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A NOTE ON QUASILINEAR PARABOLIC SYSTEMS IN GENERALIZED SPACES

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Abstract. We study the existence of solutions for quasilinear parabolic systems of the form

$$\partial_t u - \operatorname{div} \sigma(x,t,Du) = f \quad \text{in } Q = \Omega \times (0,T),$$

whose the right hand-side belongs to $W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$, supplemented with the conditions $u = 0$ on $\partial\Omega \times (0,T)$ and $u(x,0) = u_0(x)$ in Ω. By using a mild monotonicity condition for σ, namely, strict quasimonotone, and the theory of Young measures, we deduce the needed result.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and let an N-function M and its conjugate $\overline{M}$, both satisfy the Δ_2-condition (see Section 2 for the details). In this paper, we establish the existence of weak solutions in the framework of Orlicz–Sobolev spaces for the following quasilinear parabolic system:

$$\frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,Du) = f \quad \text{in } Q, \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on } \partial Q, \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{1.3}$$

where $Q = \Omega \times (0,T)$, $\partial Q = \partial\Omega \times (0,T)$, and its boundary and the function $\sigma : Q \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ will be assumed to satisfy some conditions. The notation $\mathbb{M}^{m\times n}$ stands for the real vector space of $m \times n$ matrices equipped with the

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inner product $A : B = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ij}$. We will prove the existence of a vector-valued function $u : Q \to \mathbb{R}^m$, $m \in \mathbb{N}$, solution to the problem (1.1)–(1.3) for every f belonging to $W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$.

Problem (1.1)–(1.3) was considered in [4, 8]. In [8], for $f \in W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$, we proved the existence and uniqueness results by using the theory of Young measures under the following assumptions:

(H0) (Continuity) $\sigma : Q \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, that is, $(x,t) \mapsto \sigma(x,t,A)$ is measurable for every $A \in \mathbb{M}^{m\times n}$ and $A \mapsto \sigma(x,t,A)$ is continuous for almost every $(x,t) \in Q$.

(H1) (Growth and coercivity) There exist $0 \leq d_1(x,t) \in E_{\overline{M}}(Q)$, $d_2(x,t) \in L^1(Q)$, and $\alpha, \beta, \eta > 0$ such that

$$|\sigma(x,t,A)| \leq d_1(x,t) + \overline{M}^{-1} M(\gamma|A|)$$

$$\sigma(x,t,A) : A \geq \alpha M\big(\frac{|A|}{\beta}\big) - d_2(x,t).$$

(H2) (Monotonicity) σ satisfies one of the conditions:

(a) For all $(x,t) \in Q$, $A \mapsto \sigma(x,t,A)$ is a C^1-function and is monotone, that is, for all $(x,t) \in Q$, we have

$$\big(\sigma(x,t,A) - \sigma(x,t,B)\big) : (A - B) \geq 0.$$

(b) There exists a function $W : Q \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,t,A) = \frac{\partial W}{\partial A}(x,t,A)$ and $A \to W(x,t,A)$ is convex and C^1 for all $(x,t) \in Q$.

(c) σ is strictly monotone, that is, σ is monotone and

$$\big(\sigma(x,t,A) - \sigma(x,t,B)\big) : (A - B) = 0 \Rightarrow A = B.$$

(d) σ is strictly M-quasimonotone, that is,

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,\lambda) - \sigma(x,t,\overline{\lambda})\big) : (\lambda - \overline{\lambda}) d\nu_{(x,t)}(\lambda) dx dt > 0,$$

where $\overline{\lambda} = \langle \nu_{(x,t)}, id\rangle$, $\nu = \{\nu_{(x,t)}\}_{(x,t)\in Q}$ is any family of Young measures generated by a sequence in $L_M(Q)$ and not a Dirac measure for a.e. $(x,t) \in Q$.

In [4], we considered (1.1)–(1.3) for f belongs to some $X'(Q)$, with

$$X(Q) = \big\{u \in L^2(Q;\mathbb{R}^m)/\, Du \in L_M(Q;\mathbb{M}^{m\times n});$$
$$u(t) := u(\cdot,t) \in W_0^1 L_M(\Omega;\mathbb{R}^m) \text{ a.e. } t \in [0,1]\big\},$$

and proved the existence of weak solutions under conditions (H0)–(H2).

Our aim in this paper, is to establish an existence result for the problem (1.1)–(1.3) in Orlicz–Sobolev spaces, where σ satisfies conditions (H0) and (H1), but without assuming non of the conditions listed in (H2). To this purpose, we will assume another mild monotonicity condition, namely, strictly quasimonotone and we proceed the proof differently than [4, 8].

For related topics and similar problems to (1.1)–(1.3), we refer the reader to [6, 10, 11, 13, 17] and other references therein. See also [2, 3, 5, 7] for the steady case, where the theory of Young measures is applied.

Now, instead of condition (H2), we assume the following:
(H2′) σ is strictly quasimonotone, that is, there exist a constant $\alpha_0 > 0$ and $\gamma > 0$ such that

$$\int_Q \big(\sigma(x,t,Du) - \sigma(x,t,Dv)\big) : (Du - Dv)dxdt \geq \alpha_0 \int_Q M(\gamma|Du - Dv|)dxdt$$

for all $u, v \in W_0^{1,x} L_M(Q;\mathbb{R}^m)$.

Let us make some light on this condition and its relation to the previous condition (H2)(d). Indeed, let $\eta : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ be a function satisfying the growth condition

$$|\eta(A)| \leq \overline{M}^{-1} M(\gamma|A|) \tag{1.4}$$

and the structure condition

$$\int_Q \big(\eta(A + D\varphi) - \eta(A)\big) : D\varphi dxdt \geq \alpha_1 \int_Q M(\gamma|D\varphi|)dxdt$$

for a constant $\alpha_1 > 0$ and for all $\varphi \in C_0^\infty(Q)$ and all $A \in \mathbb{M}^{m\times n}$. Note that the above structure condition was investigated by Zhang [18]. We know that for every $W^{1,x}L_M$ gradient Young measure ν, there exists a sequence $(D\varphi_k)$ generating ν for which $\{M(\gamma|D\varphi_k|)\}$ is equi-integrable (see [8, Lemma 4]). Hence, there holds

$$\int_Q \big(\eta(A + D\varphi_k) - \eta(A)\big) : D\varphi_k dxdt \geq \alpha_1 \int_Q M(\gamma|D\varphi_k|)dxdt.$$

By the Hölder inequality and (1.4), it follows that $\big\{\big(\eta(A+D\varphi_k) - \eta(A)\big) : D\varphi_k\big\}$ is equi-integrable. According to the fundamental theorem on Young measures (see [9]), we get

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\eta(A+\lambda) - \eta(\lambda)\big) : \lambda d\nu(\lambda)dxdt \geq \alpha_1 \int_Q \int_{\mathbb{M}^{m\times n}} M(\gamma|\lambda|)d\nu(\lambda)dxdt.$$

We choose first $A + \lambda = \overline{\lambda}$ and then $A = \lambda$. We conclude since $L_M \subset L^1$ that

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\eta(\overline{\lambda}) - \eta(\lambda)\big) : (\overline{\lambda} - \lambda)d\nu(\lambda) \geq \alpha_1 \int_Q \int_{\mathbb{M}^{m\times n}} M(\gamma|\overline{\lambda} - \lambda|)d\nu(\lambda) > 0,$$

which is exactly condition (H2)(d). Thus (H2′) implies (H2)(d). Consequently, the above arguments indicate that our problem (1.1)–(1.3) may admits a solution.

Our main result reads as follows.

Theorem 1.1. *If σ satisfies conditions (H0), (H1), and (H2′), then* (1.1)–(1.3) *has a weak solution $u \in W_0^{1,x} L_M(Q;\mathbb{R}^m) \cap C(0,T;L^2(\Omega;\mathbb{R}^m))$ for every f in $W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$ and every u_0 in $L^2(\Omega;\mathbb{R}^m)$.*

Remark 1.2. The above result holds also when f belongs to the dual of $X(Q)$ stated in [4].

To prove Theorem 1.1, we will use the Galerkin method, which will serve us to obtain the approximating solutions. Let us recall first basic properties of Orlicz–Sobolev spaces and Young measures.

2. Preliminaries

In this section, we recall the necessary and sufficient overview on Orlicz–Sobolev spaces and Young measures. For more details, one could see [1,15,16] and [9,12,14] and references therein.

2.1. A function $M : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an N-function if it is a continuous, nonnegative, and convex function, which has superlinear growth near zero and infinity, that is, $\lim_{\tau\to 0} M(\tau)/\tau = 0$, $\lim_{\tau\to\infty} M(\tau)/\tau = \infty$, and $M(\tau) = 0$ if and only if $\tau = 0$. Its complementary $\overline{M}$ is also an N-function and defined by

$$\overline{M}(s) = \sup_{\tau\in\mathbb{R}^+} \big(\tau s - M(\tau)\big) \quad \text{for } s \in \mathbb{R}^+.$$

The N-function M is said to satisfy the Δ_2-condition ($M \in \Delta_2$) if for some $\epsilon > 0$,

$$M(2\tau) \leq \epsilon M(\tau) \quad \text{for all } \tau \geq 0. \tag{2.1}$$

When the above inequality holds for $\tau \geq$ some $\tau_0 > 0$, then $M \in \Delta_2$ near infinity. Clearly, $\overline{\overline{M}} = M$.

2.2. Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. The class $W^1L_M(\Omega;\mathbb{R}^m)$ or $W^1E_M(\Omega;\mathbb{R}^m)$ consists of all functions u in the Orlicz spaces

$$L_M(\Omega;\mathbb{R}^m) = \Big\{u : \Omega \to \mathbb{R}^m \text{ measurable} \,/\, \int_\Omega M\Big(\frac{u(x)}{\beta}\Big)dx < \infty, \text{ for some } \beta > 0\Big\}$$

or $E_M(\Omega;\mathbb{R}^m)$, such that $Du \in L_M(\Omega;\mathbb{M}^{m\times n})$ or $Du \in E_M(\Omega;\mathbb{M}^{m\times n})$, respectively, where $E_M(\Omega;\mathbb{R}^m)$ denotes the closure of all measurable, simple functions in $L_M(\Omega;\mathbb{R}^m)$. Note that $E_M = L_M$ if and only if (2.1) satisfies. The classes $W^1L_M(\Omega;\mathbb{R}^m)$ and $W^1E_M(\Omega;\mathbb{R}^m)$ are Banach spaces under the norm

$$\|u\|_{1,M} = \|u\|_M + \|Du\|_M,$$

where $\|u\|_M$ is the norm of $L_M(\Omega;\mathbb{R}^m)$ and defined by

$$\|u\|_M = \inf\Big\{\beta > 0 / \int_\Omega M\Big(\frac{u(x)}{\beta}\Big) \leq 1\Big\}.$$

If M satisfies (2.1), then $W^1L_M(\Omega;\mathbb{R}^m)$ is separable, and it is reflexive if further $\overline{M} \in \Delta_2$. Let $u_k, u \in L_M(\Omega;\mathbb{R}^m)$. Then we say that $u_k \to u$ modularly in $L_M(\Omega;\mathbb{R}^m)$ if for some $\beta > 0$, $\int_\Omega M\big(\frac{u_k-u}{\beta}\big)dx \to 0$ as $k \to \infty$. This convergence is equivalent to the one in norm if $M \in \Delta_2$. The following Hölder's inequality holds for $u \in L_M(\Omega;\mathbb{R}^m)$ and $v \in L_{\overline{M}}(\Omega;\mathbb{R}^m)$:

$$\int_\Omega u(x)v(x)dx \leq 2\|u\|_M\|v\|_{\overline{M}},$$

where $\|.\|_{\overline{M}}$ is defined as $\|.\|_M$ and $L_{\overline{M}}(\Omega;\mathbb{R}^m)$ is the dual space of $E_M(\Omega;\mathbb{R}^m)$.

2.3. The symbol $C_0^\infty(\Omega;\mathbb{R}^m) = D(\Omega;\mathbb{R}^m)$ means the space of all C^∞-functions $u : \Omega \to \mathbb{R}^m$ with compact support in Ω. If $|\Omega| < \infty$ (finite measure) and M satisfies (2.1) near infinity, then $W_0^1L_M(\Omega;\mathbb{R}^m)$ is the (norm) closure of $D(\Omega;\mathbb{R}^m)$ in $W^1L_M(\Omega;\mathbb{R}^m)$ and $W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m) = \big(W_0^1L_M(\Omega;\mathbb{R}^m)\big)^*$. Moreover, $W^1L_M(\Omega;\mathbb{R}^m)$ and $W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m)$ are reflexive and separable if M and $\overline{M}$ satisfy the Δ_2-condition near infinity. Remark that $W_0^1L_M(\Omega;\mathbb{R}^m) = W_0^{1,p}(\Omega;\mathbb{R}^m)$

(Sobolev space), for some $p \in (1, \infty)$, when the N-function M is given as $M(\tau) = |\tau|^p$.

2.4. Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $T > 0$, and set $Q = \Omega \times (0, T)$. Let M be an N-function. The inhomogeneous Orlicz–Sobolev spaces are defined for each $\alpha \in \mathbb{N}^n$ as

$$W^{1,x}L_M(Q;\mathbb{R}^m) = \big\{u \in L_M(Q;\mathbb{R}^m) :\ D_x^\alpha u \in L_M(Q;\mathbb{M}^{m\times n}),\ \forall |\alpha| \leq 1\big\},$$

$$W^{1,x}E_M(Q;\mathbb{R}^m) = \big\{u \in E_M(Q;\mathbb{R}^m) :\ D_x^\alpha u \in E_M(Q;\mathbb{M}^{m\times n}),\ \forall |\alpha| \leq 1\big\},$$

where D_x^α is the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^n$. The above spaces are Banach spaces under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D_x^\alpha u\|_M.$$

The space $W_0^{1,x}E_M(Q;\mathbb{R}^m)$ is defined as the (norm) closure in $W^{1,x}E_M(Q;\mathbb{R}^m)$ of $D(Q;\mathbb{R}^m)$. Its dual space is defined as

$$W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m) = \big\{f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha;\ f_\alpha \in L_{\overline{M}}(Q;\mathbb{R}^m)\big\}.$$

If $u \in W^{1,x}L_M(Q;\mathbb{R}^m)$, then the function $t \mapsto u(t) = u(\cdot, t)$ is defined on $(0, T)$ with values in $W^1L_M(\Omega;\mathbb{R}^m)$. If further $u \in W^{1,x}E_M(Q;\mathbb{R}^m)$, then the concerned function is $W^1E_M(\Omega;\mathbb{R}^m)$-valued and is strongly measurable. Moreover, the following embedding holds:

$$W^{1,x}E_M(Q;\mathbb{R}^m) \subset L^1(0,T;W^1E_M(\Omega;\mathbb{R}^m)).$$

2.5. By $C_0(\mathbb{R}^m)$, we denote the closure of the space of continuous functions on $\mathbb{R}^m$ with compact support with respect to the $\|.\|_\infty$-norm. Its dual is the space of signed Radon measures with finite mass and denoted by $\mathcal{M}(\mathbb{R}^m)$. The duality pairing is defined for $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ by

$$\langle \nu, \varphi\rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu(\lambda).$$

When $\varphi \equiv id$, $\langle \nu, id\rangle = \int_{\mathbb{R}^m} \lambda d\nu(\lambda)$.

Lemma 2.1. *Let $\{z_j\}_{j\geq 1}$ be a measurable sequence in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exist a subsequence $\{z_k\}_k \subset \{z_j\}_j$ and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such that for almost each $\varphi \in C_0(\mathbb{R}^m)$ we have*

$$\varphi(z_k) \rightharpoonup^* \overline{\varphi} \quad \textit{weakly in } L^\infty(\Omega;\mathbb{R}^m),$$

where $\overline{\varphi}(x) = \langle \nu_x, \varphi\rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu_x(\lambda)$ for a.e. $x \in \Omega$.

Definition 2.2. We call $\nu = \{\nu_x\}_{x\in\Omega}$ the family of Young measures associated with the subsequence $\{z_k\}_k$.

Remark 2.3. (1) In [9], it was shown that for any Carathéodory function $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}$ and $\{z_k\}_k$ generating a Young measure ν_x, we have

$$\varphi(x, z_k) \rightharpoonup \langle \nu_x, \varphi(x, \cdot)\rangle = \int_{\mathbb{R}^m} \varphi(x,\lambda) d\nu_x(\lambda)$$

weakly in $L^1(\Omega')$ for all measurable $\Omega' \subset \Omega$, provided that the negative part $\varphi^-(x, z_k)$ is equi-integrable.
(2) The above properties remain true if $z_k = Du_k$ for $u_k : \Omega \to \mathbb{R}^m$.

Lemma 2.4 (see [8])**.** *If $\{Du_k\}_k$ is bounded in $L_M(Q; \mathbb{R}^m)$, then the Young measure $\nu_{(x,t)}$ generated by Du_k has the following properties:*

(i) *$\nu_{(x,t)}$ is a probability measure, that is, $\|\nu_{(x,t)}\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for a.e. $(x,t) \in Q$.*
(ii) *The weak L^1-limit of Du_k is given by $\langle \nu_{(x,t)}, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda)$.*
(iii) *$\nu_{(x,t)}$ satisfies $Du(x,t) = \langle \nu_{(x,t)}, id\rangle$ for a.e. $(x,t) \in Q$.*

3. Proof of the main result

We intend to build solutions of problem (1.1)–(1.3) as the limit of finite-dimensional approximations by the well-known Galerkin method. To this purpose, we choose an $L^2(\Omega; \mathbb{R}^m)$-orthonormal base $\{w_i\}_{i\geq 1}$, such that

$$\{w_i\}_{i\geq 1} \subset C_0^\infty(\Omega; \mathbb{R}^m) \subset \overline{\bigcup_{k\geq 1} W_k}^{C^1(\Omega;\mathbb{R}^m)},$$

where $W_k = \operatorname{span}\{w_1, \ldots, w_k\}$. We make the following sequence for approximating solutions of (1.1)–(1.3):

$$u_k(x,t) = \sum_{i=1}^{k} c_{ki}(t) w_i(x) \tag{3.1}$$

where $c_{ki} : [0, T) \to \mathbb{R}$ are supposed to be measurable-bounded functions. Clearly, each u_k satisfies the boundary condition (1.2), by the construction of u_k in the sense that $u_k \in W_0^{1,x} L_M(Q; \mathbb{R}^m)$. For the initial condition (1.3), we choose the coefficient

$$c_{ki}(0) := (u_0, w_i)_{L^2} = \int_\Omega u_0(x) w_i(x) dx$$

such that

$$u_0(\cdot, 0) = \sum_{i=1}^{k} c_{ki}(0) w_i(\cdot) \longrightarrow u_0 \text{ in } L^2(\Omega; \mathbb{R}^m) \text{ as } k \to \infty.$$

We recover the results of [8, Assertion 4] as follows:

(i) u_k defined in (3.1) is the desired solution.
(ii) The local solution constructed in (i) can be extended to the whole interval $[0, T)$.
(iii) The sequence (u_k) is bounded in $W_0^{1,x} L_M(Q; \mathbb{R}^m) \cap L^\infty(0, T; L^2(\Omega; \mathbb{R}^m))$. Furthermore, $u(\cdot, 0) = u_0$ and $u_k(\cdot, T) \rightharpoonup u(\cdot, T)$ in $L^2(\Omega)$.

Now, the following lemma can be seen as the second main result, and it is the key ingredient to pass to the limit in the approximating equations.

Lemma 3.1. *If σ satisfies (H0), (H1), and (H2′), then the following inequality holds:*

$$\liminf_{k\to\infty} \int_Q \big(\sigma(x,t,Du_k) - \sigma(x,t,Du)\big) : (Du_k - Du) dx dt \leq 0.$$

Proof. Let us consider the sequence

$$\begin{aligned} I_k &:= \big(\sigma(x,t,Du_k) - \sigma(x,t,Du)\big) : (Du_k - Du) \\ &= \sigma(x,t,Du_k) : (Du_k - Du) - \sigma(x,t,Du) : (Du_k - Du) \\ &=: I_{k,1} + I_{k,2}, \end{aligned}$$

for arbitrary $u \in W_0^{1,x} L_M(Q;\mathbb{R}^m)$.
According to the growth condition in (H1),

$$\int_Q \overline{M}(|\sigma(x,t,Du)|)dxdt \le c \int_Q \big(\overline{M}(d_1(x,t)) + M(\gamma|Du|)\big)dxdt < \infty$$

since $d_1 \in E_{\overline{M}}(Q)$. Hence $\sigma(\cdot) \in L_{\overline{M}}(Q;\mathbb{M}^{m\times n})$. Note that, since (u_k) is bounded in $W_0^{1,x} L_M(Q;\mathbb{R}^m) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, by Lemma 2.1, it follows the existence of a Young measure $\nu_{(x,t)}$ generated by Du_k in $L_M(Q;\mathbb{M}^{m\times n})$, which satisfies the properties of Lemma 2.4. By virtue of the weak limit in Lemma 2.4, it follows that

$$\begin{aligned} \liminf_{k\to\infty} \int_Q I_{k,2} dxdt &= \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,Du) : (\lambda - Du) d\nu_{(x,t)}(\lambda) dxdt \\ &= \int_Q \sigma(x,t,Du) : \big(\underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda)}_{=:Du(x,t)} - Du\big) dxdt = 0. \end{aligned}$$

Thanks to [8, Assertion 3], the sequence $\{-\operatorname{div} \sigma(x,t,Du_k)\}$ is bounded in $W^{-1,x} L_{\overline{M}}(Q;\mathbb{R}^m)$. Hence

$$-\operatorname{div} \sigma(x,t,Du_k) \rightharpoonup \chi \quad \text{in } W^{-1,x} L_{\overline{M}}(Q;\mathbb{R}^m)$$

for $\chi \in W^{-1,x} L_{\overline{M}}(Q;\mathbb{R}^m)$. Hence the first property of χ is the following energy equality:

$$\frac{1}{2}\|u(\cdot,T)\|_{L^2}^2 + \int_Q \chi.u dxdt = \frac{1}{2}\|u(\cdot,0)\|_{L^2}^2 + \langle f, u\rangle. \tag{3.2}$$

It follows that

$$\begin{aligned} \liminf_{k\to\infty} -\int_Q \sigma(x,t,Du_k) : Du dxdt &= -\int_Q \chi.u dxdt \\ &\overset{(3.2)}{=} \frac{1}{2}\|u(\cdot,T)\|_{L^2}^2 - \frac{1}{2}\|u(\cdot,0)\|_{L^2}^2 - \langle f, u\rangle. \end{aligned} \tag{3.3}$$

By virtue of the Galerkin equations, we can write

$$\begin{aligned} \int_Q \sigma(x,t,Du_k) : Du_k dxdt &= \langle f, u_k\rangle - \int_Q u_k \frac{\partial u_k}{\partial t} dxdt \\ &= \langle f, u_k\rangle - \frac{1}{2}\|u_k(\cdot,T)\|_{L^2}^2 + \frac{1}{2}\|u_k(\cdot,0)\|_{L^2}^2. \end{aligned}$$

Applying the weak limit defined in the property (iii) above, it follows that

$$\liminf_{k\to\infty} \int_Q \sigma(x,t,Du_k) : Du_k dxdt \le \langle f, u\rangle - \frac{1}{2}\|u(\cdot,T)\|_{L^2}^2 + \frac{1}{2}\|u(\cdot,0)\|_{L^2}^2. \tag{3.4}$$

By the combination of (3.3) and (3.4), we deduce that

$$\liminf_{k\to\infty} \int_Q \sigma(x,t,Du_k) : (Du_k - Du)dxdt \leq 0.$$

Since

$$\liminf_{k\to\infty} \int_Q \sigma(x,t,Du) : (Du_k - Du)dxdt = 0,$$

and the desired inequality of Lemma 3.1 follows. □

Remark 3.2. In this paper, we do not need to prove the div-curl inequality, which was necessary in the proof of the main result of [8] and [4].

We are now in a position to show the existence of solutions for (1.1)–(1.3).

Proof of Theorem 1.1. Remark that for a positive constant c,

$$\int_Q M(\gamma|Du_k - Du|)dxdt \leq c\int_Q \big(\sigma(x,t,Du_k) - \sigma(x,t,Du)\big) : (Du_k - Du)dxdt$$

by condition (H2′). By virtue of Lemma 3.1 and the embedding $L_M \subset L^1$, it follows that the limit inf of the above inequality gives

$$\lim_{k\to\infty} \int_Q M(\gamma|Du_k - Du|)dxdt = 0.$$

Let $E_{k,\epsilon} = \{(x,t) \in Q : |Du_k - Du| \geq \epsilon\}$. Thus

$$\begin{aligned}\int_Q M(\gamma|Du_k - Du|)dxdt &\geq \int_{E_{k,\epsilon}} M(\gamma|Du_k - Du|)dxdt \\ &\geq c\int_{E_{k,\epsilon}} |Du_k - Du|dxdt \\ &\geq c\epsilon|E_{k,\epsilon}|,\end{aligned}$$

where c is the constant of the embedding $L_M \subset L^1$. Therefore

$$|E_{k,\epsilon}| \leq \frac{1}{c\epsilon}\int_Q M(\gamma|Du_k - Du|)dxdt \to 0 \quad \text{as } k\to\infty.$$

Hence $Du_k \to Du$ in measure and almost everywhere (up to a subsequence). The continuity property of σ implies that

$$\sigma(x,t,Du_k) \to \sigma(x,t,Du) \quad \text{almost everywhere.}$$

On the other hand, since $\|Du_k\|_M$ is bounded by a constant c, for a measurable $Q' \subset Q$, we have

$$\int_{Q'} |\sigma(x,t,Du_k) : D\varphi|dxdt \leq c\big(\|d_1\|_{\overline{M}} + \underbrace{\|Du_k\|_M}_{\leq c}\big)\Big(\int_{Q'} M(|D\varphi|)dxdt,$$

for all $\varphi \in W_0^{1,x}L_M(Q;\mathbb{R}^m)$. Since $\int_{Q'} M(|D\varphi|)dxdt$ is arbitrary small if the measure of Q' is chosen small enough, then $\big(\sigma(x,t,Du_k) : D\varphi\big)$ is equi-integrable.

The Vitali theorem implies

$$\int_Q \big(\sigma(x,t,Du_k) - \sigma(x,t,Du)\big) : D\varphi dxdt \longrightarrow 0 \quad \text{as } k \to \infty.$$

Now, we take a test function $w \in \cup_{k\in\mathbb{N}} W_k$ and $\varphi \in C_0^\infty([0,T])$ in

$$\big(\partial_t u_k, w_j\big)_{L^2} + \int_\Omega \sigma(x,t,Du_k) : Dw_j dx = \langle f(t), w_j\rangle$$

(with $j = 1,\dots,k$) and integrate over interval $(0,T)$ and pass to the limit $k \to \infty$. The resulting equation is

$$\int_Q \partial_t u(x)\varphi(t)w(x)dxdt + \int_Q \sigma(x,t,Du) : Dw(x)\varphi(t)dxdt = \langle f, \varphi w\rangle\rangle,$$

for arbitrary $w \in \cup_{k\in\mathbb{N}} W_k$ and $\varphi \in C^\infty([0,T])$. By the density of the linear span of these functions in $W_0^{1,x}L_M(Q;\mathbb{R}^m)$, this proves, that u is in fact a weak solution. □

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