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TOPOLOGICAL CHARACTERIZATION OF CHAINABLE SETS AND CHAINABILITY VIA CONTINUOUS FUNCTIONS

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Abstract. In the last decade, the notions of function-f-ϵ-chainability, uniformly function-f-ϵ-chainability, function-f-ϵ-chainable sets, and locally function-f-chainable sets were studied in some papers. We show that the notions of function-f-ϵ-chainability and uniformly function-f-ϵ-chainability are equivalent to the notion of nonultrapseudocompactness in topological spaces. Also, all of these are equivalent to the condition that each pair of nonempty subsets (resp., subsets with nonempty interiors) is function-f-ϵ-chainable (resp., locally function-f-chainable). Further, we provide a criterion for connectedness with covers. In the paper [Indian J. Pure Appl. Math. 33 (2002), no. 6, 933–940], the chainability of a pair of subsets in a metric space has been defined wrongly, and consequently Theorems 1 and 5 are not valid. We rectify their definition appropriately and consequently, we give appropriate results and counterexamples.

1. Introduction and preliminaries

Throughout this paper, a map means a continuous function and the word space means topological space. Cantor [3, p. 575] introduced the notion of connectedness of some subsets of Euclidean spaces $\mathbb{R}^n$ in 1883 in the following way: A set is connected, if for any elements x and y of the set and any $\epsilon > 0$, a finite set of points $x = x_0, x_1, \ldots, x_n = y$ can be found with the property that $d(x_i, x_{i+1}) < \epsilon$ for every $0 \leq i \leq n-1$. A metric space (X, d) with the above property is called *chainable* and the collection $\{x_0, x_1, \ldots, x_n\}$ is an ϵ-*chain of length* n from x to

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y, whereas (X, d) is said to be ϵ-*chainable* if any pair of elements of X can be connected by a finite-length ϵ-chain.

From the beginning of 21st-century, three topics of chainability: finitely chainable metric spaces, chainable subsets of metric spaces, and chainability through functions, have been studied seriously. The notion of finitely chainable metric space was introduced by Atsuji [2] as follows.

Definition 1.1. Let (X, d) be a metric space. If for any $\epsilon > 0$, there exist a finite collection of points $C \subset X$ and $n \in \mathbb{N}$ such that each element of X can be connected with some element of C by an n-length ϵ-chain, then (X, d) is called *finitely chainable*.

Following [2, Theorem 2], one can see that there exists an unbounded real-valued uniformly continuous function on a metric space X if and only if X is not finitely chainable. Garrido and Meroño [6] defined cofinally Bourbaki–Cauchy and Bourbaki–Cauchy sequences in metric spaces and characterized finitely chainable metric spaces (under the name of Bourbaki-bounded metric spaces) in terms of these sequences. Recently, Kundu, Aggarwal, and Hazra [9] collected equivalent conditions for finite chainability in metric spaces.

The concept of ϵ-chain between two points of a metric space (X, d) extended to ϵ-chain between two subset of X as follows.

Definition 1.2 (see [12])**.** Let (X, d) be a metric space, let ϵ be a positive number, and let $A, B \subset X$ be given. An ϵ-*chain of length* n from A to B is a finite sequence $A_0, A_1, \ldots, A_n$ of subsets of X such that $A = A_0$, $A_n = B$ and

$$A_i \subset V_\epsilon(A_{i+1}), \qquad A_{i+1} \subset V_\epsilon(A_i), \qquad \text{for any } \ 0 \leq i \leq n-1,$$

where $V_\epsilon(A) = \bigcup_{x \in A} B(x, \epsilon)$ and $B(x, \epsilon) = \{y \in X : \mathrm{d}(x, y) < \epsilon\}$.

Using the above definition, Shrivastava and Agrawal [12] tried to define chainability of a pair $\langle A, B \rangle$ of subsets of a metric space X, as this new concept characterizes the chainable metric spaces. In Section 2, a counterexample is presented, which shows that their attempt has failed. In fact, this example illustrates that some of the main results of [12] including Theorems 1 and 5 are not valid. They considered the following definition.

($*$) The pair $\langle A, B \rangle$ is ϵ-chainable if there is an ϵ-chain between A and B, whereas $\langle A, B \rangle$ is chainable if it is ϵ-chainable for each $\epsilon > 0$.

Moreover, in Section 2, we present an equivalent condition for connectedness with covers. For every open cover $\mathcal{V}$ of space $(X, \mathcal{T})$ and $H \subset X$, the *star of* H *with respect to* $\mathcal{V}$, denoted by $\mathrm{Star}(H, \mathcal{V})$, is defined as follows:

$$\mathrm{Star}(H, \mathcal{V}) = \bigcup \{V \in \mathcal{V} : V \cap H \neq \emptyset\}.$$

We will also denote

$$\begin{aligned} H^0 &= H, \\ H^n &= \bigcup \{V \in \mathcal{V} : V \cap H^{n-1} \neq \emptyset\} = \mathrm{Star}(H^{n-1}, \mathcal{V}), \quad \text{for } n \geq 1, \\ H^\infty &= \bigcup_{n \in \omega} H^n. \end{aligned} \tag{1.1}$$

Recall that a topological space X is said to be *ultrapseudocompact* (UPC for short) if every real-valued map on X is constant. The class of UPC spaces was studied in [10]. More generally, spaces on which every map into a given space R is constant, were examined in [7]. Every UPC space X is connected. In fact, every real-valued map on X has the intermediate value property. Note that a real-valued function f on a space X has the *intermediate value property* if the following condition holds:

($\Diamond$) If $y_1, y_2 \in f(X)$ and $y_1 < y_2$, then the entire interval $y_1 \leq y \leq y_2$ is contained in the set $f(X)$.

In the third section, it is shown that the notions of function-f-ϵ-chainability and uniformly function-f-ϵ-chainability introduced in [8] are equivalent to non-UPCness in topological spaces. We recall the following definition.

Definition 1.3 (see [8])**.** A space X is called

- *function-f-ϵ-chainable* if for $\epsilon > 0$, there exists a nonconstant map $f : X \to [0, \infty)$ such that for every pair of elements x and y of X, a finite set of points $x = x_0, x_1, \ldots, x_n = y$ can be found such that
$$|f(x_{i+1}) - f(x_i)| < \epsilon, \qquad \text{for any } 0 \leq i \leq n-1;$$
- *uniformly function-f-ϵ-chainable* if for $\epsilon > 0$, there exist $l_\epsilon \in \mathbb{N}$ and a nonconstant map $f : X \to [0, \infty)$ with the property that for each two elements x and y in X, there exists a finite set $\{x = x_0, x_1, \ldots, x_n = y\}$ of points of X such that $n \leq l_\epsilon$ and $|f(x_{i+1}) - f(x_i)| < \epsilon$ for every $0 \leq i \leq n-1$.

The last result of Section 3, illustrates the equality of the concepts of function-f-ϵ-chainable sets and locally function-f-chainable sets with non-UPCness.

Definition 1.4. Let X be a topological space and let $A, B \subset X$. The pair $\langle A, B\rangle$ is called

- (see [13]) *function-f-ϵ-chainable* if for $\epsilon > 0$, there exists a nonconstant map $f : X \to [0, \infty)$ such that there is a finite sequence $A = A_0, A_1, \ldots, A_n = B$ of subsets of X with
$$A_i \subset V_{f\epsilon}(A_{i+1}), \qquad A_{i+1} \subset V_{f\epsilon}(A_i), \qquad \text{for any } 0 \leq i \leq n-1,$$
where $V_{f\epsilon}(A) = f^{-1}\big(V_\epsilon\big(f(A)\big)\big)$;
- (see [4]) *locally function-f-chainable* at point $(a, b) \in A \times B$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that the pair
$$\langle V_{f\delta}(\{a\}) \cap \operatorname{int}(A), V_{f\delta}(\{b\}) \cap \operatorname{int}(B)\rangle$$
is function-f-ϵ-chainable and
$$V_{f\delta}(\{a\}) \cap \operatorname{int}(A) \subset V_{f\epsilon}(\{a\}), \qquad V_{f\delta}(\{b\}) \cap \operatorname{int}(B) \subset V_{f\epsilon}(\{b\}),$$
where the sets $V_{f\delta}(\{a\}) \cap \operatorname{int}(A)$ and $V_{f\delta}(\{b\}) \cap \operatorname{int}(B)$ are nonempty. The pair $\langle A, B\rangle$ is *locally function-f-chainable* if it is locally function-f-chainable at each point of $A \times B$.

2. Chainable sets in metric spaces

We start with a lemma, which is an immediate consequence of (1.1).

Lemma 2.1. *Let $(X, \mathcal{T})$ be a space and let $A, B \subset X$. Then the following statements hold:*

(1) *For any $n \in \omega$, we have*
  (i) $A^n \subset A^{n+1}$;
  (ii) $(A \cup B)^n = A^n \cup B^n$;
  (iii) $(A \cap B)^n \subset A^n \cap B^n$;
(2) *If $A \subset B$, then $A^n \subset B^n$ for any $n \in \omega$;*
(3) *If $W \in \mathcal{T}$, then $W^{n^\infty} = W^\infty$ for any $n \in \omega$;*
(4) *If $U, W \in \mathcal{T}$ and $U \cap W \neq \emptyset$, then $U^\infty = W^\infty$;*
(5) *If $U \in \mathcal{T}$, then for any $W \in \mathcal{T}$ satisfying $W \cap U^n \neq \emptyset$ for some $n \in \omega$, we have $U^\infty = W^\infty$.*

Proof. Only (5) is nontrivial. For the case $n = 0$, the result is clear by 4. For $m = n + 1$, let $W \in \mathcal{T}$ satisfying $W \cap U^m \neq \emptyset$. There exists $V \in \mathcal{T}$ such that $W \cap V \neq \emptyset$ and $V \cap U^n \neq \emptyset$. Hence $W^\infty = V^\infty$ and $V^\infty = U^\infty$. □

For a metric space (X, d), [12, Theorem 1] says that the pair $\langle A, B \rangle$ of subsets of X is ϵ-chainable if and only if there exists an ϵ-chain from every point of A to some point of B and vice versa. The following example shows the sufficient part does not hold in general.

Example 2.2. Consider subsets $A = (-\infty, 0] \cap \mathbb{Q}$ and $B = [1, \infty) \cap \mathbb{Q}$ of the space $\mathbb{Q}$ with the topology induced from $\mathbb{R}$. For any positive number ϵ, there exists an ϵ-chain from every member of A to some member of B and vice versa. Now fix $\epsilon > 0$. There is no finite-length ϵ-chain from A to B. Therefore $\langle A, B \rangle$ is not ϵ-chainable.

In this perspective, we suggest to redefine $(*)$ independent from Definition 1.2. Before proceeding, we set a notation. For a subset A of a metric space (X, d), the set $\bigcup_{n \in \omega} A_\epsilon^n$ will be designated by A_ϵ^∞, where $A_\epsilon^0 = A$ and for every $n \geq 1$, we have

$$A_\epsilon^n = \bigcup \{B(x, \epsilon) : x \in X, B(x, \epsilon) \cap A_\epsilon^{n-1} \neq \emptyset\}.$$

Definition 2.3. Let (X, d) be a metric space, let $A, B \subset X$, and let ϵ be a positive number. The pair $\langle A, B \rangle$ is said to be *ϵ-chainable* if $A_\epsilon^\infty = B_\epsilon^\infty$. Further, if for each $\epsilon > 0$, $\langle A, B \rangle$ is ϵ-chainable, then $\langle A, B \rangle$ is called *chainable*.

The following proposition is required to prove the next theorem.

Proposition 2.4. *The pair $\langle A, B \rangle$ of subsets of a metric space (X, d) is ϵ-chainable if and only if there exists an ϵ-chain from every point of A to some point of B and vice versa.*

Proof. Suppose $A_\epsilon^\infty = B_\epsilon^\infty$ for $\epsilon > 0$. Then for every $x \in A$, there is $n \in \omega$ such that $x \in B_\epsilon^n$. Hence there is a finite set $\{x = x_0, x_1, \dots, x_{2n}\}$ of elements of metric space X such that $x_{2n} \in B$ and the inequality $\mathrm{d}(x_i, x_{i+1}) < \epsilon$ holds for any $0 \leq i \leq 2n-1$. Similarly, there exists an ϵ-chain from every point of B to some point of A.

Vice versa, for every $x \in A$, there is $n_x \in \omega$ such that $x \in B_\epsilon^{n_x} \subset B_\epsilon^\infty$. Therefore $A_\epsilon^0 \subset B_\epsilon^\infty$ and $A_\epsilon^\infty \subset (B_\epsilon^\infty)^\infty = B_\epsilon^\infty$. Likewise, $B_\epsilon^\infty \subset A_\epsilon^\infty$, and the proof is complete. □

Now [12, Theorem 5] is immediately obtained from Definition 2.3 and Proposition 2.4.

Theorem 2.5 (see [12]). *A metric space (X, d) is ϵ-chainable if and only if $\langle A, B\rangle$ is ϵ-chainable for every nonempty subsets A and B of X.*

Further, we have the following proposition.

Proposition 2.6. *A metric space (X, d) is chainable if and only if for every positive number ϵ, there exists an element x of X with $B(x,\epsilon)^\infty = X$.*

Proof. Consider a chainable metric space (X, d), and fix $x \in X$. Now let $\epsilon > 0$ be given. For every $y \in X$, there is a finite set $\{x = y_0, y_1, \ldots, y_n = y\}$ of elements of X in such a way that $\mathrm{d}(y_i, y_{i+1}) < \epsilon$ for $0 \leq i \leq n-1$. Hence $B(x,\epsilon) \cap B(y_1,\epsilon) \neq \emptyset$ implies that $B(y_1,\epsilon) \subset B(x,\epsilon)^1$. Similarly, $B(y_2,\epsilon) \subset B(x,\epsilon)^2$. Therefore

$$y \in B(y_n,\epsilon) \subset B(x,\epsilon)^n \subset B(x,\epsilon)^\infty,$$

proving that $X = B(x,\epsilon)^\infty$.

Conversely, take $y, z \in X$ and $\epsilon > 0$. Then there exist a point x of the space X and $n_y, n_z \in \omega$ such that $y \in B(x,\epsilon)^{n_y}$ and $z \in B(x,\epsilon)^{n_z}$. Therefore the collection $\{y = y_0, y_1, \ldots, y_{2n_y+1} = x = z_0, z_1, \ldots, z_{2n_z+1} = z\}$ is an ϵ-chain from y to z, which implies that (X, d) is chainable. □

A metric space (X, d) is called *uniformly connected*, if X cannot be partitioned into two subsets A and B of X with $\inf\{\mathrm{d}(x,y) : x \in A, y \in B\} > 0$. Note that (X, d) is uniformly connected if and only if it is chainable; see [5].

The next statement obviously follows from the results we have already obtained above.

Corollary 2.7. *In a metric space X, the following conditions are equivalent:*

(1) *X is chainable;*
(2) *X is uniformly connected;*
(3) *For every $\epsilon > 0$, there is a point $x \in X$ such that $B(x,\epsilon)^\infty = X$;*
(4) *For every $\epsilon > 0$, the equality $A_\epsilon^\infty = B_\epsilon^\infty$ holds for every nonempty subsets A and B of X.*

Now we define a relation $\sim$ on a space $(X, \mathcal{T})$ by $x \sim y$ if and only if $x, y \in U^\infty$ for some $U \in \mathcal{T}$. We have to make sure that $\sim$ is an equivalence relation.

Proposition 2.8. *The relation $\sim$ is an equivalence relation on X.*

Proof. Let $U, V \in \mathcal{T}$ and let $x \in U^\infty \cap V^\infty$. There exist $n, m \in \omega$ such that $x \in V^n \cap U^m$. Therefore there exist $O, W \in \mathcal{T}$ such that

$$x \in O \cap W, \quad O \cap V^{n-1} \neq \emptyset, \quad W \cap U^{m-1} \neq \emptyset.$$

Consequently, part 5 of Lemma 2.1 implies that $U^\infty = V^\infty$. □

The quotient set of X under the relation $\sim$ will be denoted by $\mathrm{C}_{\mathcal{T}}(X)$. Recall that a *chain* in a space $(X, \mathcal{T})$ is a finite family of sets $\{C_1, C_2, \ldots, C_n\}$ in $\mathcal{T}$ such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$ for all $1 \leq i, j \leq n$, and is also called a *chain from x to y* in $\mathcal{T}$ for $x \in C_1$ and $y \in C_n$. The following theorem can be used as an equivalent characterization of connected topological spaces.

Theorem 2.9 (Connectivity covering criterion)**.** *Let $(X, \mathcal{T})$ be a topological space. The following statements are equivalent:*

1. *$(X, \mathcal{T})$ is connected;*
2. *For every subcover $\mathcal{V}$ of $\mathcal{T}$, there exists $V \in \mathcal{V}$ such that $V^\infty = X$;*
3. *$|\,\mathrm{C}_{\mathcal{V}}(X)| = 1$ for every subcover $\mathcal{V}$ of $\mathcal{T}$;*
4. *There exists a chain in $\mathcal{V}$ between any pair of points of X for every subcover $\mathcal{V}$ of $\mathcal{T}$.*

Proof. (1) ⇒ (2). Choose a subcover $\mathcal{V}$ of $\mathcal{T}$ such that $V^\infty \neq X$ for all $V \in \mathcal{V}$. For a fixed $x \in X$, there exists $V_x \in \mathcal{V}$ such that $x \in V_x$ and $V_x^\infty \neq X$. For any $y \in X \setminus V_x^\infty$, the set V_y^∞ satisfies $y \in V_y \in \mathcal{V}$ and $V_y^\infty \cap V_x^\infty = \emptyset$. Also X cannot be written as the union of two disjoint nonempty elements of $\mathcal{T}$; therefore $V_x^\infty \cup V_y^\infty \neq X$. Continue this process until $X \setminus (\bigcup_{j \in J \subseteq X} V_j^\infty) = \emptyset$. The sets $U = V_j^\infty$ and $V = \bigcup_{j \neq i \in J} V_i^\infty$ are disjoint nonempty elements of $\mathcal{T}$ with $X = U \cup V$, a contradiction.

The implications (2) ⇒ (3) and (4) ⇒ (1) are trivial. Let a subcover $\mathcal{V}$ of $\mathcal{T}$ be given. For every $x, y \in X$, there exists $V \in \mathcal{V}$ such that $x, y \in V^\infty$. Hence $y \in V^n$ and $x \in V^m$ for some $n, m \in \omega$. Fix any point z in V, there exists a finite sequence $U_1, U_2, \ldots, U_n$ of elements of $\mathcal{V}$ such that $y \in U_n$, $V \cap U_1 \neq \emptyset$, and $U_i \cap U_{i+1} \neq \emptyset$ for all $1 \leq i \leq n-1$. That is, $\{V = U_0, U_1, U_2, \ldots, U_n\}$ is a chain from z to y in $\mathcal{V}$. Similarly for x, there exists a chain $\{W_0, W_1, \ldots, W_m = V\}$ from x to z in $\mathcal{V}$, and (3) ⇒ (4) is proved. □

3. **Chainability of spaces through maps**

At the outset, we note that the use of f in Definitions 1.3 and 1.4 is not required. Although, we use these definitions to avoid ambiguity. The following theorem is the key to our results in this section.

Theorem 3.1. *Every non-UPC space is function-f-ϵ-chainable for all $\epsilon > 0$.*

Proof. Let X be any non-UPC space, and fix $\epsilon > 0$. There exists a nonconstant real-valued map on X, say ρ. Now define a map $\varphi^\epsilon : X \to (0, \epsilon)$ as follows:

$$\varphi^\epsilon(x) = \tfrac{\epsilon e^{\rho(x)}}{1 + e^{\rho(x)}}.$$

For every pair of elements x and y in X, the inequality $|\varphi^\epsilon(x) - \varphi^\epsilon(y)| < \epsilon$ implies that X is function-f-ϵ-chainable. □

Since for $\epsilon > 0$, every non-UPC space X is uniformly function-f-ϵ-chainable, where $l_\epsilon = 1$, the above statement implies the next one.

Corollary 3.2. *Let X be a space. The following statements are equivalent:*

(1) *X is non-UPC;*
(2) *X is uniformly function-f-ϵ-chainable for every $\epsilon > 0$.*

A space X is *function-f-chainable* if there exists a nonconstant map $f : X \to [0, \infty)$ such that for any two points x and y of X and any $\epsilon > 0$, there exists a finite sequence $x = x_0, x_1, \ldots, x_n = y$ of elements of X such that $|f(x_{i+1}) - f(x_i)| < \epsilon$ for $i = 0, 1, \ldots, n-1$ (see [8]). Every function-f-chainable space is non-UPC, but the converse need not hold, as the following example illustrates.

Example 3.3. The two-point discrete space $\{0, 1\}$, say $\mathbb{D}$, is non-UPC, which is not function-f-chainable. In fact, let $f : \mathbb{D} \to [0, \infty)$ be a nonconstant map. Now fix $\epsilon > 0$ with $\epsilon < |f(0) - f(1)|$. For $x_0 = 0$ and $x_n = 1$, there is no finite sequence $\{x_0, x_1, \ldots, x_n\}$ of elements of $\mathbb{D}$ such that $|f(x_{i+1}) - f(x_i)| < \epsilon$ for $i = 0, 1, \ldots, n-1$.

Proposition 3.4. *A space X is function-f-chainable if and only if X can be continuously mapped onto a chainable subset of $[0, \infty)$ with more than one point.*

Proof. Let $\epsilon > 0$. For any pair of distinct points $y_0, y_n \in f(X)$, there exist $x_0, x_n \in X$ such that $f(x_0) = y_0$ and $f(x_n) = y_n$. Then a finite set of points $x_0, x_1, \ldots, x_n$ of X can be found such that $|f(x_{i+1}) - f(x_i)| < \epsilon$ for $i = 0, 1, \ldots, n-1$. That is, the collection $\{y_0, y_1 = f(x_1), \ldots, y_n\}$ is an ϵ-chain from y_0 to y_n, and so $f(X)$ is a chainable subset of $[0, \infty)$. Moreover, since f is nonconstant, $|f(X)| > 1$.

Conversely, suppose that $f : X \to [0, \infty)$ is a map and that $f(X)$ is a chainable subset of the real line with more than one point. Fix $\epsilon > 0$. For every two elements x and y of X, there exists an ϵ-chain from $f(x)$ to $f(y)$, say $\{f(x) = y_0, y_1, \ldots, y_n = f(y)\}$. Now consider the finite sequence $\{x = x_0, x_1, \ldots, x_n = y\}$ such that $f(x_i) = y_i$ for $1 \leq i \leq n-1$. To finish, it suffices to remark that

$$|f(x_{i+1}) - f(x_i)| = d(y_{i+1}, y_i) < \epsilon, \qquad \text{for any } 0 \leq i \leq n-1.$$

□

Corollary 3.5. *Every denumerable discrete space is function-f-chainable.*

It is clear that any UPC space X is connected but not function-f-chainable. Also, the space $\mathbb{Q}$ with the topology induced from $\mathbb{R}$ is function-f-chainable, but not connected. In fact, every chainable metric space is function-f-chainable and the discrete metric on $[0, \infty)$ is function-f-chainable, but not chainable. For non-UPC connected space, we have the following result.

Theorem 3.6. *Every non-UPC, connected space is function-f-chainable.*

Proof. Suppose that the space X is non-UPC. This implies the existence of a nonconstant real-valued map on X, say ρ. Then the function $f : X \to \mathbb{R}^{>0}$ defined by $f(x) = e^{\rho(x)}$ is continuous and has the intermediate value property (see [11, Corollary 3.6]). We claim that X is function-f-chainable. Let $\epsilon > 0$ and let $a, b \in X$. Without loss of generality, we can assume that $r = f(b) - f(a) > 0$. Choose $n \in \mathbb{N}$ such that $\frac{r}{n} < \epsilon$. Now define $y_i = y_0 + \frac{ir}{n}$ for all $i = 1, 2, \ldots, n$ and $y_0 = f(a)$. Consider the finite sequence $\{x_0, x_1, \ldots, x_n\}$ in X such that $x_0 = a, x_n = b$, and $f(x_i) = y_i$ for all $i = 1, 2, \ldots, n-1$. Then

$$|f(x_{i+1}) - f(x_i)| = |y_{i+1} - y_i| = \frac{r}{n} < \epsilon \quad \text{for all } i = 0, 1, \ldots, n-1.$$

□

Since every topological group is a Tychonoff space [1, Theorem 3.3.11], we have the following.

Corollary 3.7. *Every nontrivial connected group is function-f-chainable.*

We also obtain from Theorem 3.1 the following.

Theorem 3.8. *For a topological space X, the following conditions are equivalent:*

(1) *X is non-UPC;*
(2) *For any nonempty $A, B \subset X$, the pair $\langle A, B\rangle$ is function-f-ϵ-chainable for every positive number ϵ;*
(3) *The pair $\langle A, B\rangle$ is locally function-f-chainable, for every nonempty subsets A and B of X with nonempty interiors.*

Proof. To prove (1) ⇒ (2), it is enough to notice that for each $\epsilon > 0$, the relations $A \subset V_{\varphi^\epsilon \epsilon}(B)$ and $B \subset V_{\varphi^\epsilon \epsilon}(A)$ are true for any nonempty $A, B \subset X$.

(1) ⇒ (3). Suppose A and B are nonempty subsets of X with nonempty interiors. Let $\epsilon > 0$ and put $\delta = \epsilon$. The pair $\langle V_{\varphi^\epsilon \epsilon}(\{a\}) \cap \text{int}(A), V_{\varphi^\epsilon \epsilon}(\{b\}) \cap \text{int}(B)\rangle$ is function-f-ϵ-chainable for every $(a, b) \in A \times B$. □

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