_\tau)}(\mathcal{B})$. Thus, $y_\eta = \Psi(x_\tau)$ by Lemma 2.8. Therefore, Ψ is surjective.

Part 7. *Let $x \in X$ and let $y \in \Psi(x_\tau)$. Then*

$$|S_1(f)(x)S_2(g)(x)| = |T_1(f)(y)T_2(g)(y)|,$$

for all $f, g \in \mathcal{A}$.

Let $f, g \in \mathcal{A}$. If any of $S_1(f)$, $S_2(g)$, $T_1(f)$, or $T_2(g)$ is identically 0, then the result follows by (3.1). So we assume that none of $S_1(f)$, $S_2(g)$, $T_1(f)$, and $T_2(g)$ is identically 0. Now, suppose that $S_1(f)(x)S_2(g)(x) = 0$. Then either $S_1(f)(x) = 0$ or $S_2(g)(x) = 0$. Without loss of generality, we can assume that $S_1(f)(x) = 0$. To prove $|T_1(f)(y)T_2(g)(y)| = 0$, let $\varepsilon > 0$ be given. By Theorem 2.5(ii), there exists a function $h \in \mathcal{P}_x(\mathcal{A})$ such that

$$\|S_1(f)h\|_X < \frac{\varepsilon}{\|S_2(g)\|_X}. \tag{3.14}$$

The surjectivity of S_1 and S_2 implies that there exist $k_1, k_2 \in \mathcal{A}$ such that $h = S_1(k_1) = S_2(k_2)$. Thus, $S_1(k_1)S_2(k_2) = h^2 \in F_{x_\tau}(\mathcal{A})$. By Part 4, $T_1(k_1)T_2(k_2) \in F_{y_\eta}(\mathcal{B})$. Therefore, by (3.1) and (3.14), we have

$$|T_1(f)(y)T_2(g)(y)| = |T_1(f)(y)T_2(k_2)(y)T_1(k_1)(y)T_2(g)(y)|$$

$$
\begin{aligned}
&\le \|T_1(f)T_2(k_2)\|_Y \|T_1(k_1)T_2(g)\|_Y \\
&= \|S_1(f)S_2(k_2)\|_X \|S_1(k_1)S_2(g)\|_X \\
&= \|S_1(f)h\|_X \|S_2(g)h\|_X \\
&\le \|S_1(f)h\|_X \|S_2(g)\|_X \|h\|_X \\
&= \|S_1(f)h\|_X \|S_2(g)\|_X \\
&< \varepsilon.
\end{aligned}
$$

As ε was chosen arbitrary, $|T_1(f)(y)T_2(g)(y)| = 0$. Applying a similar argument, we can show that if $T_1(f)(y)T_2(g)(y) = 0$, then $|S_1(f)(x)S_2(g)(x)| = 0$. Since $T_1(f)(y)T_2(g)(y) = 0$ if and only if $S_1(f)(x)S_2(g)(x) = 0$, we deduce that $S_1(f)(x)S_2(g)(x) \neq 0$ if and only if $T_1(f)(y)T_2(g)(y) \neq 0$. Suppose that $S_1(f)(x)S_2(g)(x) \neq 0$. Then $S_1(f)(x) \neq 0$ and $S_2(g)(x) \neq 0$. By Theorem 2.5(i), there exist $h_1, h_2 \in \mathcal{P}_x(\mathcal{A})$ such that

$$\mathrm{Ran}_{\pi,X}(S_1(f)h_1) = \{S_1(f)(x), S_1(f)(\tau(x))\}$$

and

$$\mathrm{Ran}_{\pi,X}(S_2(g)h_2) = \{S_2(g)(x), S_2(g)(\tau(x))\}.$$

The surjectivity of S_1 and S_2 implies that there exist $k_1, k_2 \in \mathcal{A}$ such that $S_1(k_1) = h_2$ and $S_2(k_2) = h_1$. Thus, $S_1(k_1)S_2(k_2) = h_2h_1 \in \mathcal{P}_x(\mathcal{A})$ and so $S_1(k_1)S_2(k_2) \in F_{x_\tau}(\mathcal{A})$. By Part 4, $T_1(k_1)T_2(k_2) \in F_{y_\eta}(\mathcal{B})$. Hence,

$$
\begin{aligned}
|T_1(f)(y)T_2(g)(y)| &= |T_1(f)(y)T_2(k_2)(y)||T_1(k_1)(y)T_2(g)(y)| \\
&\le \|T_1(f)T_2(k_2)\|_Y \|T_1(k_1)T_2(g)\|_Y \\
&= \|S_1(f)S_2(k_2)\|_X \|S_1(k_1)S_2(g)\|_X \\
&= \|S_1(f)h_1\|_X \|S_2(g)h_2\|_X \\
&= |S_1(f)(x)||S_2(g)(x)| \\
&= |S_1(f)(x)S_2(g)(x)|.
\end{aligned}
$$

By the argument above, $T_1(f)(y)T_2(g)(y) \neq 0$. An analogous argument gives that

$$|S_1(f)(x)S_2(g)(x)| \le |T_1(f)(y)T_2(g)(y)|.$$

Hence, $|S_1(f)(x)S_2(g)(x)| = |T_1(f)(y)T_2(g)(y)|$.

Part 8. *Let $f, g \in \mathcal{A}$, let $x \in X$, and let $y \in \Psi(x_\tau)$. Then $x \in M(S_1(f)S_2(g))$ if and only if $y \in M(T_1(f)T_2(g))$.*

By Part 7 and Lemma 3.1, the proof is clear.

The proof of the theorem is now complete. □

4. Peripherally multiplicative maps

In this section, we study surjective jointly peripherally multiplicative spectrum preserving maps between real Lipschitz algebra with involution and prove that such mappings are essentially weighted composition operators. Throughout this section, assume that (X, d) and (Y, ρ) are compact metric spaces, that $\tau : X \to X$ and $\eta : Y \to Y$ are Lipschitz involutions on (X, d) and (Y, ρ), respectively, that $\mathcal{A} = \mathrm{Lip}(X, d, \tau)$, and that $\mathcal{B} = \mathrm{Lip}(Y, \rho, \eta)$.

Theorem 4.1. *Let $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ be surjective jointly peripherally multiplicative spectrum preserving or, equivalently, jointly peripherally multiplicative range preserving maps. Then there are two functions $\kappa_1, \kappa_2 \in \mathcal{B}$ with $\kappa_1\kappa_2 = 1_Y$ and a Lipschitz homeomorphism φ from (Y, ρ) to (X, d) with $\tau \circ \varphi = \varphi \circ \eta$ on Y such that*

$$T_j(f) = \kappa_j \cdot (S_j(f) \circ \varphi),$$

for all $f \in \mathcal{A}$ and $j = 1, 2$.

Proof. We divide the proof into several steps. By Lemma 2.10 and the hypotheses, we have

$$\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) = \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)), \tag{4.1}$$

for all $f, g \in \mathcal{A}$. This implies that

$$\|T_1(f)T_2(g)\|_Y = \|S_1(f)S_2(g)\|_X, \tag{4.2}$$

for all $f, g \in \mathcal{A}$. Then by Theorem 3.2, there exists a bijection mapping $\Psi : X_\tau \to Y_\eta$ such that

$$|T_1(f)(y)T_2(g)(y)| = |S_1(f)(x)S_2(g)(x)|,$$

for all $f, g \in \mathcal{A}$, $x \in X$ and $y \in \Psi(x_\tau)$.

Step 1. *If $f, g \in \mathcal{A}$ with $S_1(f)S_2(g) = 1_X$, then $T_1(f)T_2(g) = 1_Y$.*
Let $f, g \in \mathcal{A}$ with $S_1(f)S_2(g) = 1_X$. Then we have

$$\mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)) = \{1\} \tag{4.3}$$

and

$$M(S_1(f)S_2(g)) = X. \tag{4.4}$$

By (4.1) for f, g and (4.3), we have

$$\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) = \{1\}. \tag{4.5}$$

On the other hand, by (4.2), we have

$$\|T_1(f)T_2(g)\|_Y = \|S_1(f)S_2(g)\|_X = \|1_X\|_X = 1.$$

Let $y \in Y$. By the surjectivity of Ψ, there exists $x \in X$ such that

$$\Psi(x_\tau) = y_\eta. \tag{4.6}$$

Applying (4.6) and (4.4), we deduce that $y \in M(T_1(f)T_2(g))$. It follows that

$$1 = \|T_1(f)T_2(g)\|_Y = |T_1(f)(y)T_2(g)(y)|. \tag{4.7}$$

By (4.7), we have $T_1(f)(y)T_2(g)(y) \in \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g))$ and so $T_1(f)(y)T_2(g)(y) = 1$, by (4.5). Since $y \in Y$ was chosen arbitrarily, we deduce that $T_1(f)T_2(g) = 1_Y$.

Step 2.

(i) *If $f_1, g_1 \in \mathcal{A}$ with $S_1(f_1) = S_1(g_1) = 1_X$, then $T_1(f_1) = T_1(g_1)$.*
(ii) *If $f_2, g_2 \in \mathcal{A}$ with $S_2(f_2) = S_2(g_2) = 1_X$, then $T_2(f_2) = T_2(g_2)$.*

Let $f_1, g_1 \in \mathcal{A}$ with $S_1(f_1) = S_1(g_1) = 1_X$. By the surjectivity of S_2, there exists $h \in \mathcal{A}$ such that $S_2(h) = 1_X$. By Step 1, we have

$$T_1(f_1)T_2(h) = 1_Y = T_1(g_1)T_2(h).$$

This implies that $T_1(f_1) = T_1(g_1)$ and so (i) holds.

Similarly, we can show that (ii) holds. Step 2 allows us to define two functions $\kappa_1, \kappa_2 \in \mathcal{B}$ by

$$\kappa_1 = T_1(h) \quad (h \in \mathcal{A},\ S_1(h) = 1_X), \quad \kappa_2 = T_2(h) \quad (h \in \mathcal{A},\ S_2(h) = 1_X). \tag{4.8}$$

By Step 1, we have $\kappa_1\kappa_2 = 1_Y$.

Step 3. *Define the maps $\tilde{T}_1, \tilde{T}_2 : \mathcal{A} \to \mathcal{B}$ by*

$$\tilde{T}_1(f) = T_1(f)\kappa_2 \quad (f \in \mathcal{A}), \qquad \tilde{T}_2(f) = T_2(f)\kappa_1 \quad (f \in \mathcal{A}).$$

Then the following properties hold:

(i) *$\tilde{T}_1$ and $\tilde{T}_2$ are surjective mappings.*
(ii) *$\tilde{T}_1(f)\tilde{T}_2(g) = T_1(f)T_2(g)$ for all $f, g \in \mathcal{A}$.*
(iii) *$\mathrm{Ran}_{\pi,Y}(\tilde{T}_1(f)\tilde{T}_2(g)) = \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g))$ for all $f, g \in \mathcal{A}$.*
(iv) *$\mathrm{Ran}_{\pi,Y}(\tilde{T}_j(f)) = \mathrm{Ran}_{\pi,X}(S_j(f))$ for all $f \in \mathcal{A}$ and $j \in \{1, 2\}$.*
(v) *For $f \in \mathcal{A}$, $S_j(f) \in \mathcal{P}(\mathcal{A})$ if and only if $\tilde{T}_j(f) \in \mathcal{P}(\mathcal{B})$, where $j \in \{1, 2\}$.*
(vi) *For $f \in \mathcal{A}$, $S_j(f) \in i\mathcal{P}(\mathcal{A})$ if and only if $\tilde{T}_j(f) \in i\mathcal{P}(\mathcal{B})$, where $j \in \{1, 2\}$.*
(vii) *$|\tilde{T}_j(f)(y)| = |S_j(f)(x)|$ for all $f \in \mathcal{A}$ and $j \in \{1, 2\}$, where $x \in X$ and $y \in \Psi(x_\tau)$.*
(viii) *$S_j(f) \in \mathcal{P}_x(\mathcal{A})$ if and only if $\tilde{T}_j(f) \in \mathcal{P}_y(\mathcal{B})$, where $j \in \{1, 2\}$, $f \in \mathcal{A}$, $x \in X$ and $y \in \Psi(x_\tau)$.*

To prove (i), let $h \in \mathcal{B}$. Then $h\kappa_1 \in \mathcal{B}$. The surjectivity of T_1 implies that there exists $f \in \mathcal{A}$ such that $T_1(f) = h\kappa_1$. Hence,

$$\tilde{T}_1(f) = T_1(f)\kappa_2 = h\kappa_1\kappa_2 = h1_Y = h.$$

This shows that $\tilde{T}_1$ is surjective.

Similarly, we can show that $\tilde{T}_2$ is surjective. Hence, (i) holds.

(ii) Let $f, g \in \mathcal{A}$. Then

$$\tilde{T}_1(f)\tilde{T}_2(g) = T_1(f)\kappa_2T_2(g)\kappa_1 = \kappa_1\kappa_2T_1(f)T_2(g) = 1_YT_1(f)T_2(g) = T_1(f)T_2(g).$$

(iii) It follows from (ii).

(iv) Assume that $j \in \{1, 2\}$. Let $f \in \mathcal{A}$. The surjectivity of S_j implies that $S_j(h_j) = 1_X$ for some $h_j \in \mathcal{A}$. Then $T_j(h_j) = \kappa_j$. Hence,

$$\begin{aligned}\mathrm{Ran}_{\pi,Y}(\tilde{T}_1(f)) &= \mathrm{Ran}_{\pi,Y}(T_1(f)\kappa_2) = \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(h_2)) \\ &= \mathrm{Ran}_{\pi,X}(S_1(f)S_2(h_2)) = \mathrm{Ran}_{\pi,X}(S_1(f)1_X) \\ &= \mathrm{Ran}_{\pi,X}(S_1(f)).\end{aligned}$$

Similarly, we can show that $\mathrm{Ran}_{\pi,Y}(\tilde{T}_2(f)) = \mathrm{Ran}_{\pi,X}(S_2(f))$.

(v) Let $f \in \mathcal{A}$ and $j \in \{1, 2\}$. By (iv), we have $\mathrm{Ran}_{\pi,Y}(\tilde{T}_j(f)) = \mathrm{Ran}_{\pi,X}(S_j(f))$. Therefore,

$$S_j(f) \in \mathcal{P}(\mathcal{A}) \Leftrightarrow \mathrm{Ran}_{\pi,X}(S_j(f)) = \{1\} \Leftrightarrow \mathrm{Ran}_{\pi,Y}(\tilde{T}_j(f)) = \{1\} \Leftrightarrow \tilde{T}_j(f) \in \mathcal{P}(\mathcal{B}).$$

(vi) Let $f\in\mathcal{A}$ and $j\in\{1,2\}$. By (iv), we have $\mathrm{Ran}_{\pi,Y}(\tilde{T}_j(f))=\mathrm{Ran}_{\pi,X}(S_j(f))$. Therefore,

$$\begin{aligned} S_j(f)\in i\mathcal{P}(\mathcal{A}) &\Leftrightarrow \mathrm{Ran}_{\pi,X}(S_j(f))=\{-i,i\}\Leftrightarrow \mathrm{Ran}_{\pi,Y}(\tilde{T}_j(f))\\ &=\{-i,i\}\Leftrightarrow \tilde{T}_j(f)\in i\mathcal{P}(\mathcal{B}). \end{aligned}$$

(vii) Let $x\in X$ and let $y\in\Psi(x_\tau)$. The surjectivity of S_j implies that $S_j(h_j)=1_X$ for some $h_j\in\mathcal{A}$. Then $T_j(h_j)=\kappa_j$. Hence, by Theorem 3.2, for f and h_2,

$$\begin{aligned} |\tilde{T}_1(f)(y)| &= |T_1(f)(y)\kappa_2(y)|=|T_1(f)(y)T_2(h_2)(y)|\\ &=|S_1(f)(x)S_2(h_2)(x)|=|S_1(f)(x)|. \end{aligned}$$

Similarly, we can show that $|\tilde{T}_2(f)(y)|=|S_2(f)(x)|$.

(viii) Let $j\in\{1,2\}$, let $f\in\mathcal{A}$, $x\in X$, and let $y\in\Psi(x_\tau)$. We first assume that $S_j(f)\in\mathcal{P}_x(\mathcal{A})$. Then $S_j(f)\in\mathcal{P}(\mathcal{A})$ and $S_j(f)(x)=1$. Hence, $\tilde{T}_j(f)\in\mathcal{P}(\mathcal{B})$ by (v) and $|\tilde{T}_j(f)(y)|=|S_j(f)(x)|$ by (vii). Therefore, $\mathrm{Ran}_{\pi,Y}(\tilde{T}_j(f))=\{1\}$ and $|\tilde{T}_j(f)(y)|=1$. Hence, $\tilde{T}_j(f)(y)=1$. Thus, $\tilde{T}_j(f)\in\mathcal{P}_y(\mathcal{B})$.

Similarly, we can show that $\tilde{T}_j(f)\in\mathcal{P}_y(\mathcal{B})$ implies that $S_j(f)\in\mathcal{P}_x(\mathcal{A})$.

Step 4. *$\tau(x)=x$ if and only if $\eta(y)=y$, whenever $x\in X$ and $y\in\Psi(x_\tau)$.*

Let $x\in X$ with $\tau(x)=x$ and let $y\in\Psi(x_\tau)$. To prove $\eta(y)=y$, it is sufficient to show that $g(\eta(y))=g(y)$ for all $g\in\mathcal{B}$, since $\mathcal{B}$ separates the points of Y. Let $g\in\mathcal{B}$. If $g(y)=0$, then $g(\eta(y))=\overline{g(y)}=g(y)$. Suppose that $g(y)\neq 0$. By Theorem 2.5, there exists a function $k\in\mathcal{P}_y(\mathcal{B})$ such that

$$\mathrm{Ran}_{\pi,Y}(gk)=\{g(y),g(\eta(y))\}. \tag{4.9}$$

The surjectivity of $\tilde{T}_1$ and $\tilde{T}_2$ implies that there exist $f_1,f_2\in A$ such that $\tilde{T}_1(f_1)=g$ and $\tilde{T}_2(f_2)=k$. By part (iii) of Step 3 and (4.9), we have

$$\begin{aligned} \mathrm{Ran}_{\pi,X}(S_1(f_1)S_2(f_2)) &= \mathrm{Ran}_{\pi,Y}(\tilde{T}_1(f_1)\tilde{T}_2(f_2))=\mathrm{Ran}_{\pi,Y}(gk)\\ &=\{g(y),g(\eta(y))\}. \end{aligned}\tag{4.10}$$

On the other hand, by (4.9) and $k\in\mathcal{P}_y(\mathcal{B})$ we have $\|gk\|_Y=|g(y)|=|g(y)k(y)|$, which implies that $y\in M(gk)=M(\tilde{T}_1(f_1)\tilde{T}_2(f_2))=M(T_1(f_1)T_2(f_2))$. Hence, $x\in M(S_1(f_1)S_2(f_2))$ by Theorem 3.2. Thus,

$$S_1(f_1)(x)S_2(f_2)(x)\in\mathrm{Ran}_{\pi,X}(S_1(f_1)S_2(f_2))$$

and so by (4.10), we have

$$S_1(f_1)(x)S_2(f_2)(x)\in\{g(y),g(\eta(y))\}. \tag{4.11}$$

Since $\tau(x)=x$, we have $S_1(f_1)(x)S_2(f_2)(x)\in\mathbb{R}$. Hence, by (4.11) we have $g(y)\in\mathbb{R}$ or $g(\eta(y))\in\mathbb{R}$. It follows that $g(y)=g(\eta(y))$.

Similarly, we can show that $\tau(x)=x$ wherever, $\eta(y)=y$.

Step 5. *Let $x\in X$ with $\tau(x)\neq x$ and let $y\in\Psi(x_\tau)$. Then $\tilde{T}_1(h)(y)\tilde{T}_2(k)(y)=-1$ for all $h\in S_1^{-1}(i\mathcal{P}_x(\mathcal{A}))$ and $k\in S_2^{-1}(i\mathcal{P}_x(\mathcal{A}))$.*

By Step 4, we have $\eta(y)\neq y$. Therefore, $i\mathcal{P}_x(\mathcal{A})\neq\emptyset$ and $i\mathcal{P}_y(\mathcal{B})\neq\emptyset$ by Corollary 2.4. Assume that $h\in S_1^{-1}(i\mathcal{P}_x(\mathcal{A}))$ and $k\in S_2^{-1}(i\mathcal{P}_x(\mathcal{A}))$. Then $S_1(h),S_2(k)\in i\mathcal{P}_x(\mathcal{A})$ and so $S_1(h),S_2(k)\in i\mathcal{P}(\mathcal{A})$. Thus, $\tilde{T}_1(h),\tilde{T}_2(k)\in i\mathcal{P}(\mathcal{B})$

by part (vi) of Step 3. Since $x \in X$ and $y \in \Psi(x_\tau)$, we have $|\tilde{T}_1(h)(y)| = |S_1(h)(x)|$ and $|\tilde{T}_2(k)(y)| = |S_2(k)(x)|$ by part (vii) of Step 3. On the other hand, $|S_1(h)(x)| = 1$ and $|S_2(k)(x)| = 1$ since $S_1(h), S_2(k) \in i\mathcal{P}_x(\mathcal{A})$. Therefore,

$$|\tilde{T}_1(h)(y)| = 1 = |\tilde{T}_2(k)(y)|.$$

This implies that $\tilde{T}_1(h)(y), \tilde{T}_2(k)(y) \in \{-i, i\}$ since $\tilde{T}_1(h), \tilde{T}_2(k) \in i\mathcal{P}(\mathcal{B})$. Hence, $\tilde{T}_1(h)(y)\tilde{T}_2(k)(y) \in \{-1, 1\}$. Since $(\tilde{T}_1(h)\tilde{T}_2(k))(y) \in \mathbb{R} \setminus \{0\}$, by Theorem 2.5(i), there exists $g \in \mathcal{P}_y(\mathcal{B})$ such that

$$\operatorname{Ran}_{\pi,Y}(\tilde{T}_1(h)\tilde{T}_2(k)g) = \{\tilde{T}_1(h)(y)\tilde{T}_2(k)(y)\}. \tag{4.12}$$

We claim that $\operatorname{Ran}_{\pi,Y}(\tilde{T}_1(h)\tilde{T}_2(k)g^2) = \{\tilde{T}_1(h)(y)\tilde{T}_2(k)(y)\}$. Since $g(y) = 1$, by (4.12) we have

$$\tilde{T}_1(h)(y)\tilde{T}_2(k)(y) \in \operatorname{Ran}_{\pi,Y}(\tilde{T}_1(h)\tilde{T}_2(k)g^2). \tag{4.13}$$

Let $z \in Y \setminus \{y, \eta(y)\}$. Then

$$\begin{aligned}|\tilde{T}_1(h)(z)\tilde{T}_2(k)(z)g^2(z)| &\le \|\tilde{T}_1(h)\tilde{T}_2(k)\|_Y |g(z)|^2 = |g(z)|^2 \\ &< 1 = |\tilde{T}_1(h)(y)\tilde{T}_2(k)(y)|. \end{aligned}\tag{4.14}$$

By (4.13) and (4.14), we deduce that

$$\operatorname{Ran}_{\pi,Y}(\tilde{T}_1(h)\tilde{T}_2(k)g^2) = \{\tilde{T}_1(h)(y)\tilde{T}_2(k)(y)\}, \tag{4.15}$$

since $\tilde{T}_1(h)(y)\tilde{T}_2(k)(y) \in \mathbb{R}$. Hence, our claim is justified.

The surjectivity of $\tilde{T}_1$ and $\tilde{T}_2$ implies that there exist $f_1, f_2 \in \mathcal{A}$ such that $\tilde{T}_1(f_1) = \tilde{T}_1(h)g$ and $\tilde{T}_2(f_2) = \tilde{T}_2(k)g$. From (4.12) and (4.15), we have

$$\begin{aligned}\operatorname{Ran}_{\pi,X}(S_1(f_1)S_2(f_2)) = \operatorname{Ran}_{\pi,Y}(\tilde{T}_1(f_1)\tilde{T}_2(f_2)) &= \operatorname{Ran}_{\pi,Y}(\tilde{T}_1(h)\tilde{T}_2(k)g^2) \\ &= \{\tilde{T}_1(h)(y)\tilde{T}_2(k)(y)\}. \end{aligned}\tag{4.16}$$

By (4.12), we get

$$\begin{aligned}\operatorname{Ran}_{\pi,X}(S_1(h)S_2(f_2)) = \operatorname{Ran}_{\pi,Y}(\tilde{T}_1(h)\tilde{T}_2(f_2)) &= \operatorname{Ran}_{\pi,Y}(\tilde{T}_1(h)\tilde{T}_2(k)g) \\ &= \{\tilde{T}_1(h)(y)\tilde{T}_2(k)(y)\}. \end{aligned}\tag{4.17}$$

Part (iii) of Step 3 implies that

$$\begin{aligned}\operatorname{Ran}_{\pi,X}(S_1(f_1)S_2(k)) = \operatorname{Ran}_{\pi,Y}(\tilde{T}_1(f_1)\tilde{T}_2(k)) &= \operatorname{Ran}_{\pi,Y}(\tilde{T}_1(h)g\tilde{T}_2(k)) \\ &= \operatorname{Ran}_{\pi,Y}(\tilde{T}_1(h)\tilde{T}_2(f_2)) = \operatorname{Ran}_{\pi,X}(S_1(h)S_2(f_2)). \end{aligned}\tag{4.18}$$

By (4.17) and (4.18), $y \in M(\tilde{T}_1(h)\tilde{T}_2(f_2)) \cap M(\tilde{T}_1(f_1)\tilde{T}_2(k))$. It follows that $x \in M(S_1(h)S_2(f_2)) \cap M(S_1(f_1)S_2(k))$ by Theorem 3.2. Hence,

$$iS_1(f_1)(x) = S_1(f_1)(x)S_2(k)(x) \in \operatorname{Ran}_{\pi,X}(S_1(f_1)S_2(k)) \tag{4.19}$$

and

$$iS_2(f_2)(x) = S_1(h)(x)S_2(f_2)(x) \in \operatorname{Ran}_{\pi,X}(S_1(h)S_2(f_2)). \tag{4.20}$$

From (4.17)–(4.20), we get

$$iS_1(f_1)(x) = \tilde{T}_1(h)(y)\tilde{T}_2(k)(y) = iS_2(f_2)(x). \tag{4.21}$$

By (4.21) and $\tilde{T}_1(h)(y)\tilde{T}_2(k)(y) \in \{-1, 1\}$, we obtain

$$S_1(f_1)(x)S_2(f_2)(x) = -1. \tag{4.22}$$

From (4.16), we have $y \in M(\tilde{T}_1(f_1)\tilde{T}_2(f_2))$. Hence, $x \in M(S_1(f_1)S_2(f_2))$ by Theorem 3.2. Therefore, by (4.22), we have $-1 \in \mathrm{Ran}_{\pi,X}(S_1(f_1)S_2(f_2))$. Hence, $\tilde{T}_1(h)(y)\tilde{T}_2(k)(y) = -1$ by (4.16).

Step 6. *Let $x \in X$ with $\tau(x) \neq x$ and let $y \in \Psi(x_\tau)$. Then there is a unique $y' \in \Psi(x_\tau)$ such that $\tilde{T}_j(h)(y') = i$ for all $h \in S_j^{-1}(i\mathcal{P}_x(\mathcal{A}))$ where $j \in \{1, 2\}$.*

Since $\tau(x) \neq x$, by Corollary 2.4, we have $i\mathcal{P}_x(\mathcal{A}) \neq \emptyset$. The surjectivity of S_1 and S_2 implies that $S_1^{-1}(i\mathcal{P}_x(\mathcal{A})) \neq \emptyset$ and $S_2^{-1}(i\mathcal{P}_x(\mathcal{A})) \neq \emptyset$.

We claim that for each $h \in S_1^{-1}(i\mathcal{P}_x(\mathcal{A}))$, there exist $y_h \in \Psi(x_\tau)$ such that

$$\tilde{T}_1(h)(y_h) = i.$$

Let $h \in S_1^{-1}(i\mathcal{P}_x(\mathcal{A}))$. Then $S_1(h) \in i\mathcal{P}_x(\mathcal{A})$. This implies that $\mathrm{Ran}_{\pi,X}(S_1(h)) = \{i, -i\}$ and $S_1(h)(x) = i$. By part (iv) of Step 3, $\mathrm{Ran}_{\pi,Y}(\tilde{T}_1(h)) = \mathrm{Ran}_{\pi,X}(S_1(h))$. Thus, $\mathrm{Ran}_{\pi,Y}(\tilde{T}_1(h)) = \{-i, i\}$. Since $y \in \Psi(x_\tau)$, we have

$$|\tilde{T}_1(h)(y)| = |S_1(h)(x)| = |i| = 1.$$

Hence, $\tilde{T}_1(h)(y) \in \mathrm{Ran}_{\pi,Y}(\tilde{T}_1(h))$ and so $\tilde{T}_1(h)(y) \in \{-i, i\}$. Choose $y_h = y$ if $\tilde{T}_1(h)(y) = i$ and $y_h = \eta(y)$ if $\tilde{T}_1(h)(y) = -i$. Then $y_h \in \Psi(x_\tau)$ and $\tilde{T}_1(h)(y_h) = i$. Hence, our claim is justified.

Let $h, k \in S_1^{-1}(i\mathcal{P}_x(\mathcal{A}))$. Then $\tilde{T}_1(h)(y), \tilde{T}_1(k)(y) \in \{-i, i\}$. By the argument above, there exist $y_h, y_k \in \Psi(x_\tau)$ such that

$$\tilde{T}_1(h)(y_h) = i, \qquad \tilde{T}_1(k)(y_k) = i.$$

Assume that $f_2 \in S_2^{-1}(i\mathcal{P}_x(\mathcal{A}))$. By Step 5, we have

$$\tilde{T}_1(h)(y)\tilde{T}_2(f_2)(y) = -1 = \tilde{T}_1(k)(y)\tilde{T}_2(f_2)(y).$$

This implies that

$$\tilde{T}_1(h)(y) = \tilde{T}_1(k)(y).$$

If $y_h = y$, then $\tilde{T}_1(h)(y) = i$ and so $\tilde{T}_1(k)(y) = i$. We claim that $y_k = y$. Otherwise, $y_k = \eta(y)$ since $y_k, y \in \Psi(x_\tau)$ and $\Psi(x_\tau) = \{y, \eta(y)\}$. Hence,

$$i = \tilde{T}_1(k)(y_k) = \tilde{T}_1(k)(\eta(y)) = \overline{\tilde{T}_1(k)(y_k)} = \bar{i} = -i,$$

which is a contradiction. Hence, our claim is justified. Therefore, $y_k = y$ and so $y_h = y_k$. If $y_h = \eta(y)$, then $\tilde{T}_1(h)(y) = \tilde{T}_1(h)(\eta(y_h)) = \overline{\tilde{T}_1(h)(y_h)} = -i$ and so $\tilde{T}_1(k)(y) = -i$. Hence, $y_k = \eta(y)$ and so $y_h = y_k$. Therefore, there exists a unique element $y' \in \Psi(x_\tau)$ such that $\tilde{T}_1(h)(y') = i$ for all $h \in S_1^{-1}(i\mathcal{P}_x(\mathcal{A}))$. Similarly, we can show that there exists a unique element $y'' \in \Psi(x_\tau)$ such that $\tilde{T}_2(h)(y'') = i$ for all $h \in S_2^{-1}(i\mathcal{P}_x(\mathcal{A}))$. We claim that $y' = y''$. Otherwise, $y'' = \eta(y')$ since $y', y'' \in \Psi(x_\tau)$. Let $h \in S_1^{-1}(i\mathcal{P}_x(\mathcal{A}))$ and let $k \in S_2^{-1}(i\mathcal{P}_x(\mathcal{A}))$. Then $\tilde{T}_1(h)(y') = i$ and $\tilde{T}_2(k)(y'') = i$. Hence, $\tilde{T}_1(h)(y'')\tilde{T}_2(k)(y'') = (-i)(i) = 1$. On the other hand, $\tilde{T}_1(h)(y'')\tilde{T}_2(k)(y'') = -1$ by Step 5 since $x \in X$ and $y'' \in \Psi(x_\tau)$. This contradiction implies that our claim is justified.

We now define the map $\psi : X \to Y$ as follows:

$$\begin{aligned} &\{\psi(x)\} = \Psi(x_\tau) && \text{if } \tau(x) = x, \\ &\psi(x) \in \Psi(x_\tau) \text{ such that } \tilde{T}_j(h)(\psi(x)) = i && \\ &\text{for all } h \in S_j^{-1}(i\mathcal{P}_x(\mathcal{A})) \quad (j \in \{1,2\}) && \text{if } \tau(x) \neq x. \end{aligned} \tag{4.23}$$

Note that the map ψ is well-defined by Steps 4 and 6.

Step 7. *Let $j \in \{1,2\}$ and let $f \in \mathcal{A}$. Then $\tilde{T}_j(f) \circ \psi = S_j(f)$.*

Take $l \in \{1,2\}$ with $l \neq j$. Let $x \in X$. Since $\psi(x) \in \Psi(x_\tau)$, we have

$$|\tilde{T}_j(f)(\psi(x))| = |S_j(f)(x)|, \tag{4.24}$$

by part (vii) of Step 3. If $S_j(f)(x) = 0$, then $\tilde{T}_j(f)(\psi(x)) = 0$ by (4.24) and so $(\tilde{T}_j(f) \circ \psi)(x) = S_j(f)(x)$. We now assume that $S_j(f)(x) \neq 0$. By Theorem 2.5, there exists a function $h \in \mathcal{P}_x(\mathcal{A})$ such that

$$\mathrm{Ran}_{\pi,X}(S_j(f)h) = \{S_j(f)(x), S_j(f)(\tau(x))\}. \tag{4.25}$$

The surjectivity of S_l implies that there exists $g_1 \in \mathcal{A}$ such that $S_l(g_1) = h$. Since $S_l(g_1) \in \mathcal{P}_x(\mathcal{A})$, we have $\tilde{T}_l(g_1) \in \mathcal{P}_{\psi(x)}(\mathcal{B})$ by part (viii) of Step 3. By (4.25), $x \in M(S_j(f)h) = M(S_j(f)S_l(g_1))$ and so $\psi(x) \in M(\tilde{T}_j(f)\tilde{T}_l(g_1))$ by Theorem 3.2. This implies that

$$\tilde{T}_j(f)(\psi(x)) = \tilde{T}_j(f)(\psi(x))\tilde{T}_l(g_1)(\psi(x)) \in \mathrm{Ran}_{\pi,Y}(\tilde{T}_j(f)\tilde{T}_l(g_1)), \tag{4.26}$$

since $\tilde{T}_l(g_1) \in \mathcal{P}_{\psi(x)}(\mathcal{B})$. On the other hand,

$$\mathrm{Ran}_{\pi,Y}(\tilde{T}_j(f)\tilde{T}_l(g_1)) = \mathrm{Ran}_{\pi,X}(S_j(f)S_l(g_1)), \tag{4.27}$$

by part (iii) of Step 3. Hence,

$$\tilde{T}_j(f)(\psi(x)) \in \{S_j(f)(x), S_j(f)(\tau(x))\}, \tag{4.28}$$

by (4.26),(4.27), and (4.25) since $S_l(g_1) = h$. If either $\tau(x) = x$ or $S_j(f)(x) = S_j(f)(\tau(x))$, then $(\tilde{T}_j(f) \circ \psi)(x) = S_j(f)(x)$. Suppose that $S_j(f)(x) \neq S_j(f)(\tau(x))$. By Theorem 2.6, there exists a function $k \in i\mathcal{P}_x(\mathcal{A})$ such that

$$\mathrm{Ran}_{\pi,X}(S_j(f)k) = \{iS_j(f)(x), -iS_j(f)(\tau(x))\}. \tag{4.29}$$

The surjectivity of S_l implies that there exists $g_2 \in \mathcal{A}$ such that $S_l(g_2) = k$. Hence, $g_2 \in S_l^{-1}(i\mathcal{P}_x(\mathcal{A}))$. By the definition of ψ, we have $\tilde{T}_l(g_2)(\psi(x)) = i$. Thus, $\tilde{T}_l(g_2) \in i\mathcal{P}_{\psi(x)}(\mathcal{B})$. From (4.29), we have $x \in M(S_j(f)k) = M(S_j(f)S_l(g_2))$. By Theorem 3.2, $\psi(x) \in M(\tilde{T}_j(f)\tilde{T}_l(g_2))$ since $\psi(x) \in \Psi(x_\tau)$. Hence,

$$i\tilde{T}_j(f)(\psi(x)) = (\tilde{T}_j(f)\tilde{T}_l(g_2))(\psi(x)) \in \mathrm{Ran}_{\pi,Y}(\tilde{T}_j(f)\tilde{T}_l(g_2)).$$

By part (iii) of Step 3, we have

$$\mathrm{Ran}_{\pi,Y}(\tilde{T}_j(f)\tilde{T}_l(g_2)) = \mathrm{Ran}_{\pi,X}(S_j(f)S_l(g_2)).$$

Hence, by Theorem 2.6

$$\tilde{T}_j(f)(\psi(x)) \in \{S_j(f)(x), -S_j(f)(\tau(x))\}. \tag{4.30}$$

We claim that $\tilde{T}_j(f)(\psi(x)) \neq S_j(f)(\tau(x))$. Otherwise, by (4.30), we have $S_j(f)(\tau(x)) \in \{S_j(f)(x), -S_j(f)(\tau(x))\}$, which implies that $S_j(f)(\tau(x)) =$

$-S_j(f)(\tau(x))$ since it was assumed that $S_j(f)(x) \neq S_j(f)(\tau(x))$. Therefore, $S_j(f)(\tau(x)) = 0$ and so $S_j(f)(x) = 0$, which contradicts to $S_j(f)(x) \neq 0$. Hence, our claim is justified. Therefore, $(\tilde{T}_j(f) \circ \psi)(x) = S_j(f)(x)$ by (4.28).

Step 8. *The map $\psi : X \to Y$ defined by (4.23) satisfies $\psi \circ \tau = \eta \circ \psi$ and it is a bijection.*

Let $x \in X$. Assume that $g \in \mathcal{B}$. The surjectivity of $\tilde{T}_1 : \mathcal{A} \to \mathcal{B}$ implies that there exists a function $f \in \mathcal{A}$ such that $g = \tilde{T}_1(f)$. Thus, by Step 7, we have

$$\begin{aligned} g((\eta \circ \psi)(x)) &= (g \circ \eta)(\psi(x)) = \overline{g(\psi(x))} = \overline{\tilde{T}_1(f)(\psi(x))} \\ &= \overline{S_1(f)(x)} = S_1(f)(\tau(x)) = \tilde{T}_1(f)(\psi(\tau(x)) \\ &= g((\psi \circ \tau)(x)). \end{aligned} \tag{4.31}$$

Since $\mathcal{B}$ separates the points of Y and (4.31) holds for all $g \in \mathcal{B}$, we deduce that

$$(\eta \circ \psi)(x) = (\psi \circ \tau)(x). \tag{4.32}$$

Since (4.32) holds for all $x \in X$, we conclude that $\eta \circ \psi = \psi \circ \tau$.

To prove the injectivity of ψ, let $x, z \in X$ with $\psi(x) = \psi(z)$. Assume that $h \in \mathcal{A}$. The surjectivity of $S_1 : \mathcal{A} \to \mathcal{A}$ implies that $h = S_1(f)$ for some $f \in \mathcal{A}$. Thus, by Step 7, we have

$$h(x) = S_1(f)(x) = \tilde{T}_1(f)(\psi(x)) = \tilde{T}_1(f)(\psi(z)) = S_1(f)(z) = h(z). \tag{4.33}$$

Since $\mathcal{A}$ separates the points of X and (4.33) holds for all $h \in \mathcal{A}$, we deduce that $x = z$. Hence, ψ is injective.

To prove the surjectivity of ψ, let $y \in Y$. The surjectivity of the map $\Psi : X_\tau \to Y_\eta$ defined by (3.12) implies that there exists $x \in X$ such that $\Psi(x_\tau) = y_\eta$. If $\tau(x) = x$, then by Step 4, we have $\eta(y) = y$ and so $\psi(x) = y$ by the definition of ψ. Suppose that $\tau(x) \neq x$. Since $y \in \Psi(x_\tau)$, by Step 6, there exists $y' \in \Psi(x_\tau)$ such that $\tilde{T}_j(h)(y') = i$ for all $h \in S_j^{-1}(iP_x(\mathcal{A}))$, where $j \in \{1, 2\}$. If $y' = y$, then by the definition of ψ, we have $\psi(x) = y' = y$. Suppose that $y' = \eta(y)$. Then $\tilde{T}_j(h)(\eta(y)) = i$ for all $h \in S_j^{-1}(iP_x(\mathcal{A}))$ where $j \in \{1, 2\}$. Hence, by the definition of ψ we have $\eta(y) = \psi(x)$ and so

$$y = \eta(\psi(x)) = (\eta \circ \psi)(x) = (\psi \circ \tau)(x) = \psi(\tau(x)).$$

Therefore, ψ is surjective.

Step 9. *Define the map $\varphi : Y \to X$ by $\varphi = \psi^{-1}$. Then $\varphi \circ \eta = \tau \circ \varphi$ and $\tilde{T}_j(f) = S_j(f) \circ \varphi$ for all $f \in \mathcal{A}$, where $j \in \{1, 2\}$.*

By Step 8, we have $\eta \circ \psi = \psi \circ \tau$. This implies that $\psi^{-1} \circ \eta = \tau \circ \psi^{-1}$ and so $\varphi \circ \eta = \tau \circ \varphi$.

By Step 7, we have $\tilde{T}_j(f) \circ \psi = S_j(f)$ for all $f \in \mathcal{A}$, where $j \in \{1, 2\}$. This implies that $\tilde{T}_j(f) = S_j(f) \circ \psi^{-1} = S_j(f) \circ \varphi$ for all $f \in \mathcal{A}$ where $j \in \{1, 2\}$.

Step 10. *φ is a Lipschitz homeomorphism from (Y, ρ) to (X, d).*

Let $h \in \mathcal{A}$. The surjectivity of S_1 implies that $S_1(f) = h$ for some $f \in \mathcal{A}$. By Step 9, we have $\tilde{T}_1(f) = S_1(f) \circ \varphi = h \circ \varphi$. Hence, $h \circ \varphi \in \mathcal{B}$. Therefore, φ is a Lipschitz mapping from (Y, ρ) to (X, d) by Lemma 2.11.

We now show that φ^{-1} is a Lipschitz mapping from (X,d) to (Y,ρ). Let $g \in \mathcal{B}$. The surjectivity of $\tilde{T}_1$ implies that $\tilde{T}_1(f) = g$ for some $f \in \mathcal{A}$. By Step 7, we have

$$g \circ \psi = \tilde{T}_1(f) \circ \psi = S_1(f).$$

Hence, $g \circ \psi \in \mathcal{A}$. By Lemma 2.11, ψ is a Lipschitz mapping from (X,d) to (Y,ρ). That is, φ^{-1} is a Lipschitz mapping from (X,d) to (Y,ρ).

Step 11. *There exist two functions $\kappa_1, \kappa_2 \in B$ with $\kappa_1\kappa_2 = 1_Y$ such that*

$$T_j(f) = \kappa_j \cdot (S_j(f) \circ \varphi),$$

for all $f \in \mathcal{A}$ and $j \in \{1,2\}$.

Let $\kappa_1, \kappa_2 \in \mathcal{B}$ be the functions that are defined in (4.8). By Step 3, we have $T_j(f) = \kappa_j \tilde{T}_j(f)$ for all $f \in \mathcal{A}$, where $j \in \{1,2\}$. By Step 9, we have $\tilde{T}_j(f) = S_j(f) \circ \varphi$ for all $f \in \mathcal{A}$, where $j \in \{1,2\}$. Hence, $T_j(f) = \kappa_j \cdot (S_j(f) \circ \varphi)$ for all $f \in \mathcal{A}$, where $j \in \{1,2\}$.

The proof of the theorem is now complete. □

5. Jointly weakly peripherally multiplicative maps

In this section, we study surjective jointly weakly peripherally multiplicative spectrum preserving maps between real Lipschitz algebras with involution and prove that such mappings are essentially weighted composition operators.

Throughout this section, we assume that (X,d) and (Y,ρ) are compact metric spaces, that $\tau : X \to X$ and $\eta : Y \to Y$ are Lipschitz involutions on (X,d) and (Y,ρ), respectively, that $\mathcal{A} = \mathrm{Lip}(X,d,\tau)$, that $\mathcal{B} = \mathrm{Lip}(Y,\rho,\eta)$, and that four mappings $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ are surjective jointly weakly peripherally multiplicative spectrum preserving or, equivalently, jointly weakly peripherally multiplicative range preserving.

Proposition 5.1. (i) *Four mappings $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ are jointly uniform norm multiplicative.*

(ii) *There is a bijection mapping $\Psi : X_\tau \to Y_\eta$ such that*

$$|T_1(f)(y)T_2(g)(y)| = |S_1(f)(x)S_2(g)(x)|,$$

for all $f, g \in \mathcal{A}$, $x \in X$, and $y \in \Psi(x_\tau)$.

Proof. (i) Let $f, g \in \mathcal{A}$. Assume that $\lambda \in \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) \cap \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g))$. Then $\|T_1(f)T_2(g)\|_Y = |\lambda| = \|S_1(f)S_2(g)\|_X$. Hence, (i) holds.

(ii) By (i) and Theorem 3.2, (ii) holds. □

Lemma 5.2. *Let $\Psi : X_\tau \to Y_\eta$ be a bijection mapping such that*

$$|T_1(f)(y)T_2(g)(y)| = |S_1(f)(x)S_2(g)(x)|,$$

for all $f, g \in \mathcal{A}$, $x \in X$ and $y \in \Psi(x_\tau)$. Suppose that $f, g \in \mathcal{A}$ and $x \in X$. Then $M(S_1(f)S_2(g)) = x_\tau$ if and only if $M(T_1(f)T_2(g)) = \Psi(x_\tau)$.

Proof. We first assume that $M(S_1(f)S_2(g)) = x_\tau$. Let $y \in \Psi(x_\tau)$. By Lemma 3.1, we have $y \in M(T_1(f)T_2(g))$. Hence,

$$\Psi(x_\tau) \subseteq M(T_1(f)T_2(g)). \tag{5.1}$$

Let $y \in M(T_1(f)T_2(g))$. Then y_η is a subset of $M(T_1(f)T_2(g))$. The surjectivity of $\Psi : X_\tau \to Y_\eta$ implies that there exists $z \in X$ such that $y_\eta = \Psi(z_\tau)$. By the hypotheses, we have

$$|T_1(f)(y)T_2(g)(y)| = |S_1(f)(z)S_2(g)(z)|.$$

By Lemma 3.1, $z \in M(S_1(f)S_2(g))$. Hence, $z \in x_\tau$ and so $z_\tau = x_\tau$. Therefore, $\Psi(z_\tau) = \Psi(x_\tau)$ and so $y \in \Psi(x_\tau)$. Hence,

$$M(T_1(f)T_2(g)) \subseteq \Psi(x_\tau). \tag{5.2}$$

By (5.1) and (5.2), we deduce that

$$M(T_1(f)T_2(g)) = \Psi(x_\tau).$$

We now assume that $M(T_1(f)T_2(g)) = \Psi(x_\tau)$. Let $z \in M(S_1(f)S_2(g))$. Then z_τ is a subset of $M(S_1(f)S_2(g))$. By hypothesis, we have

$$|T_1(f)(y)T_2(g)(y)| = |S_1(f)(z)S_2(g)(z)|,$$

for all $y \in \Psi(z_\tau)$. By Lemma 3.1, $\Psi(z_\tau)$ is a subset of $M(T_1(f)T_2(g))$. Hence, $\Psi(z_\tau) \subseteq \Psi(x_\tau)$ and so $z_\tau \subseteq x_\tau$. Therefore, $z \in x_\tau$. Thus,

$$M(S_1(f)S_2(g)) \subseteq x_\tau. \tag{5.3}$$

Let $z \in x_\tau$. Then $z_\tau = x_\tau$ and so $\Psi(z_\tau) = \Psi(x_\tau)$. Hence, $\Psi(z_\tau) = M(T_1(f)T_2(g))$. By Lemma 3.1, we have $z \in M(S_1(f)S_2(g))$. Hence,

$$x_\tau \subseteq M(S_1(f)S_2(g)). \tag{5.4}$$

By (5.3) and (5.4), we have $M(S_1(f)S_2(g)) = x_\tau$. □

Lemma 5.3. *Let $\Psi : X_\tau \to Y_\eta$ be a bijection mapping such that*

$$|T_1(f)(y)T_2(g)(y)| = |S_1(f)(x)S_2(g)(x)|,$$

for all $f, g \in \mathcal{A}$, $x \in X$, and $y \in \Psi(x_\tau)$. Let $f, g \in \mathcal{A}$ and let $x \in X$.

- (i) *If either $M(S_1(f)S_2(g)) = x_\tau$ or $M(T_1(f)T_2(g)) = \Psi(x_\tau)$, then* $\mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)) = \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g))$.
- (ii) *$S_1(f)S_2(g) \in \mathcal{P}(\mathcal{A})$ with $M(S_1(f)S_2(g)) = x_\tau$ if and only if $T_1(f)T_2(g) \in \mathcal{P}(\mathcal{B})$ with $M(T_1(f)T_2(g)) = \Psi(x_\tau)$.*
- (iii) *$S_1(f)S_2(g) \in i\mathcal{P}(\mathcal{A})$ with $M(S_1(f)S_2(g)) = x_\tau$ if and only if $T_1(f)T_2(g) \in i\mathcal{P}(\mathcal{B})$ with $M(T_1(f)T_2(g)) = \Psi(x_\tau)$.*
- (iv) *If $y \in \Psi(x_\tau)$ and $T_1(f), T_2(g) \in i\mathcal{P}_y(\mathcal{B})$ with $M(T_1(f)T_2(g)) = \Psi(x_\tau)$, then* $\mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)) = \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) = \{-1\}$ *and* $M(S_1(f)S_2(g)) = x_\tau$.
- (v) *If $S_1(f), S_2(g) \in i\mathcal{P}_x(\mathcal{A})$ with $M(S_1(f)S_2(g)) = x_\tau$, then $M(T_1(f)T_2(g)) = \Psi(x_\tau)$ and* $\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) = \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)) = \{-1\}$.

Proof. (i) By Lemma 5.2, it is sufficient to prove the assertion for the case $M(S_1(f)S_2(g)) = x_\tau$. Let $M(S_1(f)S_2(g)) = x_\tau$. Then

$$\mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)) = \{S_1(f)(x)S_2(g)(x), S_1(f)(\tau(x))S_2(g)(\tau(x))\}. \tag{5.5}$$

By the hypotheses, either $S_1(f)(x)S_2(g)(x) \in \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g))$ or

$$S_1(f)(\tau(x))S_2(g)(\tau(x)) = \overline{S_1(f)(x)S_2(g)(x)} \in \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)),$$

since $\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g))$ is closed under the conjugate. Hence,

$$\{S_1(f)(x)S_2(g)(x), S_1(f)(\tau(x))S_2(g)(\tau(x))\} \subseteq \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)). \tag{5.6}$$

By (5.5) and (5.6), we have

$$\mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)) \subseteq \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)). \tag{5.7}$$

Let $\lambda \in \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g))$. Then there exists $y \in M(T_1(f)T_2(g))$ such that $\lambda = T_1(f)(y)T_2(g)(y)$. On the other hand, by Lemma 5.2, we have

$$M(T_1(f)T_2(g)) = \Psi(x_\tau).$$

Let $w \in \Psi(x_\tau)$. Then $M(T_1(f)T_2(g)) = \{w, \eta(w)\} = w_\eta$. Hence, $y \in \{w, \eta(w)\}$ and so $y_\eta = w_\eta$. Thus, $M(T_1(f)T_2(g)) = \{y, \eta(y)\}$. Therefore, $\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) = \{\lambda, \bar{\lambda}\}$. By the hypotheses either $\lambda \in \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g))$ or $\bar{\lambda} \in \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g))$. Since $\mathrm{Ran}_{\pi,X}(S_1(f)S_2(g))$ is closed under the conjugate, we deduce that

$$\{\lambda, \bar{\lambda}\} \subseteq \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)).$$

Hence,

$$\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) \subseteq \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)). \tag{5.8}$$

By (5.7) and (5.8), we have

$$\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) = \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)).$$

(ii) We first assume that $S_1(f)S_2(g) \in \mathcal{P}(\mathcal{A})$ and $M(S_1(f)S_2(g)) = x_\tau$. By (i), we have

$$\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) = \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)) = \{1\}.$$

This implies that $T_1(f)T_2(g) \in \mathcal{P}(\mathcal{B})$. On the other hand, $M(T_1(f)T_2(g)) = \Psi(x_\tau)$ by Lemma 5.2.

We now assume that $T_1(f)T_2(g) \in \mathcal{P}(\mathcal{B})$ and that $M(T_1(f)T_2(g)) = \Psi(x_\tau)$. By (i), we have

$$\mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)) = \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) = \{1\}.$$

This implies that $S_1(f)S_2(g) \in \mathcal{P}(\mathcal{A})$. On the other hand, $M(S_1(f)S_2(g)) = x_\tau$ by Lemma 5.2. Hence, (ii) holds.

(iii) It follows from (i) and Lemma 5.2, by a similar argument as in part (ii).

(iv) Let $T_1(f), T_2(g) \in i\mathcal{P}_y(\mathcal{B})$, where $y \in \Psi(x_\tau)$ and $M(T_1(f)T_2(g)) = \Psi(x_\tau)$. Then $\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) = \{-1\}$. On the other hand, by Lemma 5.2, we have $\mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)) = \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g))$ and so $M(S_1(f)S_2(g) = x_\tau$.

(v) It follows from Lemma 5.2, by a similar argument as in part (iv). □

Lemma 5.4. *Let $\Psi : X_\tau \to Y_\eta$ be a bijection mapping such that*

$$|T_1(f)(y)T_2(g)(y)| = |S_1(f)(x)S_2(g)(x)|,$$

for all $f, g \in \mathcal{A}$, $x \in X$ and $y \in \Psi(x_\tau)$. Let $h_1, h_2, k_1, k_2 \in \mathcal{A}$, let $x \in X$, and let $y \in Y$ with $\Psi(x_\tau) = y_\eta$.

(i) *If $\eta(y) \neq y$, $T_1(h_1), T_2(h_2) \in \mathcal{P}_y(\mathcal{B})$, and $T_1(k_1), T_2(k_2) \in i\mathcal{P}_y(\mathcal{B})$ with $M(T_1(h_1)) = M(T_2(h_2)) = M(T_1(k_1)) = M(T_2(k_2)) = y_\eta$, then there is $x_0 \in x_\tau$ such that $S_1(k_1)S_2(h_2)$, $S_1(h_1)S_2(k_2) \in i\mathcal{P}_{x_0}(\mathcal{A})$.*

(ii) *If* $\tau(x) \neq x$, $S_1(h_1), S_2(h_2) \in \mathcal{P}_x(\mathcal{A})$, *and* $S_1(k_1), S_2(k_2) \in i\mathcal{P}_x(\mathcal{A})$ *with* $M(S_1(h_1)) = M(S_2(h_2)) = M(S_1(k_1)) = M(S_2(k_2)) = x_\tau$, *then there is* $y_0 \in \Psi(x_\tau)$ *such that* $T_1(k_1)T_2(h_2)$, $T_1(h_1)T_2(k_2) \in i\mathcal{P}_{y_0}(\mathcal{B})$.

Proof. (i) Let $\eta(y) \neq y$, let $T_1(h_1), T_2(h_2) \in \mathcal{P}_y(\mathcal{B})$, and let $T_1(k_1), T_2(k_2) \in i\mathcal{P}_y(\mathcal{B})$ with $M(T_1(h_1)) = M(T_2(h_2)) = M(T_1(k_1)) = M(T_2(k_2)) = y_\eta$. It is easy to see that $T_1(h_1)T_2(k_2)$, $T_1(k_1)T_2(h_2) \in i\mathcal{P}_y(\mathcal{B})$ with $M(T_1(h_1)T_2(k_2)) = M(T_1(k_1)T_2(h_2)) = y_\eta = \Psi(x_\tau)$. By part (iii) of Lemma 5.3, $S_1(k_1)S_2(h_2)$, $S_1(h_1)S_2(k_2) \in i\mathcal{P}(\mathcal{A})$. Since $T_1(h_1), T_2(h_2) \in \mathcal{P}_y(\mathcal{B})$, we have $T_1(h_1)T_2(h_2) \in \mathcal{P}_y(\mathcal{B})$. According to $M(T_1(h_1)T_2(h_2)) = \Psi(x_\tau)$, we have $S_1(h_1)S_2(h_2) \in \mathcal{P}(\mathcal{A})$ and $M(S_1(f)S_2(g)) = x_\tau$. Hence, $S_1(h_1)S_2(h_2) \in \mathcal{P}_x(\mathcal{A})$. Since $T_1(k_1), T_2(k_2) \in i\mathcal{P}_y(\mathcal{B})$ and $y_\eta = \Psi(x_\tau)$, we deduce that $\mathrm{Ran}_\pi(S_1(k_1)S_2(k_2)) = \{-1\}$ by part (iv) of Lemma 5.3. Thus,

$$\begin{aligned}&[S_1(k_1)(x)S_2(h_2)(x)][S_1(h_1)(x)S_2(k_2)(x)]\\&= [S_1(h_1)(x)S_2(h_2)(x)][S_1(k_1)(x)S_2(k_2)(x)]\\&= (1)(-1)\\&= -1.\end{aligned}$$

It follows that $S_1(k_1)(x)S_2(h_2)(x) = S_1(h_1)(x)S_2(k_2)(x) = i$ or $S_1(k_1)(x)S_2(h_2)(x) = S_1(h_1)(x)S_2(k_2)(x) = -i$. Therefore, there exists $x_0 \in x_\tau$ such that $S_1(k_1)S_2(h_2)$, $S_1(h_1)S_2(k_2) \in i\mathcal{P}_{x_0}(A)$.

(ii) The proof is analogous to the proof of (i). □

Using Lemmas 5.2–5.4, we show that any jointly weakly peripherally multiplicative spectrum preserving mappings are jointly peripherally multiplicative spectrum preserving.

Theorem 5.5. (i) *Four mappings* $S_1, S_2 : \mathcal{A} \to \mathcal{A}$ *and* $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ *are jointly peripherally multiplicative spectrum preserving or, equivalently, jointly peripherally multiplicative range preserving.*

(ii) *There are two functions* $\kappa_1, \kappa_2 \in \mathcal{B}$ *with* $\kappa_1\kappa_2 = 1_Y$ *and a Lipschitz homeomorphism* φ *from* (Y, ρ) *to* (X, d) *with* $\tau \circ \varphi = \varphi \circ \eta$ *on* Y *such that*

$$T_j(f) = \kappa_j \cdot (S_j(f) \circ \varphi),$$

for all $f \in \mathcal{A}$ *and* $j = 1, 2$.

Proof. (i) By part (ii) of Proposition 5.1, there is a bijection mapping $\Psi : X_\tau \to Y_\eta$ such that

$$|T_1(f)(y)T_2(g)(y)| = |S_1(f)(x)S_2(g)(x)|,$$

for all $f, g \in \mathcal{A}$, $x \in X$ and $y \in \Psi(x_\tau)$. Let $f, g \in \mathcal{A}$. If $S_1(f)S_2(g) = 0_X$, then $\mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)) = \{0\}$ and

$$\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) = \{0\} = \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)).$$

Assume that $T_1(f)T_2(g) \neq 0_Y$ and that $S_1(f)S_2(g) \neq 0_X$. Then $\mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g)) \subseteq \mathbb{C} \setminus \{0\}$ and $\mathrm{Ran}_{\pi,X}(S_1(f)S_2(g)) \subseteq \mathbb{C} \setminus \{0\}$. Let $\lambda \in \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(g))$. Then $\lambda \in \mathbb{C}\setminus\{0\}$ and there exists $w \in M(T_1(f)T_2(g))$ such that $\lambda = T_1(f)(w)T_2(g)(w)$. Let $z \in X$ with $\Psi(z_\tau) = w_\eta$. By Lemma 3.1, $z \in M(S_1(f)S_2(g))$. This implies that $S_1(f)(z)S_2(g)(z) \in \mathrm{Ran}_{\pi,X}(S_1(f)S_2(g))$. We claim that $S_1(f)(z)S_2(g)(z) \in \{\lambda, \bar{\lambda}\}$. Take $\alpha = T_1(f)(w)$ and $\beta = T_2(g)(w)$. Then $\alpha\beta = \lambda$ and so $\alpha \neq 0$ and $\beta \neq 0$. By part (i) of Theorem 2.5 and the surjectivity of T_1 and T_2, there exist $h_1, h_2 \in \mathcal{A}$ with $T_1(h_1), T_2(h_2) \in \mathcal{P}_w(\mathcal{B})$ such that

$$M(T_1(h_1)T_2(g)) = M(T_1(h_1)) = \{w, \eta(w)\}, \tag{5.9}$$

$$M(T_1(f)T_2(h_2)) = M(T_2(h_2)) = \{w, \eta(w)\}. \tag{5.10}$$

By (5.9) and (5.10), we deduce that $M(T_1(h_1)T_2(h_2)) = \{w, \eta(w)\}$. Since $T_1(h_1), T_2(h_2) \in \mathcal{P}_w(\mathcal{B})$, we have $T_1(h_1)T_2(h_2) \in \mathcal{P}_w(\mathcal{B})$. Hence, by part (ii) of Lemma 5.3, we have $S_1(h_1)S_2(h_2) \in \mathcal{P}_z(\mathcal{A})$ with $M(S_1(h_1)S_2(h_2)) = z_\tau$ since $w_\eta = \Psi(z_\tau)$. By (5.9), (5.10), and part (i) of Lemma 5.3, we get

$$M(S_1(f)S_2(h_2)) = M(S_1(h_1)S_2(g)) = z_\tau,$$
$$\mathrm{Ran}_{\pi,X}(S_1(f)S_2(h_2)) = \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(h_2)) = \{\alpha, \bar{\alpha}\},$$
$$\mathrm{Ran}_{\pi,X}(S_1(h_1)S_2(g)) = \mathrm{Ran}_{\pi,Y}(T_1(h_1)T_2(g)) = \{\beta, \bar{\beta}\}.$$

Thus $S_1(f)(z)S_2(h_2)(z) \in \{\alpha, \bar{\alpha}\}$ and $S_1(h_1)(z)S_2(g)(z) \in \{\beta, \bar{\beta}\}$. Since $S_1(h_1)S_2(h_2) \in \mathcal{P}_z(\mathcal{A})$, we have

$$S_1(f)(z)S_2(g)(z) = [S_1(f)(z)S_2(h_2)(z)][S_1(h_1)(z)S_2(g)(z)].$$

Hence, $S_1(f)(z)S_2(g)(z) \in \{\alpha\beta, \alpha\bar{\beta}, \bar{\alpha}\beta, \bar{\alpha}\bar{\beta}\}$. If $\eta(w) = w$, then $\bar{\alpha} = \alpha$ and $\bar{\beta} = \beta$. Thus, $S_1(f)(z)S_2(g)(z) = \alpha\beta = \lambda$. Now, suppose that $\eta(w) \neq w$. By Theorem 2.6 and the surjectivity of T_1 and T_2, there exist $k_1, k_2 \in \mathcal{A}$ such that $T_1(k_1), T_2(k_2) \in i\mathcal{P}_w(\mathcal{B})$,

$$M(T_1(f)T_2(k_2)) = M(T_2(k_2)) = \{w, \eta(w)\}, \tag{5.11}$$

$$M(T_1(k_1)T_2(g)) = M(T_1(k_1)) = \{w, \eta(w)\}. \tag{5.12}$$

By (5.11), (5.12), and part (i) of Lemma 5.3, we deduce that

$$M(S_1(f)S_2(k_2)) = M(S_1(k_1)S_2(g)) = z_\tau,$$
$$\mathrm{Ran}_{\pi,X}(S_1(f)S_2(k_2)) = \mathrm{Ran}_{\pi,Y}(T_1(f)T_2(k_2)) = \{i\alpha, -i\bar{\alpha}\},$$
$$\mathrm{Ran}_{\pi,X}(S_1(k_1)S_2(g)) = \mathrm{Ran}_{\pi,Y}(T_1(k_1)T_2(g)) = \{i\beta, -i\bar{\beta}\}.$$

According to part (i) of Lemma 5.4, there exists $x \in z_\tau$ such that $S_1(h_1)S_2(k_2)$, $S_1(k_1)S_2(h_2) \in i\mathcal{P}_x(\mathcal{A})$. Since $x \in z_\tau$, we have $S_1(f)(x)S_2(h_2)(x) \in \{\alpha, \bar{\alpha}\}$. We claim that $S_1(f)(x)S_2(h_2)(x) = \alpha$. Otherwise, $S_1(f)(x)S_2(h_2)(x) = \bar{\alpha}$ and $\bar{\alpha} \neq \alpha$. Now we have

$$iS_1(f)(x) = S_1(f)(x)S_1(k_1)(x)S_2(h_2)(x) = \bar{\alpha}S_1(k_1)(x).$$

By (5.11) and (5.12), we have

$$M(T_1(k_1)T_2(k_2)) = \{w, \eta(w)\}. \tag{5.13}$$

This implies that $\mathrm{Ran}_{\pi,Y}(T_1(k_1)T_2(k_2)) = \{-1\}$ and so by the hypotheses, we have $\mathrm{Ran}_{\pi,X}(S_1(k_1)S_2(k_2)) = \{-1\}$. By (5.13) and part (iv) of Lemma 5.3, we have $M(S_1(k_1)S_2(k_2)) = z_\tau$. Hence, $S_1(k_1)(x)S_2(k_2)(x) = -1$ since $x \in z_\tau$. Therefore,

$$iS_1(f)(x)S_2(k_2)(x) = \bar{\alpha}S_1(k_1)(x)S_2(k_2)(x) = -\bar{\alpha}.$$

This implies that

$$i\bar{\alpha} = S_1(f)(x)S_2(k_2)(x) \in \operatorname{Ran}_{\pi,X}(S_1(f)S_2(k_2)),$$

and hence $i\bar{\alpha} \in \{i\alpha, -i\bar{\alpha}\}$. It follows that $\bar{\alpha} = \alpha$ since $\alpha \neq 0$. This is a contradiction and so our claim is justified. A similar argument shows that $S_1(h_1)(x)S_2(g)(x) = \beta$. If $x = z$, then

$$\begin{aligned}\lambda = \alpha\beta &= S_1(f)(z)S_2(h_2)(z)S_1(h_1)(z)S_2(g)(z)\\ &= S_1(f)(z)S_2(g_2)(z) = S_1(f)(x)S_2(g)(x).\end{aligned}$$

If $x = \tau(z)$, then $\tau(x) = z$ and

$$\begin{aligned}\lambda = \alpha\beta &= S_1(f)(z)S_2(h_2)(z)S_1(h_1)(z)S_2(g)(z)\\ &= S_1(f)(z)S_2(g_2)(z) = S_1(f)(\tau(x))S_2(g)(\tau(x)).\end{aligned}$$

Therefore, $\lambda \in \{S_1(f)(x)S_2(g)(x), S_1(f)(\tau(x))S_2(g)(\tau(x))\}$ and our claim is justified. Thus, $\lambda \in \operatorname{Ran}_{\pi,X}(S_1(f)S_2(g))$ since $\{x, \tau(x)\} \subseteq M(S_1(f)S_2(g))$. Hence,

$$\operatorname{Ran}_{\pi,Y}(T_1(f)T_2(g)) \subseteq \operatorname{Ran}_{\pi,X}(S_1(f)S_2(g)).$$

Similarly, we can show that

$$\operatorname{Ran}_{\pi,X}(S_1(f)S_2(g)) \subseteq \operatorname{Ran}_{\pi,Y}(T_1(f)T_2(g)).$$

(ii) It follows from (i) and Theorem 4.1. □

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