Khayyam J. Math. 7 (2021), no. 1, 40–51
DOI: 10.22034/kjm.2020.224608.1753

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ALMOST AND WEAKLY NSR, NSM AND NSH SPACES

RACHID LAKEHAL[1], LJUBIŠA D.R. KOČINAC[2*] AND DJAMILA SEBA[1]

Communicated by H.R. Ebrahimi Vishki

Abstract. We introduce and study some new types of star-selection principles (almost and weakly neighborhood star-Menger, neighborhood star-Rothberger, and neighborhood star-Hurewicz). We establish some properties of these selection principles and their relations with other selection properties of topological spaces. The behavior of these classes of spaces under certain kinds of mappings is also considered.

1. Introduction and preliminaries

We give firstly definitions of notions, which are used in this paper.

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and the set of real numbers, respectively. Let X be a topological space, let $\mathcal{U}$ be a collection of subsets of X, and let $A \subset X$. Then the set $\mathrm{St}(A,\mathcal{U}) := \bigcup\{P \in \mathcal{U} : P \cap A \neq \emptyset\}$ is called the *star* of A with respect to $\mathcal{U}$; we write $\mathrm{St}(x,\mathcal{U})$ instead of $\mathrm{St}(\{x\},\mathcal{U})$. For more information about star covering properties, see [5, 18].

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of covers of a space X. Then the symbol $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that, for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$. The symbol $\mathsf{S}_{\mathsf{fin}}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an element of $\mathcal{B}$ (see [13, 22]).

Kočinac [10] introduced the star selection hypothesis similar to the previous ones. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of covers of a space X.

Date: Received: 26 March 2020; Accepted: 19 April 2020.

*Corresponding author.

2010 *Mathematics Subject Classification.* Primary 54D20; Secondary 54B05, 54C05.

Key words and phrases. Selection principles, star-Menger, star-Rothberger, almost NSM, weakly NSM.

(**A**) The symbol $\mathsf{S_{fin}}^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n\in\mathbb{N})$ of elements of $\mathcal{A}$, there exists a sequence $(\mathcal{V}_n : n\in\mathbb{N})$ such that for each $n\in\mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\{\mathrm{St}(V,\mathcal{U}_n) : V\in\mathcal{V}_n\}$ is an element of $\mathcal{B}$.

(**B**) The symbol $\mathsf{SS}^*_{\text{comp}}(\mathcal{A},\mathcal{B})$ (resp. $\mathsf{SS}^*_{\text{fin}}(A,B)$) denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n\in\mathbb{N})$ of elements of $\mathcal{A}$, there exists a sequence $(K_n : n\in\mathbb{N})$ of compact (resp. finite) subsets of X such that $\{St(K_n,\mathcal{U}_n) : n\in\mathbb{N}\}\in\mathcal{B}$.

Let $\mathcal{O}$ denote the collection of all open covers of a space X.

Definition 1.1 (see [10, 14])**.** A space X is said to be *star-Menger* (resp. *star-Rothberger*) if it satisfies the selection hypothesis $\mathsf{S_{fin}}^*(\mathcal{O},\mathcal{O})$ (resp. $\mathsf{S_1}^*(\mathcal{O},\mathcal{O})$).

Definition 1.2 (see [3, 10, 14])**.** A space X is said to be *star-Hurewicz* if for every sequence $(\mathcal{U}_n : n\in\mathbb{N})$ of open covers of X, one can choose finite $\mathcal{V}_n\subset\mathcal{U}_n$, $n\in\mathbb{N}$, such that for every $x\in X$, we have $x\in\mathrm{St}(\bigcup\mathcal{V}_n,\mathcal{U}_n)$ for all but finitely many n.

The following three generalizations of star selection properties were introduced (in a general form and under different names) in [11] and studied in details in [4].

Definition 1.3 (see [4])**.** A space X is said to be *neighborhood star-Menger* (NSM) if for every sequence $(\mathcal{U}_n : n\in\mathbb{N})$ of open covers of X, one can choose finite $F_n\subset X$, $n\in\mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}}\mathrm{St}(O_n,\mathcal{U}_n)=X$ for every open $O_n\supset F_n$, $n\in\mathbb{N}$.

Definition 1.4 (see [4])**.** A space X is said to be *neighborhood star-Rothberger* (NSR) if for every sequence $(\mathcal{U}_n : n\in\mathbb{N})$ of open covers of X, one can choose $x_n\in X$, $n\in\mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}}\mathrm{St}(O_n,\mathcal{U}_n)=X$ for every open $O_n\ni x_n$, $n\in\mathbb{N}$.

Definition 1.5 (see [4])**.** A space X is said to be *neighborhood star-Hurewicz* (NSH) if for every sequence $(\mathcal{U}_n : n\in\mathbb{N})$ of open covers of X, one can choose finite $F_n\subset X$, $n\in\mathbb{N}$, such that for every open $O_n\supset F_n$, $n\in\mathbb{N}$, each $x\in X$ belongs to $\mathrm{St}(O_n,\mathcal{U}_n)$ for all but finitely many n.

In the last few years, weaker forms of selection and star selection properties became an important area of investigation (see, for example, [1, 2, 8, 9, 12, 15, 19–21, 23–26]). In this paper, we apply the same scenario to neighborhood star selection properties.

2. About wNSL spaces

In this section, we give some facts about weaker forms of neighborhood star-Lindelöf spaces. Recall that a space X is said to be *neighborhood star-Lindelöf* (NSL) if for every open cover $\mathcal{U}$ of X, one can choose countable $A\subset X$ such that for every neighborhood O of A, we have $\mathrm{St}(O,\mathcal{U})=X$ [18].

Definition 2.1. A space X is said to be *weakly neighborhood star-Lindelöf* (wNSL) if for every open cover $\mathcal{U}$ of X, one can choose countable $A\subset X$ such that for every neighborhood O of A, we have $\overline{\mathrm{St}(O,\mathcal{U})}=X$.

Theorem 2.2. *An open F_σ-subset of a* wNSL *space is* wNSL.

Proof. Let (X,τ) be a wNSL space and let $Y = \bigcup\{H_n : n \in \mathbb{N}\}$ be an open F_σ-subset of X, where the set H_n is closed in X for each $n \in \mathbb{N}$. We show that Y is wNSL. Let $\mathcal{U}$ be an open cover of (Y,τ_Y). We have to find a countable subset F of Y such that for each τ_Y-open $O \supseteq F$, $Y \subseteq \overline{\mathrm{St}(O,\mathcal{U})}$.

For each $n \in \mathbb{N}$, consider the open cover $\mathcal{U}_n = \mathcal{U} \cup \{X \setminus H_n\}$ of X. Since X is wNSL, there exists a countable subset F_n of X such that for each (in X) open $O' \supseteq F_n$, $X = \overline{\mathrm{St}(O',\mathcal{U}_n)}$. For each $n \in \mathbb{N}$, let $M_n = F_n \cap Y$. [F_n must meet Y, because otherwise for the neighborhood $X \setminus H_n$ of F_n we have $\mathrm{St}(X \setminus H_n,\mathcal{U}_n) \subset X \setminus H_n$.] Then M_n is a countable subset of Y such that for each open $O \supseteq M_n$, $H_n \subseteq \overline{\mathrm{St}(O,\mathcal{U})}$. If we put $F = \bigcup\{M_n : n \in \mathbb{N}\}$, then F is a countable subset of Y such that for each open $O \supseteq F$, $\overline{\mathrm{St}(O,\mathcal{U})} \supseteq Y$, which shows that Y is wNSL. □

A *cozero-set* in a space X is a set of the form $f^{\leftarrow}(\mathbb{R} \setminus \{0\})$ for some real-valued continuous function f on X [6].

Since a cozero-set is an open F_σ-set, we have the following corollary of Theorem 2.2.

Corollary 2.3. *A cozero-set in a* wNSL *space is* wNSL.

Definition 2.4 (see [4])**.** Let Y be a subspace of a space X. Then

(1) Y is *relatively* NSL *in* X if for each open cover $\mathcal{U}$ of X, one can choose a countable subset A of X such that for each open $O \supset A$, $\mathrm{St}(O,\mathcal{U}) \supset Y$;

(2) Y is said to be a *relatively closed* NSL *space* if it is closed and relatively NSL in X.

Of course, every NSL space is relatively closed NSL.

Theorem 2.5. (1) *If $X = \bigcup\{Y_k : k \in \mathbb{N}\}$ and each Y_k is relatively* NSL *in X, then X is* NSL.

(2) *If $X = \bigcup_{k\in\mathbb{N}} \overline{Y_k}$ and each Y_k is relatively* NSL *in X, then X is* wNSL.

Proof. (1) Let $\mathcal{U}$ be an open cover of X. Then for each $k \in \mathbb{N}$, $\mathcal{U}$ covers Y_k, and since Y_k is relatively NSL, there is a countable set $F_k \subset X$, such that for each open set $O \supset F_k$, we have $\mathrm{St}(O,\mathcal{U}) \supset Y_k$. Put $F = \bigcup_{k\in\mathbb{N}} F_k$. Then F is a countable subset of X. Let O be any open set containing F. Using the fact that O contains all F_k, $k \in \mathbb{N}$, we conclude that $\mathrm{St}(O,\mathcal{U}) \supset \bigcup_{k\in\mathbb{N}} Y_k = X$, which means that X is NSL.

(2) Let $\mathcal{U}$ be an open cover of X. Each Y_k is covered by $\mathcal{U}$. As Y_k is relatively NSL in X, for each $k \in \mathbb{N}$, there is countable $F_k \subset X$ such that for each open $O \subset X$ containing F_k, $Y_k \subset \mathrm{St}(O,\mathcal{U})$. Let $F = \bigcup_{k\in\mathbb{N}} F_k$ and let G be an open set containing F. Then

$$X = \bigcup_{k\in\mathbb{N}} \overline{Y_k} \subset \overline{\mathrm{St}(G,\mathcal{U})},$$

that is, X is wNSL. □

Theorem 2.6. *Let X be a* wNSL *topological space and let Y be a topological space. If $f : X \to Y$ is a continuous surjection, then Y is a* wNSL.

Proof. Let $\mathcal{V}$ be an open cover of Y. Then $\mathcal{U} = f^{\leftarrow}(\mathcal{V}) = \{f^{\leftarrow}(V) : V \in \mathcal{V}\}$ is an open cover of X. Since X is wNSL, there is countable $F \subset X$ such that for each open O containing F we have $X = \overline{\mathrm{St}(O,\mathcal{U})}$. Let $K = f(F)$ and let G be an open neighborhood of K. Then $f^{\leftarrow}(G)$ is an open neighborhood of F such that $X = \overline{\mathrm{St}(f^{\leftarrow}(G),\mathcal{U})}$. We prove $Y = \overline{\mathrm{St}(G,\mathcal{V})}$.

Let $y \in Y$ and let $x \in X$ be such that $y = f(x)$. Then $x \in \overline{\mathrm{St}(f^{\leftarrow}(G),\mathcal{U})}$. It follows, $y = f(x) \in \overline{f(\mathrm{St}(f^{\leftarrow}(G),\mathcal{U}))} \subset \overline{\mathrm{St}(G,\mathcal{V})}$. Therefore, K and G witness for $\mathcal{V}$ that Y is wNSL. □

Recall that a space X is *para-Lindelöf* if every open cover $\mathcal{U}$ of X has a locally countable open refinement.

Theorem 2.7. *Every para-Lindelöf* wNSL *space X is weakly Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. There exists a locally countable open refinement $\mathcal{V}$ of $\mathcal{U}$. For each $x \in X$, there exists an open neighborhood W_x of x such that $W_x \subset V$ for some $V \in \mathcal{V}$ and $\{V \in \mathcal{V} : W_x \cap V \neq \emptyset\}$ is countable. Let $\mathcal{W} = \{W_x : x \in X\}$. Then $\mathcal{W}$ is an open refinement of $\mathcal{V}$. Since X is wNSL, there exists a countable subset A of X such that for every open $O \supset A$, $X = \overline{\mathrm{St}(O,\mathcal{V})}$. Especially, it is true for $O_A = \bigcup\{W_x \in \mathcal{W} : x \in A\} \supset A$, that is, $\overline{\mathrm{St}(O_A,\mathcal{V})} = X$. Set $\tilde{\mathcal{V}} = \{V \in \mathcal{V} : V \cap O_A \neq \emptyset\}$. Then $\overline{\tilde{\mathcal{V}}}$ is a countable cover of X. For each $V \in \tilde{\mathcal{V}}$, choose $U_V \in \mathcal{U}$ with $V \subseteq U_V$. Then $\{U_V : V \in \tilde{\mathcal{V}}\}$ is a countable subcover of $\mathcal{U}$, and $X = \overline{\bigcup_{V\in\tilde{\mathcal{V}}} U_V}$, which shows that X is weakly Lindelöf. □

3. New selection principles

In this section, we introduce weaker versions of NSM, NSR, and NSH spaces.

Definition 3.1. A space X is said to be:

(1) *almost neighborhood star-Menger* (aNSM) (resp. *weakly neighborhood star-Menger* (wNSM); *faintly neighborhood star-Menger* (fNSM)) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, one can choose finite sets $F_n \subset X$, $n \in \mathbb{N}$, such that for every open $O_n \supset F_n$, $n \in \mathbb{N}$, we have $\bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(O_n,\mathcal{U}_n)} = X$ (resp. $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n,\mathcal{U})} = X$; $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{O_n},\mathcal{U}_n) = X$).

(2) *almost neighborhood star-Rothberger* (aNSR) (resp. *weakly neighborhood star-Rothberger* (wNSR); *faintly neighborhood star-Rothberger* (fNSR)) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, one can choose the sequence $(x_n : n \in \mathbb{N})$ of elements of X such that for every open $O_n \ni x_n$, $n \in \mathbb{N}$, we have $\bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(O_n,\mathcal{U}_n)} = X$ (resp. $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n,\mathcal{U}_n)} = X$; $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{O_n},\mathcal{U}_n) = X$).

(3) *almost neighborhood star-Hurewicz* (aNSH) (resp. *faintly neighborhood star-Hurewicz* (fNSH)) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, such that for every open $O_n \supset F_n$, $n \in \mathbb{N}$, each $x \in X$ belongs to $\overline{\mathrm{St}(O_n,\mathcal{U}_n)}$ (resp. to $\mathrm{St}(\overline{O_n},\mathcal{U}_n)$) for all but finitely many n.

Remark 3.2. Of course, every NSM space is both aNSM and fNSM, and every aNSM space is wNSM, and similarly, for Rothberger-type properties. Also, every

NSH space is aNSH and fNSH. A regular space X is fNSM if and only if it is aNSM, and similarly for Rothberger and Hurewicz cases.

$$\begin{tikzcd}
\mathsf{R} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathsf{SSR} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathsf{NSR} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathsf{aNSR} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathsf{wNSR} \arrow[d, Rightarrow] \\
\mathsf{M} \arrow[r, Rightarrow] & \mathsf{SSM} \arrow[r, Rightarrow] & \mathsf{NSM} \arrow[r, Rightarrow] & \mathsf{aNSM} \arrow[r, Rightarrow] & \mathsf{wNSM} \\
\mathsf{H} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \mathsf{SSH} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \mathsf{NSH} \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \mathsf{aNSH} \arrow[u, Rightarrow] &
\end{tikzcd}$$

Diagram 1

Observe also that every wNSM space is wNSL.

$$\begin{tikzcd}
\mathsf{NSM} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathsf{aNSM} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathsf{wNSM} \arrow[d, Rightarrow] \\
\mathsf{NSL} \arrow[r, Rightarrow] & \mathsf{aNSL} \arrow[r, Rightarrow] & \mathsf{wNSL}
\end{tikzcd}$$

Diagram 2

Diagrams 1 and 2 give relations among classes of spaces defined above.

Example 3.3. (1) There is a wNSM space that is not NSM.

Let $\mathbb{P}$ be the space of irrational numbers with the usual metric topology, let $[0,\omega]$ be the ordinal space, let $\mathbb{Q}$ be the set of rational numbers, let $A = \pi + \mathbb{Q} = \{\pi + q : q \in \mathbb{Q}\}$, and let $X = (\mathbb{P} \times [0,\omega]) \setminus (\mathbb{P} \setminus A) \times \{\omega\}$ equipped with the following topology τ: A basic neighborhood of a point $< x, n >$, $x \in \mathbb{P}$, $n < \omega$, is of the form $U \times \{n\}$, where U is a neighborhood of $x \in \mathbb{P}$, while a basic neighborhood of a point $< y, \omega >$, $y \in A$, is of the form $U \times (n,\omega) \cup \{< y, \omega >\}$.

Notice that the space X is separable. Let $D = \{d_1, d_2, \ldots\}$ be a countable dense subset of X.

Claim 1. X is wNSM.

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Consider the sequence $(D_n : n \in \mathbb{N})$, where $D_n = \{d_i : i \leq n\}$, of finite subsets of X and a sequence $(O_n : n \in \mathbb{N})$ of neighborhoods of D_n, $n \in \mathbb{N}$. Then $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n)} \supset \overline{D} = X$, that is, X is wNSM.

Claim 2. X is not NSM.

The space X contains a copy of the space $\mathbb{P}$ as a closed subspace. Thus X cannot be Menger, because $\mathbb{P}$ is not Menger, and the Menger property is preserved by closed subspaces. On the other hand, in the class of paracompact spaces, the Menger property coincides with the NSM property.

(2) The Sorgenfrey line [6] is an aNSR space that is not Rothberger (it is not Menger).

Definition 3.4 (see [9])**.** We say that a topological space X is *d-paracompact* if every dense family of subsets of X has a locally finite refinement.

Theorem 3.5. *If a topological space X is* wNSM *and d-paracompact, then X is* aNSM.

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Since X is wNSM, one can choose finite $F_n \subset X$, such that for every open $O_n \supset F_n$, $n \in \mathbb{N}$, we have $\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n,\mathcal{U}_n)$ is dense in X. By the assumption, $\{\mathrm{St}(O_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ has a locally finite refinement $\mathcal{W}$. Then $\bigcup \mathcal{W} = (\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n,\mathcal{U}_n))$ and therefore $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n,\mathcal{U}_n)} = \overline{\bigcup \mathcal{W}}$, that is, $\bigcup \mathcal{W}$ is dense in X. As $\mathcal{W}$ is a locally finite family, we have that $\overline{\bigcup \mathcal{W}} = \bigcup_{W\in\mathcal{W}} \overline{W}$.

Since for every $W \in \mathcal{W}$, there is $k = k(W) \in \mathbb{N}$ such that $W \subset \mathrm{St}(O_k,\mathcal{U}_k)$. Therefore, it follows $X = \bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(O_n,\mathcal{U}_n)}$, that is, X is aNSM. □

Theorem 3.6. *Let X be an* aNSM *space and let Y be a topological space. If $f : X \to Y$ is a continuous surjection, then Y is also an* aNSM *space.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. For each $n \in \mathbb{N}$, the set $\mathcal{U}_n := \{f^{\leftarrow}(V) : V \in \mathcal{V}_n\}$ is an open cover of X. Since X is aNSM, there are finite sets $F_n \subset X$, $n \in \mathbb{N}$, such that for every open $O_n \supset F_n$, $n \in \mathbb{N}$, we have $\{\overline{\mathrm{St}(O_n,\mathcal{U}_n)} : n \in \mathbb{N}\}$ is a cover of X. The sets $f(F_n)$, $n \in \mathbb{N}$, are finite in Y. For each n, let G_n be an open neighborhood of $f(F_n)$. Then $f^{-1}(G_n) = H_n$ is an open subset of X for each $n \in \mathbb{N}$ and $H_n \supset F_n$. Thus $X = \bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(H_n,\mathcal{U}_n)}$. We prove that $Y = \bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(G_n,\mathcal{V}_n)}$.

Let $y \in Y$ and let $x \in X$ be such that $y = f(x)$. Then there is $k \in \mathbb{N}$ such that $x \in \overline{\mathrm{St}(H_k,\mathcal{U}_k)}$. Then $y = f(x) \in \overline{f(\mathrm{St}(H_k,\mathcal{U}_k)}$. Because $f(\mathrm{St}(H_k,\mathcal{U}_k)) \subset f(\mathrm{St}(f^{\leftarrow}(G_k),\mathcal{U}_k)) \subset \mathrm{St}(G_k,\mathcal{V}_k)$, we have $y \in \overline{\mathrm{St}(G_k,\mathcal{V}_k)}$. Hence,
$Y = \bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(G_k,\mathcal{V}_k)}$, that is, Y is aNSM. □

Definition 3.7 (see [6]). We say that a subset U of a space X is *regular open* (*regular closed*) if $U = \mathrm{int}(\overline{U})$ $(U = \overline{\mathrm{int}(U)})$.

Theorem 3.8. *A space X is a* wNSM *if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by regular open sets, there exist finite sets $F_n \subset X$, $n \in \mathbb{N}$, such that for every open $O_n \supset F_n$, $n \in \mathbb{N}$, it holds $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n,\mathcal{U}_n)} = X$.*

Proof. $(\Rightarrow)$: It is obvious.

$(\Leftarrow)$: Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Putting $\mathcal{V}_n =: \{\mathrm{int}(\overline{U}) : U \in \mathcal{U}_n\}$, $n \in \mathbb{N}$, we obtain a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of covers of X by regular open sets. Then there exist finite $F_n \subset X$, $n \in \mathbb{N}$, such that for every open $\mathcal{O}_n \supset F_n$, $n \in \mathbb{N}$, we have $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(\mathcal{O}_n,\mathcal{V}_n)} = X$. For every $n \in \mathbb{N}$ and every $V \in \mathcal{V}_n$, there exists $U_V \in \mathcal{U}_n$ such that $V = \mathrm{int}(\overline{U_V})$. Consider the sequence $(\mathcal{W}_n : n \in \mathbb{N})$, where $\mathcal{W}_n = \{U_V : V \in \mathcal{V}_n\}$. We claim that $\overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n,\mathcal{U}_n)} = X$.

Let $x \in X$ and let G be a neighborhood of x. There exist $k \in \mathbb{N}$ and $V \in \mathcal{V}_k$ such that $G \cap V \neq \emptyset$ and $V \cap O_k \neq \emptyset$, that is, there is $U = U_V \in \mathcal{U}_k$ such that $G \cap \mathrm{int}(\overline{U}) \neq \emptyset$ and $O_k \cap \mathrm{int}(\overline{U}) \neq \emptyset$. Then $G \cap U \neq \emptyset$ and $\mathcal{O}_k \cap U \neq \emptyset$. [Let us prove $G \cap U \neq \emptyset$. Let $y \in G \cap \mathrm{int}(\overline{U}) \neq \emptyset$. There is an open set H containing y such that $H \subset \overline{U}$, hence the neighborhood $G \cap H$ of y intersects U. It follows $G \cap U \neq \emptyset$.] Therefore, $x \in \overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n,\mathcal{U}_n)}$, that is, X is wNSM. □

Theorem 3.9. *If X is an* aNSM *space, then every clopen subset of X is* aNSM.

Proof. Let (Y, τ_Y) be a clopen subset of an aNSM space (X, τ) and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of (Y, τ_Y). As Y is clopen, $\mathcal{V}_n = \mathcal{U}_n \cup (X \setminus Y)$ is an open cover of X for every $n \in \mathbb{N}$. Since X is aNSM, one can choose finite $F_n \subset X$, such that for every open (in X) $O_n \supset F_n$, $n \in \mathbb{N}$, we have $\bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(O_n, \mathcal{V}_n)} = X$.

Define now $H_n = Y \cap F_n$ if $Y \cap F_n \neq \emptyset$; and H_n any finite subset of Y, otherwise. [Note that not all $Y \cap F_n$ can be empty, because otherwise $X \setminus Y$ is a neighborhood of all F_n intersecting only $X \setminus Y$ from $\mathcal{V}_n$.]

We claim that $(H_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that Y is aNSM.

Let G_n be an open set in (Y, τ_Y) containing H_n, $n \in \mathbb{N}$. Then $W_n = G_n \cup (X \setminus Y)$ is an open set in X containing F_n, $n \in \mathbb{N}$ and thus $\bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(W_n, \mathcal{V}_n)} = X$. Because Y is closed in X and $H_n \cap (X \setminus Y) = \emptyset$, we conclude $\bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(H_n, \mathcal{U}_n)} = Y$, which means that (Y, τ_Y) is aNSM. $\square$

Definition 3.10 (see [4])**.** Let Y be a subset of a space X. Then

(1) Y is *relatively* NSM (resp. *relatively* NSH) in X if for each $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, one can choose a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X, such that for every open $O_n \supset A_n$, $n \in \mathbb{N}$, we have $\bigcup\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \supset Y$ (resp. for each $y \in Y$, $y \in \mathrm{St}(O_n, \mathcal{U}_n)$ for all but finitely many n).

(2) Y is relatively NSR in X if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there are $x_n \in X$, $n \in \mathbb{N}$, such that for all open $O_n \ni x_n$, $n \in \mathbb{N}$, we have $\bigcup\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \supset Y$.

Proposition 3.11. *The following statements hold:*

(1) *If* $X = \bigcup\{Y_k : k \in \mathbb{N}\}$, *and each* Y_k *is relatively* NSM (*resp. relatively* NSH, *relatively* NSR) *in* X, *then* X *is* NSM (*resp.* NSH, NSR)*;*

(2) *If* $X = \bigcup\{\overline{Y_k} : k \in \mathbb{N}\}$, *and each* Y_k *is relatively* NSM (*resp. relatively* NSH, *relatively* NSR) *in* X, *then* X *is* aNSM (*resp.* aNSH, aNSR).

Proof. (1) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Then for each $k, n \in \mathbb{N}$, $\mathcal{U}_n$ covers Y_k, and since Y_k is relatively NSM, there are countable sets $F_{k,n} \subset X$, $n \in \mathbb{N}$, such that for each open set $O_{k,n} \supset F_{k,n}$, we have $\mathrm{St}(O_{k,n}, \mathcal{U}_n) \supset Y_k$. Consider the sequences $(F_{k,n} : k, n \in \mathbb{N})$ and $(G_{k,n} : k, n \in \mathbb{N})$ of neighborhoods of $F'_{k,n}s$. It is easy to conclude that $\bigcup_{k\in\mathbb{N}} \mathrm{St}(G_{k,n}, \mathcal{U}_n) \supset \bigcup_{k\in\mathbb{N}} Y_k = X$, which means that X is NSM.

(2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Each Y_k is covered by $\mathcal{U}_n$. As Y_k is relatively NSM in X, for each $k \in \mathbb{N}$, there is a sequence $(F_{k,n} : n \in \mathbb{N})$ of finite subsets of X such that for all open $O_{k,n} \supset F_{k,n}$, $Y_k \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(O_{k,n}, \mathcal{U}_n)$. Then

$$X = \bigcup_{k\in\mathbb{N}} \overline{Y_k} \subset \overline{\mathrm{St}(O_{k,n}, \mathcal{U}_n)},$$

that is, X is wNSM. $\square$

Proposition 3.12. *If* $X = \bigcup\{Y_k : k \in \mathbb{N}\}$ *and each* Y_k *is* wNSM (wNSR) *in* X*, then* X *is* aNSM (aNSR).

Proof. We shall prove the NSM case. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Rearrange this sequence to $(\mathcal{U}_{k,m} : k, m \in \mathbb{N})$. For each $k \in \mathbb{N}$, $(\mathcal{U}_{k,m} : m \in \mathbb{N})$ is a sequence of covers of Y_k by sets open in X. For each k,

Y_k is wNSM, and thus there are finite sets $F_{k,m} \subset X$, $m \in \mathbb{N}$, such that for every open $O_{k,m} \supset F_{k,m}$, $m \in \mathbb{N}$, we have $\overline{\bigcup_{m\in\mathbb{N}} \mathrm{St}(O_{k,m}, \mathcal{U}_{k,m})} \supset Y_k$. By the assumption, $X = \bigcup_{k\in\mathbb{N}} Y_k$. It follows $X = \bigcup_{k\in\mathbb{N}} \overline{\bigcup_{k,m\in\mathbb{N}} \mathrm{St}(O_{k,m}, \mathcal{U}_{k,m})}$, that is, X is aNSM. □

Lemma 3.13 (see [26]). *Let A be a regular closed subset of a space X and let B be a dense subset of X. Then $B \cap A$ is a dense subset of A.*

Lemma 3.14 (see [2]).
(1) *Let X be a topological space. If $Y \subset X$ is dense in X, and $D \subset Y$ is dense in Y, then D is dense in X;*
(2) *If $D \subset X$ is dense in X and $E \subset Y$ is dense in Y, then $D \times E$ is dense in $X \times Y$.*

Proposition 3.15. *The following properties hold:*

1. *Let F be a regular closed subset of a space X and let Y be a* wNSM (wNSR) *in X. Then $Y \cap F$ is* wNSM (wNSR) *in F.*
2. *Let $A \subset X$ be* wNSM (wNSR) *in X and let $B \subset A$ be* wNSM (wNSR) *in A. Then B is* wNSM (wNSR) *in X.*
3. *If $D \subset X$ is* wNSM (wNSR) *in X and $E \subset Y$ is* wNSM (wNSR) *in Y, then $D \times E$ is* wNSM (wNSR) *in $X \times Y$.*

Proof. By Lemmas 3.13 and 3.14, the proof is evident. □

Lemma 3.16 (see [10]). *If $\mathcal{U}$ is an ω-cover of a space X, then $\{U^2 : U \in \mathcal{U}\}$ is an ω-cover of X^2.*

An open cover $\mathcal{U}$ of a space X is said to be an *almost ω-cover* if each finite $F \subset X$ is contained in $\overline{U}$ for some $U \in \mathcal{U}$.

Denote by aNSM$(\mathcal{O}, \Omega)$ the following statement on a space X: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there are finite sets $F_n \subset X$, $n \in \mathbb{N}$, such that for each sequence $(O_n : n \in \mathbb{N})$ of open sets with $O_n \supset F_n$, $n \in \mathbb{N}$, $\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an almost ω-cover of X.

Theorem 3.17. *If all finite powers of a space X are* aNSM*, then X satisfies* aNSM$(\mathcal{O}, \Omega)$.

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ be a partition of $\mathbb{N}$ into infinite pairwise disjoint sets. For every $k \in \mathbb{N}$ and every $m \in N_k$, let $\mathcal{W}_m = (\mathcal{U}_m)^k = \{U^k : U \in \mathcal{U}_m\}$. Then $(\mathcal{W}_m : m \in N_k)$ is a sequence of open covers of X^k. Applying to this sequence the fact that X^k is aNSM, we find a sequence $(A_m : m \in N_k)$ of finite subsets of X^k such that for every open sequence $(O_m : m \in N_k)$ of neighborhoods of A_m, $m \in N_k$, $X^k = \bigcup_{m\in N_k} \overline{\mathrm{St}(O_m, \mathcal{W}_m)}$. For every $m \in N_k$, let B_m be a finite subset of X such that $B_m^k \supset A_m$. Consider the sequence of all $B_m, m \in N_k, k \in \mathbb{N}$, chosen in this way; denote it as $(B_n : n \in \mathbb{N})$. Let $(H_n : n \in \mathbb{N})$ be a sequence of neighborhoods of B_n, $n \in \mathbb{N}$. We claim that $\{\overline{\mathrm{St}(H_n, \mathcal{U}_n)} : n \in \mathbb{N}\}$ is an ω-cover of X. Let $F = \{x_1, \cdots, x_p\}$ be a finite subset of X. Then $\langle x_1, \cdots, x_p\rangle \in X^p$. There exists $n \in N_p$ such that $(H_n^p : n \in \mathbb{N})$ is a sequence of neighborhoods of A_n. Hence, there exists $n \in \mathbb{N}$ such that $\langle x_1, \cdots, x_p\rangle \in \overline{\mathrm{St}(H_n^p, \mathcal{W}_n)}$. Therefore we have $F \subset \overline{\mathrm{St}(H_n, \mathcal{U}_n)}$, that is, X satisfies aNSM$(\mathcal{O}, \Omega)$. □

The symbol $\mathcal{O}^{\mathrm{awgp}}$ denotes the collection of almost weakly groupable covers of a space. A countable open cover $\mathcal{U}$ of a space X is said to be *almost weakly groupable* if there is a partition $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ of $\mathcal{U}$ into finite, pairwise disjoint subcollections, such that for each finite subset F of X there is $n \in \mathbb{N}$ with $F \subset \overline{\bigcup \mathcal{U}_n}$.

Theorem 3.18. *For a space X, the following statements are equivalent:*

(1) *X satisfies $\mathsf{aNSM}(\mathcal{O}, \Omega)$;*
(2) *X satisfies $\mathsf{NSM}(\mathcal{O}, \mathcal{O}^{\mathrm{awgp}})$.*

Proof. (1) ⇒ (2) This implication directly follows from the facts that the selection principle aNSM is monotone in the second coordinate and each countable almost ω-cover is almost weakly groupable.

(2) ⇒ (1) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. For each $n \in \mathbb{N}$, $\mathcal{V}_n = \bigwedge_{i\le n} \mathcal{U}_i$. For each n, $\mathcal{V}_n$ is an open cover of X that refines $\mathcal{U}_i$ for all $i \le n$. By applying (2) to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$, there is a sequence A_n, $n \in \mathbb{N}$, of finite subsets of X such that for every sequence $(O_n : n \in \mathbb{N})$ of open neighborhoods of A_n, $n \in \mathbb{N}$, the cover $\{\mathrm{St}(O_n, \mathcal{V}_n)$ is an almost weakly groupable cover of X. In other words, there is an increasing sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of natural numbers such that for each finite set F in X, one has

$$F \subset \overline{\bigcup \{\mathrm{St}(O_i, , \mathcal{V}_i) : n_k < i \le n_{k+1}\}}$$

for some k. Define now

$$\begin{aligned} B_n &= \bigcup_{i<n} A_i, \ \text{for } n < n_1, \\ B_n &= \bigcup_{n_k<i\le n_{k+1}} A_i, \ \text{for } n_k < n \le n_{k+1}. \end{aligned}$$

Each B_n is a finite subset of X. For each n, take a neighborhood H_n of B_n. Then, by the construction of sets B_n and covers $\mathcal{V}_n$, we easily conclude that $\{\mathrm{St}(H_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an almost ω-cover of X, that is, X satisfies $\mathsf{aNSM}(\mathcal{O}, \Omega)$. □

4. More on spaces and mappings

In this section, we investigate the preservation of the properties, which we consider it in this article under some kinds of mappings.

Theorem 4.1. *Let X be an aNSH space and let Y be a topological space. If $f : X \to Y$ is a continuous mapping from X onto Y, then Y is also an aNSH space.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. For each $n \in \mathbb{N}$, the set $\mathcal{U}_n' := \{f^{\leftarrow}(U) : U \in \mathcal{U}_n\}$ is an open cover of X. Since X is aNSH, there are finite sets $(F_n \subset X)$, $n \in \mathbb{N}$, such that for every open set $O_n \supset F_n$, $n \in \mathbb{N}$, and each $x \in X$, we have $x \in \{\overline{\mathrm{St}(O_n, \mathcal{U}_n')}$ for all but finitely many n. The sets $f(F_n)$, $n \in \mathbb{N}$, are finite in Y. Let $G_n \supset f(F_n)$ for each n be an open set in Y. Then $f^{\leftarrow}(G_n) = H_n$ is an open subset of X for each $n \in \mathbb{N}$ and $H_n \supset F_n$. Thus for each $x \in X$, we have $x \in \overline{\mathrm{St}(H_n, \mathcal{U}'_n)}$ for all but finitely many n. We prove that for each $y \in Y$, $y \in \overline{\mathrm{St}(G_n, \mathcal{U}_n)}$ for all but finitely many n.

Let $y \in Y$ and let $x \in X$ be such that $y = f(x)$. Then there is $k_0 \in \mathbb{N}$ such that $x \in \overline{\mathrm{St}(H_k, \mathcal{U}'_k)}$ for each $k \geq k_0$. Then $y = f(x) \in f(\overline{\mathrm{St}(H_k, \mathcal{U}'_k)})$. Because $f(\mathrm{St}(H_k, \mathcal{U}'_k)) \subset f(\mathrm{St}(f^{\leftarrow}(G_k), \mathcal{U}'_k)) \subset \mathrm{St}(G_k, \mathcal{U}_k)$, we have $y \in \overline{\mathrm{St}(G_k, \mathcal{U}_k)}$ for all $k \geq k_0$, that is, Y is aNSH. □

A mapping $f : X \to Y$ is *weakly continuous* [16] (resp. *θ-continuous* [7], *strongly θ-continuous* [17]) if for each $x \in X$ and each open neighborhood V of $f(x)$, there is an open neighborhood U of x such that $f(U) \subset \overline{V}$ (resp. $f(\overline{U}) \subset \overline{V}$, $f(\overline{U}) \subset V$).

Theorem 4.2. *A θ-continuous image $Y = f(X)$ of an* fNSR *space X is also* fNSR.

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. Fix $x \in X$. For each $n \in \mathbb{N}$, pick a set $V(x,n) \in \mathcal{V}_n$ with $f(x) \in V(x,n)$. Using the fact that f is θ-continuous, take an open set $U(x,n) \subset X$ such that $x \in U(x,n)$ and $f(\overline{U(x,n)}) \subset \overline{V(x,n)}$. So, for each n, the set $\mathcal{U}_n := \{U(x,n) : x \in X\}$ is an open cover of X. Since X is fNSR, there is a sequence $(a_n : n \in \mathbb{N})$ of points in X such that for any sequence $(S_n : n \in \mathbb{N})$ of neighborhoods of a_n, $\{\mathrm{St}(\overline{S_n}, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of X.

Consider the sequence $(b_n = f(a_n) : n \in \mathbb{N})$ of points in Y and a sequence $(T_n : n \in \mathbb{N})$ of neighborhoods of b_n, $n \in \mathbb{N}$. For each n, there exists an open set $O_n \ni a_n$ such that $f(\overline{O_n}) \subset \overline{T_n}$. Then $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{O_n}, \mathcal{U}_n)$ implies $f(X) = Y = \bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{T_n}, \mathcal{V}_n)$; it is not hard to check. It follows that Y is an fNSR space. □

Similarly, we can prove the following.

Theorem 4.3. *A θ-continuous image $Y = f(X)$ of an* fNSM (*resp.* fNSH) *space X is also* fNSM (*resp.* fNSH).

The following results show the relationships between NSM (resp. NSH, NSR) spaces and fNSM (resp. fNSH, fNSR) spaces.

Theorem 4.4. *If a space Y is a weakly continuous image of an* NSM *space X, then Y is* fNSM.

Proof. Let $f : X \to Y$ be a weakly continuous mapping and let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. For each $x \in X$ and each $n \in \mathbb{N}$, there is $V(f(x),n) \in \mathcal{V}_n$ containing $f(x)$. Because f is weakly continuous, pick an open set $U(x,n) \subset X$ that contains x with $f(U(x,n)) \subset \overline{V(f(x),n)}$. In this way, one obtains for each $n \in \mathbb{N}$, an open cover $\mathcal{U}_n = \{U(x,n) : x \in X\}$ of X. As X is NSM, there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for any open sets $G_n \supset A_n$, $n \in \mathbb{N}$, $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(G_n, \mathcal{U}_n)$. Put $B_n = f(A_n)$, $n \in \mathbb{N}$. We have the sequence $(B_n : n \in \mathbb{N})$ of finite subsets of Y. We prove that this sequence witnesses for $(\mathcal{V}_n : n \in \mathbb{N})$ that Y is fNSM.

For each $n \in \mathbb{N}$, take an arbitrary neighborhood H_n of B_n. Since $A_n \subset X$ is finite and f is weakly continuous, there is a neighborhood O_n of A_n such that

$f(O_n) \subset \overline{H_n}$. It is easy now to prove that from the construction of the sequences $\mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n)$, it follows $Y = \bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{H_n}, \mathcal{V}_n)$. This shows that Y is an fNSM space. □

Quite similarly one can prove the following.

Theorem 4.5. *If $Y = f(X)$ is a weakly continuous image of an* NSH (*resp.* NSR) *space X, then Y is* fNSH (*resp.* fNSR).

Theorem 4.6. *A strongly θ-continuous image $Y = f(X)$ of an* fNSM (*resp.* fNSH, fNSR) *space X is an* NSM (*resp.* NSH, NSR) *space.*

Proof. (for the Rothberger case; other two cases are proved similarly) Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. Since each $\mathcal{V}_n$ covers X, for each $x \in X$ and each $n \in \mathbb{N}$, one can find a set V_x^n that contains $f(x)$. The strong θ-continuity of f implies the existence of an open neighborhood U_x^n of x with $f(\overline{U_x^n}) \subset V_x^n$. Letting $\mathcal{U}_n := \{U_x^n : x \in X\}$, we get a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X. There are points $p_1, p_2, \ldots$ in X such that for arbitrary open sets $G_1 \ni p_1, G_2 \ni p_2, \ldots$, $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{G_n}, \mathcal{U}_n)$.

Set $q_n = f(p_n)$, $n \in \mathbb{N}$, and take for each n, an open set $H_n \ni q_n$. Next, for each n, take an open set $O_n \ni p_n$ such that $f(\overline{O_n}) \subset H_n$. Then $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{O_n}, \mathcal{U}_n)$. Let $y \in Y$ and let $x \in X$ be such that $y = f(x)$. There is $k \in \mathbb{N}$ such that $x \in \mathrm{St}(\overline{O_k}, \mathcal{U}_k)$. It is easily verified that $y \in \mathrm{St}(H_k, \mathcal{V}_k)$. In other words, $Y = \bigcup_{n\in\mathbb{N}} \mathrm{St}(H_n, \mathcal{U}_n)$, that is, Y is an NSR space. □

5. Conclusion

New classes of spaces related to star versions of the classical Menger, Hurewicz, and Rothberger covering properties have been considered. This idea may be further applied to bitopological spaces and we hope it can open a new research direction. Here is a short explanation for the NSM case. A bitopological space (X, τ_1, τ_2) is a (τ_i, τ_j)-NSM-*space*, $i, j = 1, 2$, if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for any τ_j-open sets $O_n \supset F_n$, $n \in \mathbb{N}$, the set $\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a cover of X. It is worth also to study the infinitely long two-person game associated to NSM. Similarly, we define bitopological versions of other properties studied in this article.

References

1. L. Babinkostova, B.A. Pansera and M. Scheepers, *Weak covering properties and infinite games*, Topology Appl. **159** (2012), no. 17, 3644–3657.
2. L. Babinkostova, B.A. Pansera and M. Scheepers, *Weak covering properties and selection principles*, Topology Appl. **160** (2013), no. 18, 2251–2271.
3. M. Bonanzinga, F. Cammaroto and Lj.D.R. Kočinac, *Star-Hurewicz and related properties*, Appl. Gen. Topol. **5** (2004), no. 1, 79–89.
4. M. Bonanzinga, F. Cammaroto. Lj.D.R. Kočinac and M.V. Matveev, *On weaker forms of Menger, Rothberger and Hurewicz properties*, Mat. Vesnik **61** (2009), no. 1, 13–23.
5. E.K. van Douwen, G.M. Reed, A.W. Roscoe and I.J. Tree, *Star covering properties*, Topology Appl. **39** (1991), no. 1, 71–103.

6. R. Engelking, *General Topology*, Sigma Series in Pure Mathematics 6, Heldermann, 2nd edition, Berlin, 1989.
7. S.V. Fomin, *Extensions of topological spaces*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **32** (1941) 114–116.
8. D. Kocev, *Almost Menger and related spaces*, Mat. Vesnik **61** (2009), no. 2, 173–180.
9. D. Kocev, *Menger-type covering properties of topological spaces*, Filomat **29** (2015), no. 1, 99–106.
10. Lj.D.R. Kočinac, *Star-Menger and related spaces*, Publ. Math. Debrecen **55** (1999), no. 3-4, 421–431.
11. Lj.D.R. Kočinac, *Star-Menger and related spaces II*, Filomat **13** (1999), 129–140 .
12. Lj.D.R. Kočinac, *The Pixley-Roy topology and selection principles*, Questions Answers Gen. Topology **19** (2001), no. 2, 219–225.
13. Lj.D.R. Kočinac, *Selected results on selection principles*, in: Proceedings of the 3rd Seminar on Geometry and Topology, pp. 71–104, Azarb. Univ. Tarbiat Moallem, Tabriz, 2004.
14. Lj.D.R. Kočinac, *Star selection principles: A survey*, Khayyam J. Math. **1** (2015), no. 1, 82–106.
15. Lj.D.R. Kočinac, *Variations of classical selection principles: An overview*, Quaest. Math. in press, Doi: 10.2989/16073606.2019.1601646.
16. N. Levine, *Strong continuity in topological spaces*, Amer. Math. Monthly **67** (1960) 269–269.
17. P.E. Long and L.L. Herrington, *Strongly θ-continuous functions*, J. Korean Math. Soc. **18** (1981) 21–28.
18. M. Matveev, *A survey on star covering properties*, Topology Atlas, preprint No. 330, 1998.
19. B.A. Pansera, *Weaker forms of the Menger property*, Quaest. Math. **35** (2012), no. 2, 161–169.
20. M. Sakai, *The weak Hurewicz property of Pixley-Roy hyperspaces*, Topology Appl. **160** (2013), no. 8, 2531–2537.
21. M. Sakai, *Some weak covering properties and infinite games*, Cent. Eur. J. Math. **12** (2014), no. 2, 322–329.
22. M. Scheepers, *Combinatorics of open covers I: Ramsey theory*, Topology Appl. **69** (1996), no. 1, 31–62.
23. Y.-K. Song, *Remarks on neighborhood star-Lindelöf spaces*, Filomat **27** (2013), no. 1, 149–155.
24. Y.-K. Song, *Remarks on neighborhood star-Lindelöf spaces II*, Filomat **27** (2013), no.5, 875–880.
25. Y.-K. Song, *Remarks on neighborhood star-Menger spaces*, Publ. Inst. Math. (Beograd) (N.S.) **104 (118)** (2018), 183–191.
26. Y.-K. Song and R. Li, *On weakly Hurewicz spaces*, Filomat **29** (2015), no. 4, 667–671.

[1]Dynamic of Engines and Vibroacoustic Laboratory, University M'Hamed Bougara of Boumerdes, Algeria.
Email address: r.lakehal@univ-boumerdes.dz; djam_seba@yahoo.fr

[2]University of Niš, Faculty of Sciences and Mathematics, 18000 Niš, Serbia.
Email address: lkocinac@gmail.com