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NUMERICAL SIMULATION FOR A CLASS OF SINGULARLY PERTURBED CONVECTION DELAY PROBLEMS

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Abstract. This article presents a solution for a class of singularly perturbed convection with delay problems arising in the control theory. The approach of extending Taylor's series for the convection term gives to a bad approximation when the delay is not the smallest order of singular perturbation parameter. To handle the delay term, we model an interesting mesh form such that the delay term lies on mesh points. The parametric cubic spline is adapted to the continuous problem on a specially designed mesh. The truncation error for the proposed method is derived. Numerical examples are experimented to examine the effect of the delay parameter on the layer structure.

1. Introduction

Consider the following singularly perturbed convection with delay equation:

$$\mathfrak{L}^{\varepsilon,\delta}u \equiv \varepsilon u''(\mathrm{x}) + \alpha(\mathrm{x})u'(\mathrm{x}-\delta) + \beta(\mathrm{x})u(\mathrm{x}) = \gamma(\mathrm{x}), \text{ on } \Omega = (0,1), \tag{1.1}$$

with

$$u(\mathrm{x}) = \omega(\mathrm{x}) \quad , \ -\delta \leqslant \mathrm{x} \leqslant 0 \ , \ \ u(1) = \upsilon, \tag{1.2}$$

where $(0 < \varepsilon \ll 1)$ is the singular perturbation parameter and $\delta = O(\varepsilon)$ is the delay parameter. The functions $\alpha(\mathrm{x})$, $\beta(x)$, $\gamma(\mathrm{x})$, and $\omega(\mathrm{x})$ are continuously differentiable and υ is a constant. Also, it is assumed that $\beta(x) \leqslant -\Theta < 0$, where $\Theta > 0$. The solution has steep gradients or oscillatory behavior at the boundary for smaller value of singular perturbation parameter. To encounter such situations, it needs to develop suitable numerical methods to have solutions at boundaries. In general, the applications of these problems are encountered in

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various models, for instance, reaction-diffusion equations [2], diffusion in polymers [15], thermo-elasticity [6], hydrodynamics of liquid helium [8], variational problems in control theory [7], an optically bistable devices [4], and so on.

Mathematical investigation on singularly perturbed differential-difference equations (SPDDE) was initiated in [12–14]. Various numerical methods have been incorporated for solving SPDDE, for instance, the hybrid method [9], B-spline collocation method [11], finite difference schemes [10, 17], fitted tridiagonal finite difference method [18], finite difference scheme on adapted mesh [3], nonpolynomial spline with uniform mesh [16], exponentially fitted spline method for linear and nonlinear [19,21,22], tension spline on uniform mesh [20], and spline in tension with nonuniform mesh [23]. In this article, we model a special mesh discretization for convection term and the continuous problem with specially designed mesh is constructed by a parametric cubic spline.

2. Continuous problem

Since $\delta = O(\varepsilon)$, Equation (1.1) can be written as follows:

$$\mathfrak{L}^{\varepsilon,\delta}u \equiv \begin{cases} \varepsilon u''(\mathrm{x}) + \beta(\mathrm{x})u(\mathrm{x}) = \gamma(\mathrm{x}) - \alpha(\mathrm{x})\omega'(x-\delta), & \text{if } 0 < x < \delta, \\ \varepsilon u''(\mathrm{x}) + \alpha(\mathrm{x})u_1 + \beta(\mathrm{x})u(\mathrm{x}) = \gamma(\mathrm{x}) + \alpha(x)\omega_0, & \text{if } x = \delta, \\ \varepsilon u''(\mathrm{x}) + \alpha(\mathrm{x})u'(\mathrm{x}-\delta) + \beta(\mathrm{x})u(\mathrm{x}) = \gamma(\mathrm{x}), & \text{if } \delta < \mathrm{x} < 1. \end{cases}$$

The differential operator $\mathfrak{L}^{\varepsilon,\delta}$ satisfies Lemma 2.1 stated below.

Lemma 2.1. *Suppose that $\kappa(x)$ is continuously differentiable in Ω with $\kappa(0) \geqslant 0$, $\kappa(1) \geqslant 0$. Then $\mathfrak{L}^{\varepsilon,\delta}\kappa(x) \leqslant 0$, for all $x \in (0,1)$, implies that $\kappa(x) \geqslant 0$ for all $x \in [0,1]$.*

Proof. Suppose that $\kappa(t) < 0$ and that $\kappa(t) = \min_{\mathrm{x}\in\overline{\Omega}} \kappa(x)$, where $t \in \overline{\Omega}$. It is obvious that $t \notin \{0,1\}$, therefore $\kappa'(t) = 0$ and $\kappa''(t) \geqslant 0$. We have the following cases:
(i) For $0 < t < \delta$, $\mathfrak{L}^{\varepsilon,\delta}\kappa(t) = \varepsilon\kappa''(t) + \beta(t)\kappa(t) > 0$, (since $\beta(t) < 0$).
(ii) For $t = \delta$, $\mathfrak{L}^{\varepsilon,\delta}\kappa(t) = \varepsilon\kappa''(t) + a(t)u_1 + \beta(t)\kappa(t) > 0$.
(iii) For $\delta < t < 1$, $\mathfrak{L}^{\varepsilon,\delta}\kappa(t) = \varepsilon\kappa''(t) + a(t)\kappa'(t-\delta) + \beta(t)\kappa(t) > 0$. Since $t > \delta$ and $(t-\delta) \in \overline{\Omega}$, then $\mathtt{k}'(t-\delta) = 0$.
Combining the above cases (i)–(iii) contradicts the hypothesis that $\mathfrak{L}^{\varepsilon,\delta}\kappa(\mathrm{x})$ is negative. □

Lemma 2.2. *Let the analytical solution of (1.1) and (1.2) be $u(x)$; then*

$$\|u\| \leqslant \Theta^{-1}\|\gamma\| + \boldsymbol{C}_1 \max\big(\|\omega\|, |\upsilon|\big), \tag{2.1}$$

where $\|.\|$ is the l_∞-norm given by $\|x\|_\infty = \max|x_i|$ and $\boldsymbol{C}_1(\geqslant 1)$ and Θ are the positive constants.

Proof. Let $\kappa^{\pm}(\mathrm{x})$ be the barrier functions defined by

$$\kappa^{\pm}(\mathrm{x}) = \|\gamma\|\Theta^{-1} + \boldsymbol{C}_1\max(\|\omega\| + |\upsilon|) \pm u(\mathrm{x}).$$

Then

$$\begin{aligned} \kappa^{\pm}(0) &= \|\gamma\|\Theta^{-1} + \boldsymbol{C}_1\max(\|\omega\| + |\upsilon|) \pm \omega(0), \\ &= \|\gamma\|\Theta^{-1} + (\boldsymbol{C}_1\|\omega\| \pm \omega(0)) + \boldsymbol{C}_1\max|\upsilon|, \geqslant 0, \quad (\text{ since } \|\omega\| \geqslant \omega(0)) \end{aligned}$$

$$\begin{aligned}\kappa^{\pm}(1) &= \|\gamma\|\,\Theta^{-1} + \boldsymbol{C}_1 \max\left(\|\omega\| + |\upsilon|\right) \pm \upsilon,\\ &= \|\gamma\|\,\Theta^{-1} + \boldsymbol{C}_1 \|\omega\| + \left(\boldsymbol{C}_1 |\upsilon| \pm \upsilon\right) \geqslant 0.\end{aligned}$$

For $0 < \mathrm{x} \leqslant \delta$, it follows that

$$\begin{aligned}\mathfrak{L}^{\varepsilon,\delta}\kappa^{\pm} &= \varepsilon\kappa''^{\pm}(\mathrm{x}) + \beta(\mathrm{x})\kappa^{\pm}(\mathrm{x}),\\ &= \beta(\mathrm{x})\left(\|\gamma\|\,\Theta^{-1} + \boldsymbol{C}_1 \max\left(\|\omega\|,|\upsilon|\right)\right) \pm \mathfrak{L}^{\varepsilon,\mu}u,\\ &= \beta(\mathrm{x})\left(\|\gamma\|\,\Theta^{-1} + \boldsymbol{C}_1 \max\left(\|\omega\|,|\upsilon|\right)\right) \pm \left(\gamma(\mathrm{x}) - a(\mathrm{x})\omega'(\mathrm{x}-\delta)\right),\\ &= \left(-\|\gamma\| \pm \gamma(\mathrm{x})\right) + \beta(\mathrm{x})\boldsymbol{C}_1 \max\left(\|\omega\|,|\upsilon|\right) \mp \alpha(\mathrm{x})\omega'(\mathrm{x}-\delta) < 0.\end{aligned}$$

For $\delta < \mathrm{x} < 1$, we have

$$\begin{aligned}\mathfrak{L}^{\varepsilon,\delta} &= \varepsilon\kappa''^{\pm}(\mathrm{x}) + \alpha(\mathrm{x})\kappa'^{\pm}(\mathrm{x}-\delta) + \beta(\mathrm{x})\kappa^{\pm}(\mathrm{x}),\\ &= \beta(\mathrm{x})\left(\|\gamma\|\,\Theta^{-1} + \boldsymbol{C}_1 \max\left(\|\omega\|,|\upsilon|\right)\right) \pm \mathfrak{L}^{\varepsilon,\mu}u,\\ &= \beta(\mathrm{x})\left(\|\gamma\|\,\Theta^{-1} + \boldsymbol{C}_1 \max\left(\|\omega\|,|\upsilon|\right)\right) \pm \gamma(\mathrm{x}),\end{aligned}$$

$$\mathfrak{L}^{\varepsilon,\delta} = \left(-\|\gamma\| \pm \gamma(\mathrm{x})\right) + \beta(\mathrm{x})\boldsymbol{C}_1 \max\left(\|\omega\|,|\upsilon|\right) < 0 \quad (\text{since } \beta(\mathrm{x})\theta^{-1} \leqslant -1).$$

From the above inequalities, Lemma 2.2 proves the required estimate (2.1). □

2.1. **Description of the scheme.** Since $\delta = O(\varepsilon)$, extending the argument containing the delay as a Taylor's expansion can lead to a bad approximation. To resolve this situation, we model an interesting mesh after discretion such that the delay term lies on mesh levels. Let $[0,1]$ be divided to N equal subintervals by $h = \delta/l$, where $l = rs$, s is the mantissa of δ, and $r \in \mathbb{Z}^+$. Then

$$\varepsilon u''(\mathrm{x}_i) = \gamma(\mathrm{x}_i) - \alpha(\mathrm{x}_i)u'(\mathrm{x}_{i-l}) - \beta(\mathrm{x}_i)u(\mathrm{x}_i), \tag{2.2}$$

$$u_i = \omega(\mathrm{x}_i) = \omega_i\;, \qquad u(1) = \upsilon. \tag{2.3}$$

Let $\overline{\Omega} = [0,1]$ be $\mathrm{x}_i = ih$, $i = 0(1)N-1$. A function $\mathfrak{S}(\mathrm{x},\varrho)$ of $C^2(\overline{\Omega})$ interpolates $u(\mathrm{x}_i)$, which leads to the cubic spline as $\varrho \to 0$, called as parametric cubic spline function. The relation $\mathfrak{S}(\mathrm{x},\varrho) = \mathfrak{S}(\mathrm{x})$ satisfying in $[\mathrm{x}_i, \mathrm{x}_{i+1}]$,

$$\begin{aligned}\mathfrak{S}''(\mathrm{x}) + \varrho\mathfrak{S}(\mathrm{x}) = \frac{(\mathrm{x}_{i+1}-\mathrm{x})}{h}\left[\mathfrak{S}''(\mathrm{x}_i) + \varrho\mathfrak{S}(\mathrm{x}_i)\right]\\ + \frac{(\mathrm{x}-\mathrm{x}_i)}{h}\left[\mathfrak{S}''(\mathrm{x}_{i+1}) + \varrho\mathfrak{S}(\mathrm{x}_{i+1})\right],\end{aligned}$$

here $\mathfrak{S}(\mathrm{x}_i) = u_i$ and $\varrho > 0$ is called the spline in compression. Following [1], we obtain

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = \lambda_1 M_{i-1} + 2\lambda_2 M_i + \lambda_1 M_{i+1}, \quad i = 1(1)N-1, \tag{2.4}$$

where

$$\lambda_1 = \left(\lambda^{-1}\csc\lambda - \lambda^{-2}\right), \lambda_2 = \left(\lambda^{-2} - \lambda^{-1}\cot\lambda\right),\ \lambda = h\sqrt{\varrho},\ \text{and } M_i = u''(\mathrm{x}_i).$$

We have $u'_{i-1} \approx \frac{-u_{i+1}+4u_i-3u_{i-1}}{2h}$, $u'_{i+1} \approx \frac{3u_{i+1}-4u_i+u_{i-1}}{2h}$, and $u'_i \approx \frac{u_{i+1}-u_{i-1}}{2h}$. Substituting $\varepsilon M_j = \gamma(\mathrm{x}_j) - \alpha(\mathrm{x}_j)u'(\mathrm{x}_{j-l}) - \beta(\mathrm{x}_j)u(\mathrm{x}_j), j = i, i\pm 1$, in (2.4) and using the above first order approximations $u'_i, u'_{i\pm 1}$ with (2.3), we obtain

$$\psi_i^1 u_{i+1} + \psi_i^2 u_i + \psi_i^3 u_{i-1} = \varsigma_i - \psi_i^4 \omega_{i-l+1} - \psi_i^5 \omega_{i-l} - \psi_i^6 \omega_{i-l-1}$$

$$\text{for } 1 \leqslant i \leqslant l-1, \tag{2.5}$$

$$\psi_i^1 u_{i+1} + \psi_i^2 u_i + \psi_i^3 u_{i-1} + \psi_i^4 u_{i-l+1} = \varsigma_i - \psi_i^5 \omega_{i-l} - \psi_i^6 \omega_{i-l-1} \quad \text{for } i = l, \tag{2.6}$$

$$\psi_i^1 u_{i+1} + \psi_i^2 u_i + \psi_i^3 u_{i-1} + \psi_i^4 u_{i-l+1} + \psi_i^5 u_{i-l} = \varsigma_i - \psi_i^6 \omega_{i-l-1} \quad \text{for } i = l+1, \tag{2.7}$$

$$\psi_i^1 u_{i+1} + \psi_i^2 u_i + \psi_i^3 u_{i-1} + \psi_i^4 u_{i-l+1} + \psi_i^5 u_{i-l} + \psi_i^6 u_{i-l-1} = \varsigma_i \quad \text{for } l+2 \leqslant i \leqslant N-1, \tag{2.8}$$

where

$$\psi_i^1 = \varepsilon + \lambda_1 h^2 \beta_{i+1}, \qquad \psi_i^4 = \tfrac{3\lambda_1}{2} h\alpha_{i+1} + \lambda_2 h\alpha_i - \tfrac{\lambda_1}{2} h\alpha_{i-1},$$

$$\psi_i^2 = -2\varepsilon + 2\lambda_2 h^2 \beta_i, \qquad \psi_i^5 = -2\lambda_1 h\alpha_{i+1} + 2\lambda_1 h\alpha_{i-1},$$

$$\psi_i^3 = \varepsilon + \lambda_1 h^2 \beta_{i-1}, \qquad \psi_i^6 = \tfrac{\lambda_1}{2} h\alpha_{i+1} - \lambda_2 h\alpha_i - \tfrac{3\lambda_1}{2} h\alpha_{i-1},$$

$$\varsigma_i = h^2 \left(\lambda_1 \gamma_{i+1} + 2\lambda_2 \gamma_i + \lambda_1 \gamma_{i-1}\right).$$

The above system (2.5)–(2.8) with (2.3) can be solved by using the Gauss elimination process.

3. Truncation error

This section presents the truncation error ξ_i for (2.6)–(2.8) given by

$$\xi_i(u) = \xi_{1,i}(u) + \xi_{2,i}(u) + \xi_{3,i}(u) + \xi_{4,i}(u), \tag{3.1}$$

where

$$\xi_{1,i}(u) = \psi_i^1 u_{i+1} + \psi_i^2 u_i + \psi_i^3 u_{i-1} - \varsigma_i \quad , \text{ for } 1 \leqslant i \leqslant l-1, \tag{3.2}$$

$$\xi_{2,i}(u) = \psi_i^1 u_{i+1} + \psi_i^2 u_i + \psi_i^3 u_{i-1} + \psi_i^4 u_{i-l+1} - \varsigma_i, \quad \text{for } i = l, \tag{3.3}$$

$$\xi_{3,i}(u) = \psi_i^1 u_{i+1} + \psi_i^2 u_i + \psi_i^3 u_{i-1} + \psi_i^4 u_{i-l+1} + \psi_i^5 u_{i-l} - \varsigma_i, \quad \text{for } i = l+1, \tag{3.4}$$

$$\xi_{4,i}(u) = \psi_i^1 u_{i+1} + \psi_i^2 u_i + \psi_i^3 u_{i-1} + \psi_i^4 u_{i-l+1} + \psi_i^5 u_{i-l} + \psi_i^6 u_{i-l-1} - \varsigma_i, \quad \text{for } l+2 \leqslant i \leqslant N-1. \tag{3.5}$$

Using (2.2) in (3.2), we have

$$\xi_{1,i}(u) = \left(\psi_i^1 u_{i+1} + \psi_i^2 u_i + \psi_i^3 u_{i-1}\right) - h^2 \lambda_1 \left(\varepsilon u''_{i+1} + \alpha_{i+1} u'_{i-l+1} + \beta_{i+1} u_{i+1}\right) + 2\lambda_2 \left(\varepsilon u''_i + \alpha_i u'_{i-l} + \beta_i u_i\right) + \lambda_1 \left(\varepsilon u''_{i-1} + \alpha_{i-1} u'_{i-l-1} + \beta_{i-1} u_{i-1}\right). \tag{3.6}$$

Using the Taylor series expansion in (3.6) and after some simplifications with $\lambda_1 + \lambda_2 = \frac{1}{2}$, we obtain

$$\xi_{1,i}(u) \leqslant h^2 \left[\|\alpha\| \max_{\mathrm{x}_{i-m} \leqslant \mathrm{x} \leqslant \mathrm{x}_{i-m+1}} \left|u'(\mathrm{x})\right| + h^2 \varepsilon \left(\frac{1}{12} - \lambda_1\right) \max_{\mathrm{x}_{i-1} \leqslant \mathrm{x} \leqslant \mathrm{x}_{i+1}} \left|u^{(iv)}(\mathrm{x})\right| \right]. \tag{3.7}$$

In a similar way, by using (3.3)–(3.5) we get

$$\xi_{2,i}(u) \leqslant h \left[\left\{ \frac{\|\alpha\|}{2} + 2h \left\|\alpha'\right\| \right\} \max_{\mathrm{x}_{i-m} \leqslant \mathrm{x} \leqslant \mathrm{x}_{i-m+1}} |u(\mathrm{x})| \right. \tag{3.8}$$

$$+h^3\varepsilon\left(\frac{1}{12}-\lambda_1\right)\max_{\mathrm{x}_{i-1}\leqslant \mathrm{x}\leqslant \mathrm{x}_{i+1}}\left|u^{(iv)}(\mathrm{x})\right|\Bigg],$$

$$\xi_{3,i}(u)\leqslant h\left[\left\{\frac{\|\alpha\|}{2}-2\lambda_1 h\|\alpha'\|\right\}\max_{\mathrm{x}_{i-m}\leqslant \mathrm{x}\leqslant \mathrm{x}_{i-m+1}}|u(\mathrm{x})| \right. \tag{3.9}$$

$$\left. +h^3\varepsilon\left(\frac{1}{12}-\lambda_1\right)\max_{\mathrm{x}_{i-1}\leqslant \mathrm{x}\leqslant \mathrm{x}_{i+1}}\left|u^{(iv)}(\mathrm{x})\right|\right],$$

and

$$\xi_{4,i}(u)\leqslant h^3\left[\left\{\frac{\|\alpha\|}{6}-\lambda_1 h\|\alpha\|\right\}\max_{\mathrm{x}_{i-m}\leqslant \mathrm{x}\leqslant \mathrm{x}_{i-m+1}}\left|u^{(iii)}(\mathrm{x})\right| \right. \tag{3.10}$$

$$\left. +h\varepsilon\left(\frac{1}{12}-\lambda_1\right)\max_{\mathrm{x}_{i-1}\leqslant \mathrm{x}\leqslant \mathrm{x}_{i+1}}\left|u^{(iv)}(\mathrm{x})\right|\right].$$

Theorem 3.1. *Let $U(x)$ be the numerical solution to $u(x)$ of (1.1) acquired by the proposed scheme. The error estimate is given by*

$$\|e_i\| = e_i = U(x_i) - u(x_i) \leqslant \Theta^{-1}\|\xi_i\|,$$

where

$$\|\xi_i\|\leqslant h\Bigg[(\|\alpha_i\|+2h\|\alpha_i'\|(1-\lambda_1))\max_{x_{i-m}\leqslant x\leqslant x_{i-m+1}}|u(x)|$$

$$+h^3\varepsilon\left(\frac{1}{12}-\lambda_1\right)\max_{x_{i-1}\leqslant x\leqslant x_{i+1}}\left|u^{(iv)}(x)\right|\Bigg].$$

Proof. Using equations (3.7)–(3.10) in (3.1), and also Lemma 2.2, one can prove the above estimate. □

4. Computational experiments

This section presents the findings of the computational experiment to validate with the theoretical result. We use the double mesh principle [5] to calculate the maximum absolute error (MAE) and order convergence given by

$$Error^N = \max_{0\leqslant i\leqslant N}\left|u_i^N - u_{2i}^{2N}\right|, \quad Order^N = \log_2\left|\frac{Error^N}{Error^{2N}}\right|.$$

Example 4.1. Consider

$$\varepsilon u''(\mathrm{x}) - (1+\mathrm{x})u'(\mathrm{x}-\delta) - e^{-\mathrm{x}}u(\mathrm{x}) = 1,$$

with

$$u(\mathrm{x}) = 1 \quad, -\delta \leqslant \mathrm{x} \leqslant 0\,, \;\; u(1) = 1.$$

Table 1 represents the MAE for different values of λ_1, λ_2, δ, and N. It is observed that the error decreases as increasing the mesh points. Figures 1–2 reflect the behavior of the boundary layer for various delay parameter values. It can be observed in Figure 1 that the layer behavior is maintained when $\delta = o(\varepsilon)$ and also the thickness of the boundary layer behavior of the solution increases as delay increases. Furthermore, as we increase the delay parameter, the amplitude of the oscillations increases at the right end boundary, as shown in Figure 2.

Example 4.2. Consider

$$\varepsilon u''(\mathrm{x}) - e^{\mathrm{x}} u'(\mathrm{x}-\delta) - u(\mathrm{x}) = 0,$$

with

$$u(\mathrm{x}) = 1 \quad , \; -\delta \leqslant \mathrm{x} \leqslant 0 \; , \quad u(1) = 1.$$

Table 2 shows the comparison between the MAE obtained by the method proposed and the method in [11]. The errors of the proposed method can be observed to be more accurate than the method in [11]. In Figure 3, the layer behavior of the solution is preserved for $\delta = \varepsilon$ and $\delta = 2\varepsilon$, but oscillations have built up within the layer when $\delta = 3\varepsilon$.

Example 4.3. Consider

$$\varepsilon u''(\mathrm{x}) - 0.25 u'(\mathrm{x}-\delta) - u(\mathrm{x}) = 0,$$

with

$$u(\mathrm{x}) = 1 \quad , \; -\delta \leqslant \mathrm{x} \leqslant 0 \; , \quad u(1) = 0.$$

The calculated MAE for different values ε and N is presented in Table 3. It is noted in the table that the presented method gives more precise results than the methods in [10, 11]. It is evident from the numerical solution in Figure 4 that the layer behavior is destroyed and also oscillations across the whole interval.

Example 4.4. Consider

$$\varepsilon u''(\mathrm{x}) - u'(\mathrm{x}-\delta) - u(\mathrm{x}) = 0,$$

with

$$u(\mathrm{x}) = 1 \quad , \; -\delta \leqslant \mathrm{x} \leqslant 0 \; , \quad u(1) = -1.$$

The estimated MAE and orders are shown in Table 4. In Figure 5, the layer behavior of the solution is maintained throughout the interval when the delay is smaller as well as larger than the perturbation parameter.

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TABLE 1. MAE and computed order with $\varepsilon = 2^{-5}$ for Example 4.1.

	$(\lambda_1, \lambda_2) = (1/6, 1/3)$			$(\lambda_1, \lambda_2) = (1/12, 5/12)$		
$\delta \setminus N$	256	512	1024	256	512	1024
0.5ε	3.3411e-04	8.3467e-05	2.0863e-05	3.2362e-04	8.0849e-05	2.0209e-05
	2.0010	2.0003	1.9999	2.0010	2.0002	1.9999
0.8ε	1.3144e-02	4.0085e-03	7.1512e-04	1.3135e-02	4.9702e-03	8.1517e-0
	1.7133	2.4868	1.6385	1.4020	2.6081	1.3618
3ε	6.5180e-04	1.6446e-04	4.1245e-05	6.4755e-04	1.6375e-04	4.1032e-05
	1.9867	1.9954	1.9992	1.9835	1.9967	1.9992
6ε	4.9361e-03	1.2369e-03	3.0930e-04	5.3391e-03	1.3363e-03	3.3416e-04
	1.9966	1.9997	1.9999	1.9984	1.9996	1.9999
9ε	3.0907e-03	7.7267e-04	1.9317e-04	3.3057e-03	8.2647e-04	2.0664e-04
	2.0000	2.0000	2.0000	1.9999	1.9998	2.0000

TABLE 2. Comparison of MAE with $\delta = 0.03$ for Example 4.2.

$N \setminus \varepsilon$		2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
100	MAE^1	2.84e-04	7.09e-04	1.43e-03	2.71e-03	7.52e-03	3.89e-02
	MAE^2	1.84e-05	5.93e-05	1.50e-04	3.16e-04	5.75e-04	9.18e-04
	MAE^3	1.08e-05	4.32e-05	1.24e-04	2.78e-04	5.22e-04	8.47e-04
200	MAE^1	1.53e-04	3.80e-04	7.70e-04	1.47e-03	2.71e-03	8.01e-03
	MAE^2	4.60e-06	1.48e-05	3.75e-05	7.93e-05	1.44e-04	2.29e-04
	MAE^3	2.71e-06	1.08e-05	3.09e-05	6.96e-05	1.30e-04	2.12e-04
400	MAE^1	7.93e-05	1.97e-04	3.99e-04	7.64e-04	1.41e-03	2.54e-03
	MAE^2	1.15e-06	3.70e-06	9.38e-06	1.98e-05	3.59e-05	5.75e-05
	MAE^3	6.77e-07	2.70e-06	7.72e-06	1.74e-05	3.26e-05	5.31e-05

[1] Method in [11]
[2] Proposed method for $\lambda_1 = \frac{1}{6}$, $\lambda_2 = \frac{1}{3}$
[3] Proposed method for $\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{5}{12}$.

TABLE 3. Comparison of MAE with $\delta = 0.03$ for Example 4.3.

$N \setminus \varepsilon$		2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
100	MAE^1	2.08e-04	6.12e-04	1.63e-03	4.25e-03	1.17e-02	3.37e-02	9.77e-02
	MAE^2	1.76e-04	4.07e-04	9.49e-04	2.16e-03	4.67e-03	9.30e-03	1.69e-02
	MAE^3	2.37e-06	7.84e-06	2.39e-05	7.27e-05	2.52e-04	1.09e-03	5.99e-03
	MAE^4	7.26e-06	2.95e-06	1.14e-05	4.35e-05	1.87e-04	1.01e-03	6.53e-03
200	MAE^1	1.04e-04	3.07e-04	8.26e-04	2.18e-03	6.16e-03	1.88e-02	5.98e-02
	MAE^2	9.30e-05	2.15e-04	5.03e-04	1.14e-03	2.46e-03	4.89e-03	8.75e-03
	MAE^3	5.93e-07	1.96e-06	5.98e-06	1.82e-05	6.29e-05	2.74e-04	1.52e-03
	MAE^4	1.82e-07	7.38e-07	2.85e-06	1.09e-05	4.67e-05	2.52e-04	1.65e-03
400	MAE^1	5.20e-05	1.54e-04	4.15e-04	1.10e-03	3.12e-03	9.95e-03	3.32e-02
	MAE^2	4.80e-05	1.11e-04	2.59e-04	5.89e-04	1.26e-03	2.50e-03	4.44e-03
	MAE^3	1.48e-07	4.90e-07	1.49e-06	4.54e-06	1.57e-05	6.85e-05	3.80e-04
	MAE^4	4.54e-08	1.85e-07	7.13e-07	2.72e-06	1.17e-05	6.30e-05	4.13e-04

[1] Method in [10]
[2] Method in [11]
[3] Proposed method for $\lambda_1 = \frac{1}{6}$, $\lambda_2 = \frac{1}{3}$
[4] Proposed method for $\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{5}{12}$.

TABLE 4. MAE and computed order with $\delta = 1.5\varepsilon,\ \varepsilon = 2^{-7}$ for Example 4.4.

(λ_1, λ_2)	$N = 256$	$N = 512$	$N = 1024$
$(1/6, 1/3)$	1.12e-03	2.76e-04	6.90e-05
	2.0208	2.0000	2.0042
$(1/10, 2/5)$	1.08e-03	2.69e-04	6.73e-05
	2.0054	1.9989	2.0021
$(1/12, 5/12)$	1.07e-03	2.67e-04	6.68e-05
	2.0027	1.9989	2.0000
$(1/14, 3/7)$	1.06e-03	2.66e-04	6.65e-05
	1.9946	2.0000	2.0022
$(1/18, 4/9)$	1.05e-03	2.64e-04	6.61e-05
	1.9918	1.9978	2.0022
$(1/24, 11/24)$	1.05e-03	2.63e-04	6.57e-05
	1.9973	2.0011	2.0022
$(1/30, 14/30)$	1.05e-03	2.62e-04	6.55e-05
	2.0028	2.0000	1.9630

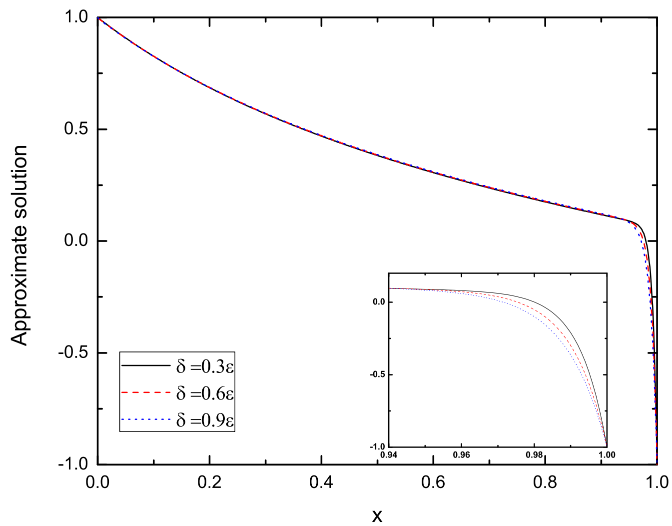

FIGURE 1. Solution profile for $\varepsilon = 0.01$ of Example 4.1.

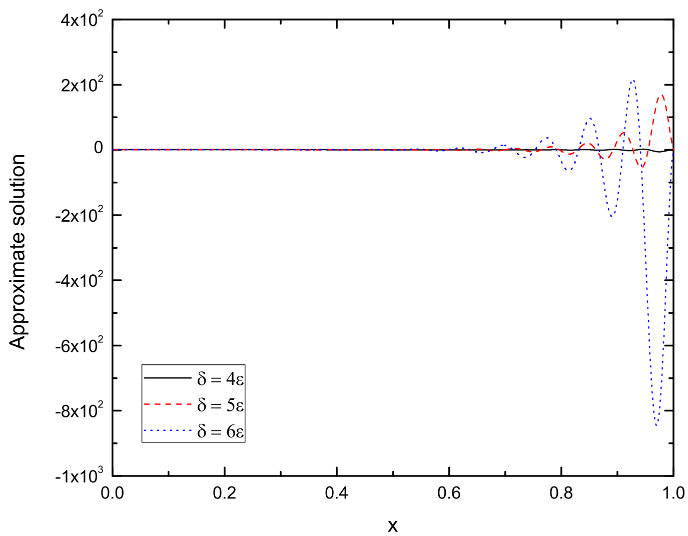

FIGURE 2. Solution profile for $\varepsilon = 0.01$ of Example 4.1.

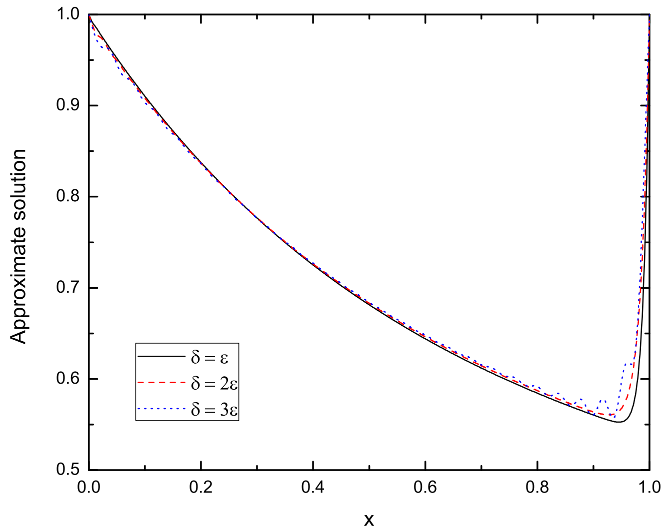

FIGURE 3. Solution for $\varepsilon = 0.01$ of Example 4.2.

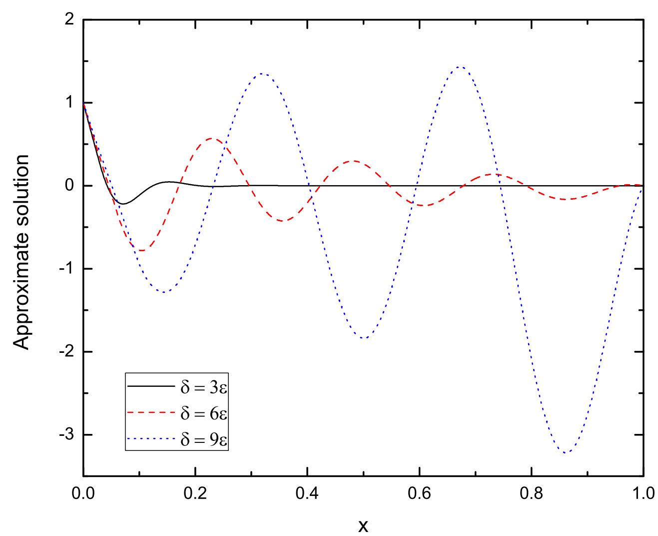

FIGURE 4. Solution profile for $\varepsilon = 0.01$ of Example 4.3.

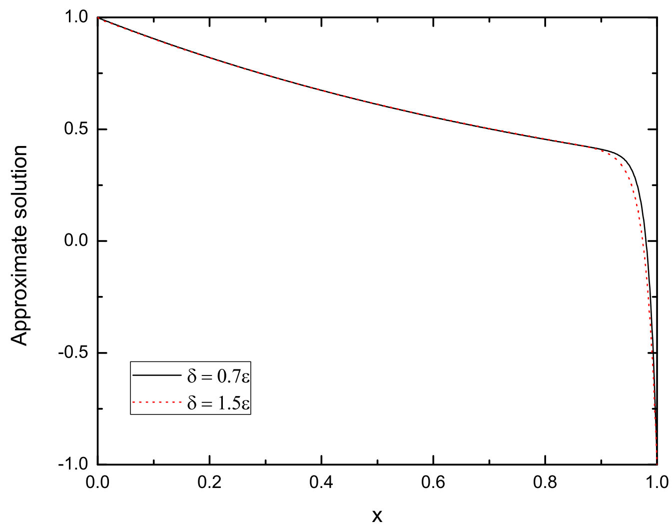

FIGURE 5. Solution profile for $\varepsilon = 0.01$ of Example 4.4.

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