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SOME NUMERICAL RADIUS INEQUALITIES FOR THE ČEBYŠEV FUNCTIONAL AND NONCOMMUTATIVE HILBERT SPACE OPERATORS

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Abstract. We prove a Grüss inequality for positive Hilbert space operators. Hence, some numerical radius inequalities are proved. On the other hand, based on a noncommutative binomial formula, a noncommutative upper bound for the numerical radius of the summand of two bounded linear Hilbert space operators is proved. A commutative version is also obtained as well.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ be the Banach algebra of all bounded linear operators defined on the complex Hilbert space $(\mathscr{H};\langle\cdot,\cdot\rangle)$ with the identity operator $1_{\mathscr{H}}$ in $\mathscr{B}(\mathscr{H})$. A bounded linear operator A defined on $\mathscr{H}$ is self-adjoint if and only if $\langle Ax,x\rangle\in\mathbb{R}$ for all $x\in\mathscr{H}$. The spectrum of an operator A is the set of all $\lambda\in\mathbb{C}$ for which the operator $\lambda I-A$ does not have a bounded linear operator inverse and is denoted by $\mathrm{sp}(A)$. Consider the real vector space $\mathscr{B}(\mathscr{H})_{sa}$ of self-adjoint operators on $\mathscr{H}$ and its positive cone $\mathscr{B}(\mathscr{H})^{+}$ of positive operators on $\mathscr{H}$. Also, $\mathscr{B}(\mathscr{H})_{sa}^{I}$ denotes the convex set of bounded self-adjoint operators on the Hilbert space $\mathscr{H}$ with spectra in a real interval I. A partial order is naturally equipped on $\mathscr{B}(\mathscr{H})_{sa}$ by defining $A\leq B$ if and only if $B-A\in\mathscr{B}(\mathscr{H})^{+}$. We write $A>0$ to mean that A is a strictly positive operator, or equivalently, $A\geq 0$ and A is invertible. When $\mathscr{H}=\mathbb{C}^n$, we identify $\mathscr{B}(\mathscr{H})$ with the algebra $\mathfrak{M}_{n\times n}$ of n-by-n complex matrices. Then $\mathfrak{M}_{n\times n}^{+}$ is just the cone of n-by-n positive semidefinite matrices.

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For a bounded linear operator T on a Hilbert space $\mathscr{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathscr{H}$ under the quadratic form $x \to \langle Tx, x\rangle$ associated with the operator. More precisely,

$$W(T) = \{\langle Tx, x\rangle : x \in \mathscr{H}, \|x\| = 1\}.$$

Also, the (maximum) numerical radius is defined by

$$w_{\max}(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup_{\|x\|=1} |\langle Tx, x\rangle| := w(T),$$

and the (minimum) numerical radius is defined to be

$$w_{\min}(T) = \inf\{|\lambda| : \lambda \in W(T)\} = \inf_{\|x\|=1} |\langle Tx, x\rangle|.$$

The spectral radius of an operator T is defined to be

$$r(T) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(T)\}.$$

We recall that, the usual operator norm of an operator T is defined to be

$$\|T\| = \sup\{\|Tx\| : x \in H, \|x\| = 1\},$$

and

$$\begin{aligned} \ell(T) :&= \inf\{\|Tx\| : x \in \mathscr{H}, \|x\| = 1\} \\ &= \inf\{|\langle Tx, y\rangle| : x, y \in \mathscr{H}, \|x\| = \|y\| = 1\}. \end{aligned}$$

It is well known that $w(\cdot)$ defines an operator norm on $\mathscr{B}(\mathscr{H})$, which is equivalent to the operator norm $\|\cdot\|$. Moreover, we have

$$\frac{1}{2}\|T\| \le w(T) \le \|T\| \tag{1.1}$$

for any $T \in \mathscr{B}(\mathscr{H})$. The inequality is sharp.

In 2003, Kittaneh [14] refined the right-hand side of (1.1), where he proved that

$$w(T) \le \frac{1}{2}\left(\|T\| + \|T^2\|^{1/2}\right)$$

for any $T \in \mathscr{B}(\mathscr{H})$.

In 2005, the same author in [15] proved that

$$\frac{1}{4}\|A^*A + AA^*\| \le w^2(A) \le \frac{1}{2}\|A^*A + AA^*\|.$$

The inequality is sharp. This inequality was also reformulated and generalized in [9], but in terms of Cartesian decomposition.

In 2007, Yamazaki [22] improved (1.1) by proving that

$$w(T) \le \frac{1}{2}\left(\|T\| + w\left(\widetilde{T}\right)\right) \le \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right),$$

where $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ with unitary U.

Dragomir [5] (see also [8]) used the Buzano inequality to improve (1.1), where he proved that

$$w^2(T) \le \frac{1}{2}\left(\|T\| + w\left(T^2\right)\right).$$

This result was also recently generalized by Sattari, Moslehian, and Yamazaki [20] and the author of this paper [3].

Dragomir [6] studied the Čebyšev functional

$$\mathcal{C}\left(f,g;A;x\right)=\left\langle f\left(A\right)g\left(A\right)x,x\right\rangle-\left\langle f\left(A\right)x,x\right\rangle\left\langle g\left(A\right)x,x\right\rangle$$

for any self-adjoint operator $A\in\mathcal{B}(H)$ and $x\in H$ with $\|x\|=1$. In particular, we have

$$\mathcal{C}\left(f,f;A;x\right)=\left\langle f^{2}\left(A\right)x,x\right\rangle-\left\langle f\left(A\right)x,x\right\rangle^{2}.$$

In a sequence of papers, Dragomir proved various bounds for the Čebyšev functional. The most popular result concerning continuous synchronous (asynchronous) functions of self-adjoint linear operators in Hilbert spaces reads as follows.

Theorem 1.1. *Let $A\in\mathscr{B}\left(\mathscr{H}\right)_{sa}$ with $\operatorname{sp}\left(A\right)\subset[\gamma,\Gamma]$ for some real numbers γ,Γ with $\gamma<\Gamma$. If $f,g:[\gamma,\Gamma]\to\mathbb{R}$ are continuous and synchronous (asynchronous) on $[\gamma,\Gamma]$, then*

$$\left\langle f\left(A\right)g\left(A\right)x,x\right\rangle\ge(\le)\left\langle g\left(A\right)x,x\right\rangle\left\langle f\left(A\right)x,x\right\rangle$$

for any $x\in H$ with $\|x\|=1$.

This result was generalized recently in [1,2]. For more related results concerning Čebyšev–Grüss type inequalities, we refer the reader to [7,16,17,19].

2. Results

The following Grüss inequality for linear bounded operators in inner product Hilbert spaces is valid.

Theorem 2.1. *Let $A\in\mathscr{B}\left(\mathscr{H}\right)^{+}$. If f and g are both measurable functions on $[0,\infty)$, then*

$$\left|\mathcal{C}\left(f,g;A;x\right)\right|\le\mathcal{C}^{1/2}\left(f,f;A;x\right)\mathcal{C}^{1/2}\left(g,g;A;x\right)\tag{2.1}$$

for any $x\in H$. In other words,

$$\begin{aligned}&\left|\left\langle f\left(A\right)g\left(A\right)x,x\right\rangle-\left\langle f\left(A\right)x,x\right\rangle\left\langle g\left(A\right)x,x\right\rangle\right|\\&\qquad\le\left(\left\langle f^{2}\left(A\right)x,x\right\rangle-\left\langle f\left(A\right)x,x\right\rangle^{2}\right)^{1/2}\left(\left\langle g^{2}\left(A\right)x,x\right\rangle-\left\langle g\left(A\right)x,x\right\rangle^{2}\right)^{1/2}.\end{aligned}$$

Proof. It is not hard to show that

$$\begin{aligned}&C\left(f,g;A;x\right)\\&\qquad=\frac{1}{2}\int_{0}^{\infty}\int_{0}^{\infty}\left(f\left(t\right)-f\left(s\right)\right)\left(g\left(t\right)-g\left(s\right)\right)d\left\langle E_{t}x,x\right\rangle d\left\langle E_{s}x,x\right\rangle.\end{aligned}\tag{2.2}$$

Utilizing the triangle inequality in (2.2) and then the Cauchy–Schwarz inequality, we get

$$\left|C\left(f,g;A;x\right)\right|=\frac{1}{2}\left|\int_{0}^{\infty}\int_{0}^{\infty}\left(f\left(t\right)-f\left(s\right)\right)\left(g\left(t\right)-g\left(s\right)\right)d\left\langle E_{t}x,x\right\rangle d\left\langle E_{s}x,x\right\rangle\right|$$

$$\begin{aligned}
&\leq \frac{1}{2}\int_0^\infty\int_0^\infty |f(t)-f(s)|\,|g(t)-g(s)|\,d\langle E_t x,x\rangle\, d\langle E_s x,x\rangle\\
&\leq \frac{1}{2}\left(\int_0^\infty\int_0^\infty |f(t)-f(s)|^2\,d\langle E_t x,x\rangle\, d\langle E_s x,x\rangle\right)^{1/2}\\
&\qquad\times\left(\int_0^\infty\int_0^\infty |g(t)-g(s)|^2\,d\langle E_t x,x\rangle\, d\langle E_s x,x\rangle\right)^{1/2}\\
&= \frac{1}{2}\left(\int_0^\infty d\langle E_s x,x\rangle\int_0^\infty f^2(t)\,d\langle E_t x,x\rangle\right.\\
&\qquad\qquad -2\int_0^\infty f(t)\,d\langle E_t x,x\rangle\int_0^\infty f(s)\,d\langle E_s x,x\rangle\\
&\qquad\qquad\qquad \left.+\int_0^\infty d\langle E_t x,x\rangle\int_0^\infty f^2(s)\,d\langle E_s x,x\rangle\right)^{1/2}\\
&\qquad\times\left(\int_0^\infty d\langle E_s x,x\rangle\int_0^\infty g^2(t)\,d\langle E_t x,x\rangle\right.\\
&\qquad\qquad -2\int_0^\infty g(t)\,d\langle E_t x,x\rangle\int_0^\infty g(x)\,d\langle E_s x,x\rangle\\
&\qquad\qquad\qquad \left.+\int_0^\infty d\langle E_t x,x\rangle\int_0^\infty g^2(s)\,d\langle E_s x,x\rangle\right)^{1/2}\\
&= \left(1_{\mathscr{H}}\cdot\int_0^\infty f^2(t)\,d\langle E_t x,x\rangle-\left(\int_0^\infty f(t)\,d\langle E_t x,x\rangle\right)^2\right)^{1/2}\\
&\qquad\times\left(1_{\mathscr{H}}\cdot\int_0^\infty g^2(t)\,d\langle E_t x,x\rangle-\left(\int_0^\infty g(t)\,d\langle E_t x,x\rangle\right)^2\right)^{1/2}\\
&= \left(\left\langle f^2(A)x,x\right\rangle-\langle f(A)x,x\rangle^2\right)^{1/2}\left(\left\langle g^2(A)x,x\right\rangle-\langle g(A)x,x\rangle^2\right)^{1/2}
\end{aligned}$$

for any $x\in\mathscr{H}$, which gives the desired result (2.1). □

Corollary 2.2. *Let $A\in\mathscr{B}(\mathscr{H})^+$. Then*

$$\begin{aligned}
&\left|\langle Ax,x\rangle-\langle A^\alpha x,x\rangle\left\langle A^{1-\alpha}x,x\right\rangle\right|\\
&\qquad\leq\left(\left\langle A^{2\alpha}x,x\right\rangle-\langle A^\alpha x,x\rangle^2\right)^{1/2}\left(\left\langle A^{2(1-\alpha)}x,x\right\rangle-\left\langle A^{1-\alpha}x,x\right\rangle^2\right)^{1/2}
\end{aligned}$$

for any $x\in\mathscr{H}$ and all $\alpha\in\left[0,\frac{1}{2}\right]$.

Proof. Setting $f(t)=t^\alpha$ and $g(t)=t^{1-\alpha}$ in (2.1), we get the desired result. □

Theorem 2.3. *Let $A\in\mathscr{B}(\mathscr{H})^+$. If f and g are both measurable functions on $[0,\infty)$, then*

$$\begin{aligned}
&w_{\max}(f(A)g(A))-w_{\min}(f(A))\cdot w_{\min}(g(A))\\
&\qquad\leq\left[\|f(A)\|^2-\ell^2\left(f^{1/2}(A)\right)\right]^{1/2}\cdot\left[\|g(A)\|^2-\ell^2\left(g^{1/2}(A)\right)\right]^{1/2}. \qquad (2.3)
\end{aligned}$$

Proof. Using the basic triangle inequality $||a|-|b|| \le |a-b|$, we have from (2.1) that

$$\begin{aligned}
&|(|\langle f(A)\,g(A)\,x,x\rangle|) - (|\langle f(A)\,x,x\rangle\,\langle g(A)\,x,x\rangle|)| \\
&\le |\langle f(A)\,g(A)\,x,x\rangle - \langle f(A)\,x,x\rangle\,\langle g(A)\,x,x\rangle| \\
&\le \left(\left\langle f^2(A)\,x,x\right\rangle - \langle f(A)\,x,x\rangle^2\right)^{1/2}\left(\left\langle g^2(A)\,x,x\right\rangle - \langle g(A)\,x,x\rangle^2\right)^{1/2}.
\end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$, we obtain

$$\begin{aligned}
&\sup_{\|x\|=1} ||\langle f(A)\,g(A)\,x,x\rangle| - |\langle f(A)\,x,x\rangle|\,|\langle g(A)\,x,x\rangle|| \\
&\le \sup_{\|x\|=1} |\langle f(A)\,g(A)\,x,x\rangle - \langle f(A)\,x,x\rangle\,\langle g(A)\,x,x\rangle| \\
&\le \sup_{\|x\|=1} |\langle f(A)\,g(A)\,x,x\rangle| - \inf_{\|x\|=1}\{|\langle f(A)\,x,x\rangle|\,|\langle g(A)\,x,x\rangle|\} \\
&\le \sup_{\|x\|=1} |\langle f(A)\,g(A)\,x,x\rangle| - \inf_{\|x\|=1}|\langle f(A)\,x,x\rangle| \cdot \inf_{\|x\|=1}|\langle g(A)\,x,x\rangle| \\
&\le \sup_{\|x\|=1}\left[\|f(A)\,x\|^2 - \langle f(A)\,x,x\rangle^2\right]^{1/2} \\
&\qquad\qquad \times \sup_{\|x\|=1}\left[\|g(A)\,x\|^2 - \langle g(A)\,x,x\rangle^2\right]^{1/2} \\
&\le \left[\sup_{\|x\|=1}\|f(A)\,x\|^2 - \inf_{\|x\|=1}\langle f(A)\,x,x\rangle^2\right]^{1/2} \\
&\qquad\qquad \times \left[\sup_{\|x\|=1}\|g(A)\,x\|^2 - \inf_{\|x\|=1}\langle g(A)\,x,x\rangle^2\right]^{1/2} \\
&= \left[\|f(A)\|^2 - \ell^2\left(f^{1/2}(A)\right)\right]^{1/2} \cdot \left[\|g(A)\|^2 - \ell^2\left(g^{1/2}(A)\right)\right]^{1/2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&w_{\max}(f(A)\,g(A)) - w_{\min}(f(A))\,w_{\min}(g(A)) \\
&\qquad \le \left[\|f(A)\|^2 - \ell^2\left(f^{1/2}(A)\right)\right]^{1/2} \cdot \left[\|g(A)\|^2 - \ell^2\left(g^{1/2}(A)\right)\right]^{1/2},
\end{aligned}$$

or equivalently, we have

$$\begin{aligned}
&w_{\max}(f(A)\,g(A)) - w_{\min}(f(A)) \cdot w_{\min}(g(A)) \\
&\qquad \le \left[\|f(A)\|^2 - \ell^2\left(f^{1/2}(A)\right)\right]^{1/2} \cdot \left[\|g(A)\|^2 - \ell^2\left(g^{1/2}(A)\right)\right]^{1/2},
\end{aligned}$$

which proves the desired result. □

Corollary 2.4. *Let $A \in \mathscr{B}(\mathscr{H})^+$. Then*

$$\begin{aligned}
&w_{\max}(A) - w_{\min}(A^\alpha) \cdot w_{\min}\left(A^{1-\alpha}\right) \\
&\qquad \le \left[\|A^\alpha\|^2 - \ell^2\left(A^{\frac{\alpha}{2}}\right)\right]^{1/2} \cdot \left[\left\|A^{1-\alpha}\right\|^2 - \ell^2\left(A^{\frac{1-\alpha}{2}}\right)\right]^{1/2}
\end{aligned}$$

for each $x \in \mathscr{H}$. *In particular,*

$$w_{\max}(A) - w_{\min}^2\left(A^{1/2}\right) \le \left\|A^{1/2}\right\|^2 - \ell^2\left(A^{1/4}\right) \tag{2.4}$$

for each $x \in \mathscr{H}$.

Proof. Setting $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ in (2.3), we get the desired result. □

Corollary 2.5. *Let* $A \in \mathscr{B}(\mathscr{H})^+$. *If* f *is a measurable function on* $[0,\infty)$, *then*

$$w_{\max}\left(f^2(A)\right) - w_{\min}^2(f(A)) \le \|f(A)\|^2 - \ell^2\left(f^{1/2}(A)\right) \tag{2.5}$$

for each $x \in \mathscr{H}$.

Proof. Setting $f = g$ in (2.1), we get the desired result. □

A generalization of (2.4) can be deduced from (2.5) as follows.

Corollary 2.6. *Let* $A \in \mathscr{B}(\mathscr{H})^+$. *Then, for any* $p > 0$, *the inequality*

$$w_{\max}\left(A^{2p}\right) - w_{\min}^2(A^p) \le \|A^p\|^2 - \ell^2\left(A^{p/2}\right)$$

holds for each $x \in \mathscr{H}$.

The Schwarz inequality for positive operators reads that if A is a positive operator in $\mathscr{B}(\mathscr{H})$, then

$$|\langle Ax, y\rangle|^2 \le \langle Ax, x\rangle \langle Ay, y\rangle, \tag{2.6}$$

for any vectors $x, y \in \mathscr{H}$.

In 1951, Reid [18] proved an inequality that in some senses considered a variant of Schwarz inequality. In fact, he proved that for all operators $A \in \mathscr{B}(\mathscr{H})$ such that A is positive and AB is self-adjoint, we have

$$|\langle ABx, y\rangle| \le \|B\| \langle Ax, x\rangle, \tag{2.7}$$

for all $x \in \mathscr{H}$. Halmos [10] presented his stronger version of the Reid inequality (2.7) by replacing $r(B)$ instead of $\|B\|$.

In 1952, Kato [11] introduced a companion inequality of (2.6), called the mixed Schwarz inequality, which asserts

$$|\langle Ax, y\rangle|^2 \le \left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle, \qquad 0 \le \alpha \le 1 \tag{2.8}$$

for all positive operators $A \in \mathscr{B}(\mathscr{H})$ and any vectors $x, y \in \mathscr{H}$, where $|A| = (A^*A)^{1/2}$.

In 1988, Kittaneh [12] proved a very interesting extension combining both the Halmos–Reid inequality (2.7) and the mixed Schwarz inequality (2.8). His result reads that

$$|\langle ABx, y\rangle| \le r(B) \|f(|A|)x\| \|g(|A^*|)y\| \tag{2.9}$$

for any vectors $x, y \in \mathscr{H}$, where $A, B \in \mathscr{B}(\mathscr{H})$ satisfy $|A|B = B^*|A|$ and f and g are nonnegative continuous functions defined on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \ge 0)$. Clearly, choose $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ with $B = 1_{\mathscr{H}}$, so we refer to (2.8). Moreover, by choosing $\alpha = \frac{1}{2}$, some manipulations refer to the Halmos version of the Reid inequality.

Theorem 2.7. *Let $A \in \mathscr{B}(\mathscr{H})$. If f and g are both positive continuous and $f(t)g(t) = t$ for all $t \in [0,\infty)$, then*

$$w_{\max}(A) - w_{\min}(f(A)) \cdot w_{\min}(g(A)) \leq \frac{1}{2}\left\|f^2(|A|) + g^2(|A^*|)\right\| - \ell^2\left(f^{1/2}(A)\right) \cdot \ell^2\left(g^{1/2}(A)\right).$$

Proof. Since $f(t)g(t) = t$ for all $t \in [0,\infty)$, then from the proof of Theorem 2.3, we have

$$\begin{aligned}
&\sup_{\|x\|=1} \left||\langle f(A) g(A) x, x\rangle| - |\langle f(A) x, x\rangle| |\langle g(A) x, x\rangle|\right| \\
&\leq \sup_{\|x\|=1} |\langle f(A) g(A) x, x\rangle| - \inf_{\|x\|=1} \{|\langle f(A) x, x\rangle| |\langle g(A) x, x\rangle|\} \\
&= \sup_{\|x\|=1} |\langle Ax, x\rangle| - \inf_{\|x\|=1} |\langle f(A) x, x\rangle| \cdot \inf_{\|x\|=1} |\langle g(A) x, x\rangle| \qquad \text{(by (2.9))} \\
&\leq \sup_{\|x\|=1} \left\langle f^2(|A|) x, x\right\rangle^{1/2} \left\langle g^2(|A^*|) x, x\right\rangle^{1/2} \\
&\qquad - \inf_{\|x\|=1} |\langle f(A) x, x\rangle| \cdot \inf_{\|x\|=1} |\langle g(A) x, x\rangle| \\
&\leq \sup_{\|x\|=1} \left\langle f^2(|A| x, x)\right\rangle^{1/2} \left\langle g^2(|A^*| x, x)\right\rangle^{1/2} \\
&\qquad - \inf_{\|x\|=1} |\langle f(A) x, x\rangle| \cdot \inf_{\|x\|=1} |\langle g(A) x, x\rangle| \\
&\leq \frac{1}{2} \sup_{\|x\|=1} \left\langle \left[f^2(|A|) + g^2(|A^*|)\right] x, x\right\rangle \\
&\qquad - \inf_{\|x\|=1} |\langle f(A) x, x\rangle| \cdot \inf_{\|x\|=1} |\langle g(A) x, x\rangle|,
\end{aligned}$$

which proves the required result. □

Corollary 2.8. *Let $A \in \mathscr{B}(\mathscr{H})^+$. If f and g are both positive continuous and $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then*

$$w_{\max}(A) - w_{\min}(A^\alpha) \cdot w_{\min}\left(A^{1-\alpha}\right) \leq \frac{1}{2}\left\||A|^{2\alpha} + |A^*|^{2(1-\alpha)}\right\| - \ell^2\left(A^{\frac{\alpha}{2}}\right) \cdot \ell^2\left(A^{\frac{1-\alpha}{2}}\right).$$

In particular,

$$w_{\max}(A) - w_{\min}^2\left(A^{1/2}\right) \leq \frac{1}{2}\||A| + |A^*|\| - \ell^4\left(A^{1/4}\right).$$

Theorem 2.9. *Let $A, B \in \mathscr{B}(\mathscr{H})$. Then*

$$w\left((A+B)^2\right) \leq w\left(A^2\right) + w\left(B^2\right) + \frac{1}{2}\min\left\{w\left(BA^2B\right) + \|AB\|^2, w\left(AB^2A\right) + \|BA\|^2\right\}. \quad (2.10)$$

Proof. Let us first note that the Dragomir refinement of Cauchy–Schwarz inequality reads that (see [4])

$$|\langle x, y\rangle| \leq |\langle x, e\rangle \langle e, y\rangle| + |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \|x\| \|y\|$$

for all $x, y, e \in \mathscr{H}$ with $\|e\| = 1$.

It is easy to deduce the inequality

$$|\langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{2}\left(|\langle x, y\rangle| + \|x\| \|y\|\right). \tag{2.11}$$

Utilizing the triangle inequality, we have

$$\left|\left\langle (A+B)^2 x, x\right\rangle\right| \leq \left|\left\langle A^2 x, x\right\rangle\right| + |\langle ABx, x\rangle| \, |\langle x, A^*B^*x\rangle| + \left|\left\langle B^2 x, x\right\rangle\right|, \tag{2.12}$$

so that by setting $e = u$, $x = ABu$, $y = A^*B^*u$ in (2.11), we get

$$|\langle ABu, u\rangle \langle u, A^*B^*u\rangle| \leq \frac{1}{2}\left(|\langle ABu, A^*B^*y\rangle| + \|ABu\| \|A^*B^*u\|\right).$$

Substituting in (2.12) and taking the supremum over all unit vector $x \in \mathscr{H}$, we get

$$w\left((A+B)^2\right) \leq w\left(A^2\right) + w\left(B^2\right) + \frac{1}{2}\left(w\left(BA^2B\right) + \|AB\|^2\right).$$

Replacing B by A and A by B in the previous inequality, we get that

$$w\left((B+A)^2\right) \leq w\left(B^2\right) + w\left(A^2\right) + \frac{1}{2}\left(w\left(AB^2A\right) + \|BA\|^2\right).$$

The desired result holds using the previous two inequalities. □

Corollary 2.10. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$w\left(A^2\right) \leq \frac{1}{4}\left(w\left(A^4\right) + \left\|A^2\right\|^2\right).$$

Proof. Setting $A = B$ in (2.10), we get the desired result. □

Let $\mathscr{U}$ be an associative algebra, not necessarily commutative, with identity $1_{\mathscr{U}}$. For two elements A and B in $\mathscr{U}$, that commute; that is, $AB = BA$. It is well known, the binomial theorem reads that

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}. \tag{2.13}$$

Wyss [21] derived an interesting noncommutative binomial formula for commutative algebra $\mathscr{U}$ with identity $1_{\mathscr{U}}$. Moreover, $\mathscr{L}(\mathscr{U})$ denotes the algebra of linear transformations from $\mathscr{U}$ to $\mathscr{U}$. Let $A, X \in \mathscr{U}$; then the element (commutator) d_A in $\mathscr{L}(\mathscr{U})$ is defined by

$$d_A(X) = [A, X] = AX - XA.$$

It follows that, A and d_A are elements of $\mathscr{L}(\mathscr{U})$. Moreover, A can be looked upon as an element in $\mathscr{L}(\mathscr{U})$ by $A(X) = AX$, which is the left multiplication.

The following properties are hold (see [21]):

(1) A and d_A commute; that is, $Ad_A(X) = d_A A(X)$.
(2) d_A is a derivation on $\mathscr{U}$; that is, $d_A(XY) = (d_A X)Y + X(d_A Y)$.
(3) $(A - d_A)X = XA$.
(4) The Jacobi identity $d_A d_B(C) + d_B d_C(A) + d_C d_A(B) = 0$ holds.

Using these properties, Wyss [21] proved the following noncommutative version of binomial theorem:

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} \left\{ (A+d_B)^k 1_{\mathscr{U}} \right\} B^{n-k} \tag{2.14}$$

for all elements A and B in the associative algebra $\mathscr{U}$ with identity $1_{\mathscr{U}}$.

We write

$$(A+d_B)^n 1_{\mathscr{U}} = A^n + D_n(B,A). \tag{2.15}$$

For a commutative algebra, $D_n(B,A)$ is identically zero. We thus call $D_n(B,A)$ the essential noncommutative part. Moreover, $D_n(B,A)$ satisfies the following recurrence relation:

$$D_{n+1}(B,A) = d_B A^n + (A+d_B) D_n(B,A), \qquad n \geq 0$$

with $D_0(B,A) = 0$.

A noncommutative upper bound for the summand of two bounded linear Hilbert space operators is proved in the following result.

Theorem 2.11. *Let $A, B \in \mathscr{B}(\mathscr{H})$. If f and g are both positive continuous and $f(t)g(t) = t$ for all $t \in [0,\infty)$, then*

$$w\left((A+B)^n\right) \leq \frac{1}{2} \sum_{k=0}^{n} \left\{ \binom{n}{k} \left\| f\left(\left| \left\{ (A+d_B)^k 1_{\mathscr{H}} \right\} B^{n-k} \right| \right) + g\left(\left| \left(B^{n-k}\right)^* \left\{ (A+d_B)^k 1_{\mathscr{H}} \right\}^* \right| \right) \right\| \right\}, \tag{2.16}$$

where $d_B(A) = [B,A] = BA - AB$ and $d_B^*(A) = [B,A]^* = A^*B^* - B^*A^*$.

Proof. By utilizing the triangle inequality in (2.14) and by employing (2.9), we have

$$\begin{aligned}
&\left|\left\langle (A+B)^n x, y \right\rangle\right| \\
&= \left| \left\langle \left(\sum_{k=0}^{n} \binom{n}{k} \left\{ (A+d_B)^k 1_{\mathscr{H}} \right\} B^{n-k} \right) x, y \right\rangle \right| \\
&\leq \sum_{k=0}^{n} \binom{n}{k} \left| \left\langle \left(\left\{ (A+d_B)^k 1_{\mathscr{H}} \right\} B^{n-k} \right) x, y \right\rangle \right| \\
&\leq \sum_{k=0}^{n} \binom{n}{k} \left\| f\left(\left| \left\{ (A+d_B)^k 1_{\mathscr{H}} \right\} B^{n-k} \right| \right) x \right\| \\
&\qquad \times \left\| g\left(\left| \left(B^{n-k}\right)^* \left\{ (A+d_B)^k 1_{\mathscr{H}} \right\}^* \right| \right) y \right\| \\
&\leq \sum_{k=0}^{n} \binom{n}{k} \left\langle f\left(\left| \left\{ (A+d_B)^k 1_{\mathscr{H}} \right\} B^{n-k} \right| \right) x, x \right\rangle^{1/2} \\
&\qquad \times \left\langle g\left(\left| \left(B^{n-k}\right)^* \left\{ (A+d_B)^k 1_{\mathscr{H}} \right\}^* \right| \right) y, y \right\rangle^{1/2}
\end{aligned}$$

$$\leq \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[\left\langle f\left(\left|\left\{(A+d_B)^k 1_{\mathscr{H}}\right\} B^{n-k}\right|\right)x,x\right\rangle\right.$$
$$\left.+\left\langle g\left(\left|\left(B^{n-k}\right)^*\left\{(A+d_B)^k 1_{\mathscr{H}}\right\}^*\right|\right)y,y\right\rangle\right],$$

where the last inequality follows by applying the AM-GM inequality. Hence, by letting $y=x$, we get

$$|\langle (A+B)^n x,x\rangle|$$
$$\leq \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[\left\langle f\left(\left|\left\{(A+d_B)^k 1_{\mathscr{H}}\right\} B^{n-k}\right|\right)x,x\right\rangle\right.$$
$$\left.+\left\langle g\left(\left|\left(B^{n-k}\right)^*\left\{(A+d_B)^k 1_{\mathscr{H}}\right\}^*\right|\right)x,x\right\rangle\right]$$
$$\leq \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left\langle\left\{f\left(\left|\left\{(A+d_B)^k 1_{\mathscr{H}}\right\} B^{n-k}\right|\right)\right.\right.$$
$$\left.\left.+g\left(\left|\left(B^{n-k}\right)^*\left\{(A+d_B)^k 1_{\mathscr{H}}\right\}^*\right|\right)\right\}x,x\right\rangle.$$

Taking the supremum over all unit vector $x\in\mathscr{H}$, we get the required result. □

Remark 2.12. Taking the supremum over all unit vectors $x,y\in\mathscr{H}$ in the proof of Theorem 2.11, we get the following power norm inequality:

$$\|(A+B)^n\| \leq \frac{1}{2}\sum_{k=0}^{n}\left\{\binom{n}{k}\left\|f\left(\left|\left\{(A+d_B)^k 1_{\mathscr{H}}\right\} B^{n-k}\right|\right)\right.\right.$$
$$\left.\left.+g\left(\left|\left(B^{n-k}\right)^*\left\{(A+d_B)^k 1_{\mathscr{H}}\right\}^*\right|\right)\right\|\right\}$$

for all $A,B\in\mathscr{B}(\mathscr{H})$.

Corollary 2.13. *Let $A,B\in\mathscr{B}(\mathscr{H})$. If f and g are both positive continuous and $f(t)g(t)=t$ for all $t\in[0,\infty)$, then*

$$w(A+B)$$
$$\leq \frac{1}{2}\left\|f(|B|)+g(|B^*|)+f(|A+d_BA|)+g(|(A^*+A^*d_B^*)|)\right\|, \quad (2.17)$$

where $d_B(A)=[B,A]=BA-AB$ and $d_B^*(A)=[B,A]^*=A^*B^*-B^*A^*$.

Proof. Setting $n=1$ in (2.16), we get that

$$w(A+B)\leq \frac{1}{2}\left\|f(|B|)+g(|B^*|)+f(|(A+d_B)1_{\mathscr{H}}|)+f(|(A+d_B)^* 1_{\mathscr{H}}|)\right\|.$$

Making use of (2.15), we have

$$(A+d_B)1_{\mathscr{H}}=A+D_1(B,A)=A+d_BA$$

and

$$(A+d_B)^* 1_{\mathscr{H}}=(A^*+d_B^*)1_{\mathscr{H}}=A^*+D_1(B^*,A^*)=A^*+A^*d_{B^*}.$$

Hence,

$$w\left(A+B\right) \le \frac{1}{2}\left\| f\left(\left|B\right|\right)+g\left(\left|B^*\right|\right)+f\left(\left|A+d_BA\right|\right)+g\left(\left|\left(A^*+A^*d_B^*\right)\right|\right)\right\|,$$

which gives the required result. □

Remark 2.14. As noted in Remark 2.12 and deduced in Corollary 2.13, we may observe that

$$\left\|A+B\right\| \le \frac{1}{2}\left\| f\left(\left|B\right|\right)+g\left(\left|B^*\right|\right)+f\left(\left|A+d_BA\right|\right)+g\left(\left|\left(A^*+A^*d_B^*\right)\right|\right)\right\|$$

for all $A,B\in\mathscr{B}\left(\mathscr{H}\right)$.

Corollary 2.15. *For $A,B\in\mathscr{B}\left(\mathscr{H}\right)$ that commute, if f and g are both positive continuous and $f(t)g(t)=t$ for all $t\in[0,\infty)$, then*

$$w\left(\left(A+B\right)^n\right) \le \frac{1}{2}\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\left\| f\left(\left|A^kB^{n-k}\right|\right)+g\left(\left|\left(B^{n-k}\right)^*\left(A^k\right)^*\right|\right)\right\|.$$

In particular,

$$w\left(A+B\right) \le \frac{1}{2}\left\| f\left(\left|B\right|\right)+g\left(\left|B^*\right|\right)+f\left(\left|A\right|\right)+g\left(\left|A^*\right|\right)\right\|.$$

Proof. Since $AB=BA$, then $d_B=0$ in (2.17). Alternatively, we may use (2.13) and proceed as in the proof of Theorem 2.11. □

Remark 2.16. As in the same way we previously remarked, for $A,B\in\mathscr{B}\left(\mathscr{H}\right)$ that commute, we have

$$\left\|\left(A+B\right)^n\right\| \le \frac{1}{2}\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\left\| f\left(\left|A^kB^{n-k}\right|\right)+g\left(\left|\left(B^{n-k}\right)^*\left(A^k\right)^*\right|\right)\right\|.$$

In particular,

$$\left\|A+B\right\| \le \frac{1}{2}\left\| f\left(\left|B\right|\right)+g\left(\left|B^*\right|\right)+f\left(\left|A\right|\right)+g\left(\left|A^*\right|\right)\right\|.$$

Setting $f\left(t\right)=t^{\alpha}$ and $g\left(t\right)=t^{1-\alpha}$ for all $\alpha\in[0,1]$, in the last inequality above, we get

$$\left\|A+B\right\| \le \frac{1}{2}\left\|\left|B\right|^{\alpha}+\left|B^*\right|^{1-\alpha}+\left|A\right|^{\alpha}+\left|A^*\right|^{1-\alpha}\right\|.$$

In special case, for $\alpha=\frac{1}{2}$, we have

$$\left\|A+B\right\| \le \frac{1}{2}\left\|\left|B\right|^{1/2}+\left|B^*\right|^{1/2}+\left|A\right|^{1/2}+\left|A^*\right|^{1/2}\right\|.$$

Corollary 2.17. *For $A\in\mathscr{B}\left(\mathscr{H}\right)$, if f and g are both positive continuous and $f(t)g(t)=t$ for all $t\in[0,\infty)$, then*

$$w\left(A^n\right) \le \frac{1}{2}\left(\left\| f\left(\left|A^n\right|\right)+g\left(\left|\left(A^n\right)^*\right|\right)\right\|\right). \tag{2.18}$$

Proof. Setting $B = 0$ in (2.16), we get the desired result. In another way, one may set $B = A$ in Corollary 2.15, so that we get

$$w(A^n) \le \frac{1}{2^{n+1}} \left\| f(|A^n|) + g(|(A^n)^*|) \right\| \cdot \sum_{k=0}^{n} \binom{n}{k},$$

but since $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, then we get the required result. □

Corollary 2.18. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$w(A^n) \le \frac{1}{2} \left(\left\| |A^n|^{\alpha} + |(A^n)^*|^{1-\alpha} \right\| \right).$$

In particular,

$$w(A) \le \frac{1}{2} \left(\left\| |A|^{\alpha} + |A^*|^{1-\alpha} \right\| \right). \tag{2.19}$$

Proof. Setting $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$ in (2.18), we get the desired result. □

Corollary 2.19. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$w(A) \le \frac{1}{2} \left(\left\| |A| + 1_{\mathscr{H}} \right\| \right) \le \frac{1}{4} \left(1 + \|A\| + \sqrt{(\|A\| - 1)^2 + 4\|A\|} \right).$$

Proof. Letting $\alpha = 1$ in (2.19), we get the first inequality. The second inequality follows by employing the norm estimates (see [13])

$$\|A + B\| \le \frac{1}{2} \left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\left\|A^{1/2}B^{1/2}\right\|^2} \right),$$

and then

$$\left\|A^{1/2}B^{1/2}\right\| \le \|AB\|^{1/2},$$

in the first inequality and use the fact that $\||A|\| = \|A\|$. In other words, we have

$$\begin{aligned} \left\| |A| + 1_{\mathscr{H}} \right\| &\le \frac{1}{2} \left(\||A|\| + \||1_{\mathscr{H}}|\| + \sqrt{(\||A^{1/2}|\| - 1)^2 + 4\left\| |A|^{1/2} 1_{\mathscr{H}} \right\|^2} \right) \\ &= \frac{1}{2} \left(1 + \|A\| + \sqrt{(\|A\| - 1)^2 + 4\|A\|} \right), \end{aligned}$$

which proves the required result. □

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