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SOME REMARKS ON CHAOS IN NONAUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. We introduce the concept of almost thick chaos and continuously almost thick transitivity for continuous maps and nonautonomous dynamical systems (NDS). We show that NDS $f_{1,\infty}$ is sensitive if it is thick transitive and syndetic. Under certain conditions, we show that NDS $(X, f_{1,\infty})$ generated by a sequence (f_n) of continuous maps on X converging uniformly to f is almost thick transitive if and only if (X, f) is almost thick transitive. Moreover, we prove that if $f_{1,\infty}$ is continuously almost thick transitive and syndetic, then it is strongly topologically ergodic. In addition, the relationship between the large deviations theorem and almost thick chaos is studied.

1. Introduction

In recent decades, chaos in dynamical systems has become very popular. Li and Yorke [18] introduced the term of chaos for the first time. A new description of chaos was proposed by Devaney [8]; a map f is said to be chaotic in the sense of Devaney if f is topologically transitive, has dense set of periodic points, and is sensitive. Later on, Banks [5] showed that the transitivity and density of periodic points imply sensitivity. Huang and Ye [14] showed that chaos in the sense of Devaney is stronger than that in the sense of Li–Yorke. Glasner and Weiss [11] got a stronger result, that is, any topologically transitive and nonminimal dynamical system whose almost all periodic points are dense in the phase space is sensitive.

Moothathu [21] proposed three stronger forms of sensitivity: Syndetic sensitivity, cofinite sensitivity, and multi-sensitivity. He proved that if f is sensitive and if the set of minimal points of f is dense in X, where X is a compact metric space,

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then f is syndetically sensitive. Furthermore, he showed that if $f : [0,1] \to [0,1]$ is sensitive, then f is cofinitely sensitive. Gu [12] studied some relationships between stochastic and topological properties of dynamical systems and showed that if a continuous map f from a compact metric space X into itself is a strongly topologically ergodic map satisfying the large deviations theorem, then it has sensitive dependence on initial conditions. According to this, Li [17] introduced the concept of ergodic sensitivity and proved that if a topologically strongly ergodic map satisfying the large deviations theorem, then it is ergodically sensitive. Improving this result, Wu and Chen [26] showed that every strongly topologically ergodic dynamical system satisfying the large deviations theorem is syndetically sensitive. Moreover, ergodicity of every transformation on a Borel probability measure space satisfying the large deviations theorem was studied.

The analogous definition of chaos in the sense of Li–Yorke for a nonautonomous dynamical system (NDS) was stated by Shi and Chen [23], and the relationship between chaos and topological entropy was studied. Dvořáková [9] considered NDS $(I, f_{1,\infty})$ given by the sequence $\{f_n\}_{n\in\mathbb{Z}_+}$ of surjective continuous maps $f_n : I \to I$ converging uniformly to a map $f : I \to I$, where I is closed unit interval $[0,1]$ and studied some aspects of chaotic behavior $f_{1,\infty}$ and f. Huang, Shi, and Zhang [13] introduced the concept of cofinitely sensitivity for nonautonomous discrete systems and proved that the topological mixing property implies the cofinite sensitivity for these systems and the strong mixing implies the cofinite sensitivity for measure preserving nonautonomous systems with full-measure.

Baliberea and Oprocha [4] devoted their study to chaotic properties of NDS such as Li–Yorke chaos and the relation between topologically weak mixing and topological entropy. Štefánková [24] showed that if f is chaotic in the sense of Li–Yorke, then NDS $f_{1,\infty}$ is Li–Yorke chaotic, when surjective continuous maps $\{f_n\}_{n\geqslant 1}$ converge uniformly to map f. Moreover NDS $f_{1,\infty}$ inherits infinite ω-limit sets of f, when f has zero topological entropy. Canovas [7] studied the limit behavior of sequences of the form $(f_n \circ \cdots \circ f_1)(x)$, $x \in [0,1]$ and showed that if $f_{1,\infty}$ is a sequence of surjective continuous interval maps converging uniformly to a map f and if the map f has positive topological entropy, then $f_{1,\infty}$ is Li–Yorke chaotic and furthermore, if the map f has the shadowing property, then $f_{1,\infty}$ is Li–Yorke chaotic if and only if f is Li–Yorke chaotic.

In this paper, we introduce new notions named almost thick chaotic and continuously almost thick transitivity for continuous maps and nonautonomous dynamical systems. The rest of this paper is organized as follows. In Section 2, we provide some basic definitions and notations and define the notion of almost thick chaos for continuous maps and NDS that we will consider. Section 3 is devoted to the proof of the existence of an almost thick chaotic NDS and theorems to achieve results like sensitivity and almost thick sensitivity for these systems. We constitute a necessary and sufficient condition for NDS $(X, f_{1,\infty})$ generated by a sequence (f_n) of continuous maps on X to be almost thick transitive. In Section 4, the concept of continuously almost thick transitivity for NDS is stated, and by an example, the existence of such systems is demonstrated. Also it is shown that if $f_{1,\infty}$ is continuously almost thick transitive and syndetic, then $f_{1,\infty}$ is strongly topologically ergodic. Also, the concept of the large deviations theorem is stated for a dynamical system (X, f) and it is shown that if f is syndetic, continuously almost thick transitive and satisfies the large deviations theorem, then f is almost thick chaotic.

2. PRELIMINARIES

Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous surjective map. A sequence $\{x_n\}_{n=0}^{\infty}$ is called an *orbit* of f, denoted by $O(x, f)$, if $x_{n+1} = f(x_n)$ for each $n \in \mathbb{Z}_+$, and we call it a *δ-pseudo-orbit* of f if

$$d(f(x_i), x_{i+1}) < \delta \quad \text{for all } i \in \mathbb{Z}_+.$$

A continuous map f is said to have the *shadowing property* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is ε-shadowed by an orbit of some point $y \in X$, that is,

$$d(f^n(y), x_n) < \varepsilon \quad \text{for all } n \in \mathbb{Z}_+ \ \text{(see [15])}.$$

Let U and V be two nonempty open subsets of X, and consider

$$N_f(U, V) = \{n \in \mathbb{Z}_+ : \ f^n(U) \cap V \neq \varnothing\}.$$

A map f is called *topologically transitive* if for any nonempty open subsets U and V of X, we have $N_f(U, V) \neq \emptyset$. Indeed f is *topologically weak mixing* if $f \times f$ is topologically transitive, and f is *topologically mixing* if $N_f(U, V)$ is cofinite. Also f is called *totally transitive*, if f^n is topologically transitive for every $n \in \mathbb{N}$; see [15].

Moreover f is called *topologically ergodic*, if for any nonempty open subsets U and V of X, $N_f(U, V)$ has positive upper density, that is,

$$D\big(N_f(U, V)\big) = \limsup_{n\to\infty} \frac{1}{n}\text{card}\big\{N_f(U, V) \cap \{0, \dots, n-1\}\big\} > 0,$$

where card(A) denotes the number of members of the finite set A.

Also f is called *strongly topologically ergodic*, if for any nonempty open subsets U and V of X, $D\big(N_f(U, V)\big) = 1$; see [17]. A subset I of $\mathbb{Z}_+$ is called *syndetic*, if there exists $m \in \mathbb{N}$ such that $[n, n+m] \cap I \neq \emptyset$ for all $n \in \mathbb{Z}_+$.

A map f is called *syndetic*, if for any two nonempty open subsets U and V of X, $N_f(U, V)$ is syndetic; see [21]. For $\delta > 0$, put

$$S_f(U, \delta) = \{n \in \mathbb{Z}_+ \big| \exists x, y \in U \ \ s.t. \ \ d(f^n(x), f^n(y)) \geq \delta\}.$$

An *interval* in $N_f(U, V)$ denoted by $[m, n]$ is the set of all integers between $m-1$ and $n+1$ that belong to $N_f(U, V)$. An interval in $S_f(U, \delta)$ is defined similarly. The length of interval $[m, n]$ is $n - m$.

We say that f is *almost thick transitive*, if for any nonempty pair of open sets U, V, $N_f(U, V)$ contains infinitely many intervals of length greater than 2, and it is *thick transitive*, if for any $L > 0$ and any nonempty open sets $U, V, N_f(U, V)$ contains intervals of length at least L.

We say that f has *sensitive dependence on initial conditions*, if there is $\delta > 0$ such that for any nonempty open set $U \subset X$, $S_f(U, \delta) \neq \emptyset$; see [11]. Also f is said to be *almost thick sensitive*, if there exists $\delta > 0$ such that for every nonempty open set $U, S_f(U, \delta)$ contains infinitely many intervals of length greater than 2, and it is *almost thick chaotic* if it is almost thick transitive and almost thick sensitive. Here we generalize the above notations and concepts to nonautonomous dynamical systems.

Let X be a compact metric space and let $\{f_i\}_{i=1}^{\infty}$ be a sequence of continuous maps on X.

For $k, n \in \mathbb{N}$, we write

$$f_k^0 := id, \quad f_k^n := f_{k+n} \circ f_{k+n-1} \circ \cdots \circ f_{k+1} \circ f_k.$$

The *orbit* of a point $x \in X$ is the sequence $\{f_1^n(x)\}_{n\in\mathbb{N}}$. Denote by $f_{1,\infty}$ the sequence $\{f_i\}_{i=1}^{\infty}$ and we call $(X, f_{1,\infty})$ an NDS (*nonoutonomous dynamical system*); see [6]. We say that an NDS $f_{1,\infty}$ is constructed by $\{g_1, g_2, \ldots, g_k\}$ if $f_i \in \{g_1, g_2, \ldots, g_k\}$ for every $i \in \mathbb{N}$.

Moreover $f_{1,\infty}$ is called *topologically transitive* if for any nonempty open sets U, V, there exists $n \in \mathbb{N}$ such that $f_1^n(U) \cap V \neq \emptyset$. put

$$N_{f_{1,\infty}}(U, V) = \{n \in \mathbb{Z}_+ \big| f_1^n(U) \cap V \neq \emptyset\}.$$

In fact, $f_{1,\infty}$ is *topologically transitive* if $N_{f_{1,\infty}}(U, V) \neq \emptyset$, for any two nonempty open sets U and V; see [23, 24]. An NDS $f_{1,\infty}$ is called *topologically mixing* if for any nonempty open sets U and V, $N_{f_{1,\infty}}(U, V)$ is cofinite [25] and is called *strongly topologically ergodic* if

$$\limsup_{n\to\infty} \frac{1}{n}\mathrm{card}\big\{N_{f_{1,\infty}}(U, V) \cap \{0, \ldots, n-1\}\big\} = 1.$$

Also $f_{1,\infty}$ is called *syndetic* if for any two nonempty open subsets U and V of X, $N_{f_{1,\infty}}(U, V)$ is syndetic [25].

For $\delta > 0$, a sequence $\{x_i\}_{i=1}^{\infty}$ is a *δ-pseudo orbit* for NDS $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$, if $d(f_i(x_i), x_{i+1}) < \delta$. We say that for $\varepsilon > 0, x$ *ε-shadows* a sequence $\{x_i\}_{i=0}^{\infty}$ if $d(f_1^n(x), x_n) < \varepsilon$ for all $n \geq 0$. An NDS $f_{1,\infty}$ is said to have the *shadowing property* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo orbit is ε-shadowed by some point of X.

For $\delta > 0$, put

$$S_{f_{1,\infty}}(U, \delta) = \{n \in \mathbb{Z}_+ \big| \exists x, y \in U \quad s.t. \quad d(f_1^n(x), f_1^n(y)) \geq \delta\}.$$

An interval in $N_{f_{1,\infty}}(U, V)$ denoted by $[m, n]$ is the set of all integers between $m-1$ and $n+1$ that belong to $N_{f_{1,\infty}}(U, V)$. An interval in $S_{f_{1,\infty}}(U, \delta)$ is defined similarly.

A nonoautonomous dynamical system $f_{1,\infty}$ is *almost thick transitive* if for any nonempty pair of open sets U, V, $N_{f_{1,\infty}}(U, V)$ contains infinitely many intervals of length greater than 2, and it is *thick transitive* if for any $L > 0$ and any nonempty open sets $U, V, N_{f_{1,\infty}}(U, V)$ contains intervals of length at least L.

An NDS $f_{1,\infty}$ has *sensitive dependence on initial conditions*, if there is $\delta > 0$ such that for any nonempty open set $U \subset X$, $S_{f_{1,\infty}}(U, \delta) \neq \emptyset$; see [25]. Moreover $f_{1,\infty}$ is said to be *almost thick sensitive* if there exists $\delta > 0$ such that for every nonempty open set $U, S_{f_{1,\infty}}(U, \delta)$ contains infinitely many intervals of length greater than 2, and it is *almost thick chaotic* if it is almost thick transitive and almost thick sensitive.

3. Almost thick chaotic and almost thick transitive NDS

Before all, the chaotic properties of autonomous dynamical systems and their relation with other topological properties are studied. There are different kinds of chaos in dynamical systems such as Devaney chaos, Auslander–Yorke chaos, Li–Yorke chaos, distributional chaos, topological chaos, and P-chaos. We can point to the relation between Devaney chaos and topological properties such as sensitivity, topologically mixing, and the specification property. Most of the authors agree in one point, chaotic dynamics must show sensitive dependence on initial conditions. The author in [20] focused on functions defined on the interval $I = [0,1]$ and showed that if f is sensitive, then f is chaotic in the sense of Devaney on a nonempty interior subset of I and also implies that the topological entropy of $f : I \to I$ is positive. Then f has topological chaos. Another important property that makes discrete dynamical systems to be chaotic, is topologically mixing property, which implies topological transitivity and sensitivity for autonomous dynamical systems. It is shown in [2] that if a map f has the specification property, then f is topologically mixing, but the converse is not true. Also, the set of all periodic points for f is dense, and f has positive topological entropy. Thus f is chaotic in the sense of Devaney. If a transitive system has the shadowing property and a fixed point, then it has the specification property and thus is topologically mixing; see [15]. Every P-chaotic map from a continuum (nondegenerate compact connected metric space) to itself is topologically mixing; see [3]. Furthermore, they are chaotic in the sense of Devaney and exhibits distributional chaos. In this section, we are interested in achieving the relation between almost thick chaotic property and other topological properties.

Here, we are interesting to study the relationship between thick transitivity and topologically mixing property for NDS. Thus naturally we have the following question.

Question: Is every thick transitive NDS $f_{1,\infty}$ with the shadowing property, topologically mixing?

In the following proposition, we answer the above question for $f_{1,\infty} = \{f_n\}_{n=1}^{\infty}$, when $f_n = f$ for every $n \in \mathbb{N}$.

Proposition 3.1. *Let f be a continuous map on a compact metric space X. If f is thick transitive and has the shadowing property, then it is topologically mixing.*

Proof. Let $k \in \mathbb{N}$ and let U and V be nonempty open subsets of X. There exists $m \in \mathbb{N}$ such that $[m, m+k+1] \subset N_f(U,V)$, since f is thick transitive. Let L be a positive integer such that $m \leq Lk \leq m+k+1$. Then $Lk \in N_f(U,V)$. This shows that f^k is topologically transitive, and hence f is totally transitive. Since f has the shadowing property, it is topologically mixing from [15, Theorem 1]. □

There is an autonomous dynamical system (X, f) that f is almost thick transitive but is not topologically mixing. Indeed, if f is topologically weak mixing, then for any two nonempty open subsets $U, V \subset X$, $N_f(U,V)$ is thick [10]. Lau and Zame [16] introduced spacing shifts to provide examples of maps that are topologically weak mixing but not topologically mixing. Hence their example is thick transitive but not topologically mixing, and as an application of the above proposition that example does not have the shadowing property. For more details about spacing shifts, we refer the reader to [1].

In the following, we state an example of almost thick chaotic nonautonomous dynamical systems.

Example 3.2. Let X be a compact metric space, let $g_0 : X \to X$ be a topologically transitive, sensitive continuous map, and let $g_1 = Id$ be the identity map. Then there exists an NDS $f_{1,\infty}$ constructed by $\{g_0, g_1\}$ that is almost thick chaotic. Indeed by the assumption, g_0 is topologically transitive and sensitive, but there maybe two open subsets U and V of X such that $N_f(U, V)$ contains no intervals. In our construction, we consider the positive integer $m \geq 2$ as the length of the intervals in the definition of almost thick chaotic. Let $\{U_i\}_{i=1}^{\infty}$ be a countable basis of X. Since g_0 is topologically transitive and sensitive, for any U_i and U_j, there exists $n_{i,j} \in \mathbb{Z}_+$ such that $n_{i,j} \in N_{g_0}(U_i, U_j)$. By induction, consider $\{n_{1,j}\}_{j=1}^{\infty}$ as follows. Choose $n_{1,1} \in N_{g_0}(U_1, U_1)$. Suppose that $n_{1,j}$ is selected. Consider $n_{1,j+1} \in N_{g_0}(U_1, U_{j+1})$ such that $n_{1,j} < n_{1,j+1}$. For every $i > 1, j \geq 1$, choose $n_{i,j} \in N_{g_0}(U_i, U_j)$ such that $n_{1,j} < n_{i,j}$. By the above construction, for any two nonempty open subsets U and V of X, there are infinitely many distinct $n_{i,j}$ such that $n_{i,j} \in N_{g_0}(U, V)$.

There exists $n_i \in \mathbb{N}$ such that $n_i \in S_{g_0}(U_i, \delta)$, where δ is sensitivity constant for g_0. We can reorder $\{n_{i,j}\}$ and $\{n_i\}$ to obtain two sequences $\{m_i\}_{i=1}^{\infty}$ and $\{m'_i\}_{i=1}^{\infty}$, respectively, such that $m_i < m_j$ and $m'_i < m'_j$, for every $i < j$. Assume that $m_1 < m'_1$. Put

$$m_{j_1} = \max\{m_j \big| m_j \leq m'_1\}.$$

To construct $f_{1,\infty}$, for $x \in X$, we have the following equation:

$$f_1^{m'_1+j_1 m}(x) := \underbrace{g_0 \circ \cdots \circ g_0}_{m'_1} \circ \cdots \circ \underbrace{g_1 \circ \cdots \circ g_1}_{m} \circ \underbrace{g_0 \circ \cdots \circ g_0}_{m_{j_1} - m_{(j_1-1)}} \circ \cdots$$
$$\circ \underbrace{g_1 \circ \cdots \circ g_1}_{m} \circ \underbrace{g_0 \circ \cdots \circ g_0}_{m_2 - m_1} \circ \underbrace{g_1 \circ \cdots \circ g_1}_{m} \circ \underbrace{g_0 \circ \cdots \circ g_0}_{m_1}(x).$$

Indeed, there are $U_{i_1}, U_{j_1} \in \{U_i\}_{i=1}^{\infty}$ such that

$$f_1^{m_1}(U_{i_1}) \cap U_{j_1} = (g_0)^{m_1}(U_{i_1}) \cap U_{j_1} \neq \emptyset,$$

and also there are $U_{i_2}, U_{j_2} \in \{U_i\}_{i=1}^{\infty}$ such that for any $1 \leq i \leq m$,

$$f_1^{m_2+i}(U_{i_2}) \cap U_{j_2} = (g_0)^{m_2}(U_{i_2}) \cap U_{j_2} \neq \emptyset.$$

so $[m_2, m_2 + m] \subset N_{f_{1,\infty}}(U_{i_2}, U_{j_2})$. Continuing this process, there are $U_{i_{j_1}}, U_{j_{j_1}}$, $U_{i'}, U_{j'} \in \{U_i\}_{i=1}^{\infty}$ such that

$$f_1^{m_{j_1}+(j_1-1)i}(U_{i_{j_1}}) \cap U_{j_{j_1}} = (g_0)^{m_{j_1}}(U_{i_{j_1}}) \cap U_{j_{j_1}} \neq \emptyset,$$

so $[m_{j_1} + (j_1 - 1)m, m_{j_1} + j_1 m] \subset N_{f_{1,\infty}}(U_{i_{j_1}}) \cap U_{j_{j_1}}$. Thus

$$f_1^{m'_1+j_1 m}(U_{i'}) \cap U_{j'} = (g_0)^{m'_1}(U_{i'}) \cap U_{j'} \neq \emptyset.$$

In general, for any natural number n, put $m_{j_n} = \max\{m_j \big| m'_{n-1} \leq m_j \leq m'_n\}$ if $[m'_{n-1}, m'_n] \cap \{m_i\} \neq \emptyset$, and $m_{j_n} = m'_n$ if $[m'_{n-1}, m'_n] \cap \{m_i\} = \emptyset$.

Let

$$f_i = \begin{cases} g_0, & \big(j_{n-1}+(n-1)\big)m + m'_{n-1} + 1 \le i \le \big(j_{n-1}+(n-1)\big)m + m_{j_{n-1}+1}, \\ g_1, & (j_{n-1}+n-1)m + m_{j_{n-1}+1} + 1 \le i \le (j_{n-1}+n)m + m_{j_{n-1}+1}, \\ \vdots & \\ g_0, & (j_n+n-2)m + m_{j_n-1} + 1 \le i \le (j_n+n-2)m + m_{j_n}, \\ g_1, & (j_n+n-2)m + m_{j_n} + 1 \le i \le (j_n+n-1)m + m_{j_n}, \\ g_0, & (j_n+n-1)m + m_{j_n} + 1 \le i \le (j_n+n-1)m + m'_n, \\ g_1, & (j_n+n-1)m + m'_n + 1 \le i \le (j_n+n)m + m'_n. \end{cases}$$

Let U and V be two open nonempty subsets of X. There exists $\{U_{i_k}, U_{j_k}\}_{k=1}^{\infty} \subseteq \{U_i\}_{i=1}^{\infty}$ such that $U_{i_k} \subset U$, $U_{j_k} \subset V$. For any k, there exists $m_{L_k} \in \{m_i\}_{i=1}^{\infty}$ such that $m_{L_k} = n_{i_k, j_k} \in N(U_{i_k}, V_{j_k})$. By the construction of $f_{1,\infty}$, $[m_{L_k}, m_{L_k}+m] \subset N_{f_{1,\infty}}(U_{i_k}, V_{j_k}) \subset N_{f_{1,\infty}}(U, V)$. Hence $f_{1,\infty}$ is almost thick transitive. The proof of almost thick sensitivity is similar. Indeed, if U is a nonempty open subset of X, there exists $\{U_{i_k}\}_{k=1}^{\infty} \subseteq \{U_i\}_{i=1}^{\infty}$ such that $U_{i_k} \subset U$. For any k, there exists $m'_{L_k} \in \{m'_i\}_{i=1}^{\infty}$ such that $m'_{L_k} = n_{i_k} \in S(U_{i_k}, \delta)$. By the construction of $f_{1,\infty}$, we have $[m'_{L_k}, m'_{L_k}+m] \subset S_{f_{1,\infty}}(U_{i_k}, \delta) \subset S_{f_{1,\infty}}(U, \delta)$. Therefore $f_{1,\infty}$ is almost thick sensitive.

Since due to the definition of almost thick transitivity, each topologically mixing NDS is almost thick transitive, and from [13, Theorem 3.3], each topologically mixing NDS is cofinitely sensitive. Hence it is almost thick chaotic, and then we can give an example of a topologically mixing NDS, which will also be an example of an almost thick chaotic NDS. In the following example from [27], authors constructed a finitely generated NDS that is topologically mixing.

Let the space $\Sigma = \{0,1\}^{\mathbb{N}} = \{x_1x_2x_3\cdots : x_i \in \{0,1\} \mathit{for\, any}\, i \in \mathbb{N}\}$ be the sequence space on two symbols, that is, the set of all infinite sequences of "0"s and "1"s with the product metric

$$d(x,y) = \sum_{n=1}^{+\infty} \frac{|\, x_n - y_n \,|}{2^n},$$

for any $x = x_1x_2\cdots$, $y = y_1y_2\cdots \in \Sigma$. Also $w = w_1w_1\cdots w_n \in \{0,1\}^n$ is called a *word of length* n, for any $n \in \mathbb{N}$. The concatenation of two words $a = a_1a_2\cdots a_n$ and $b = b_1b_2\cdots b_m$ is the word $ab = a_1a_2\cdots a_nb_1b_2\cdots b_m$. Let $A = a_1a_2\cdots a_n$ be a word. The *inverse* of A is $\overline{A} = \overline{a_1a_2\cdots a_n}$, where

$$\bar{a_i} = \begin{cases} 0, & a_i = 1, \\ 1, & a_i = 0. \end{cases}$$

Define the *shift map* $\sigma : (\Sigma, d) \to (\Sigma, d)$ by $\sigma(x_0x_1x_2\cdots) = x_1x_2x_3\cdots$.

Example 3.3. Let $f_0 = \sigma$ and let $f_1 = \sigma^2$. Take $A_0 = 0 \in \{0,1\}$, $A_1 = \bar{A}_0 = 1 \in \{0,1\}$, and $A_n = \overline{A_0A_1\cdots A_{n-1}}$ for all $n \ge 2$, and take $t = t_1t_2t_3\cdots = A_0A_1A_2\cdots A_n\cdots$ (classical Thue–Morse-sequence).

Now, for any $i \in \mathbb{N}$, let $F_i = f_{t_i}$ and let $F_{1,\infty} = \{F_i\}_{i=1}^{\infty}$. Then $(\Sigma, F_{1,\infty})$ is finitely generated NDS. Authors in [27] showed that $F_{1,\infty}$ is topologically mixing.

Example 3.4. Let S^1 be the unit circle, let the sequence $f_n : S^1 \to S^1$ be given by $f_n(e^{i\theta}) = e^{i(\frac{n+1}{n})\theta}$, for each $n \geq 1$, and let $f_{1,\infty} = \{f_n\}_{n=1}^{\infty}$. Authors in [19, Example 2.2] showed that $f_{1,\infty}$ has topological mixing property. Hence it is an almost thick chaotic NDS.

Remark 3.5. Let $\{L_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ be a sequence of positive integers such that $\lim_{i\to+\infty} L_i = +\infty$. If in Example (3.2), m is replaced by L_i in every step, then it is an example of a thick transitive NDS.

According to the definitions, we have the following implications:

$$\text{Topologically mixing} \Rightarrow \text{thick transitivity}$$
$$\Rightarrow \text{almost thick transitivity} \Rightarrow \text{topological transitivity.}$$

The following remark shows that opposite implications are not true.

Remark 3.6. Suppose that g_0 in Example 3.2 is topologically transitive but not topologically mixing. Then similar to Remark 3.5 by modifying the proof of Example 3.2, we have an example of a thick transitive NDS that is not topologically mixing. By the construction of NDS in Example 3.2, one can see that there is no interval of large length in $N_{f_{1,\infty}}(U, V)$ for some U and V, if g_0 is not almost thick transitive. Finally we will see that Remark 3.11 shows that transitivity does not imply almost thick transitivity.

Now, we investigate the relation between sensitivity and almost thick sensitivity for NDS. In the following proposition, we state and prove that sensitivity of the map f implies almost thick sensitivity. In order to prove Proposition 3.8, first we express and prove the following lemma.

Lemma 3.7. *Let f be a continuous map on a compact metric space X. If f is sensitive with sensitivity constant δ and $m \in \mathbb{N}$ is fixed, then for any open neighborhood U of $x \in X$, there exists $0 < \delta' < \delta$ such that for any $n \in S_f(U, \delta)$, we have $[n - m, n] \subset S_f(U, \delta')$.*

Proof. Since f is continuous, then for the above $\delta > 0$, there exists $0 < \delta' < \delta$ such that

$$d(x, y) < \delta' \Longrightarrow d(f^i(x), f^i(y)) < \delta, \tag{3.1}$$

for each $1 \leq i \leq m$ and any $x, y \in X$. If there exists $j \in [n - m, n]$ such that $j \notin S_f(U, \delta')$, then $d(f^j(x), f^j(y)) < \delta'$. By (3.1), we have

$$d(f^{i+j}(x), f^{i+j}(y)) < \delta,$$

for any $1 \leq i \leq m$. Since $n - m + 1 \leq i + j \leq n + m$ and $n \in [n - m + 1, n + m]$, it contradicts with $n \in S_f(U, \delta)$. Then we have $[n - m, n] \subset S_f(U, \delta')$.

□

Proposition 3.8. *Let f be a continuous map on a compact metric space X. If f is sensitive, then it is almost thick sensitive.*

Proof. Suppose that δ is the sensitivity constant of f, and let $m \in \mathbb{N}$ be fixed. By Lemma 3.7, for any open neighborhood U of $x \in X$, there exists $0 < \delta' < \delta$ such that for any $n \in S_f(U, \delta)$, we have $[n - m, n] \subset S_f(U, \delta')$. It is sufficient to prove that for any open neighborhood U of $x \in X$, the set $S_f(U,\delta)$ is infinite. Since f is continuous, then for the above $\delta > 0$, there exists $0 < \delta' < \delta$ such that (3.1) is satisfied for any $1 \leq i \leq m$. By sensitivity of f, we have for any $x \in X$ and any open neighborhood U of x with diameter less than δ', there exist $y \in U$ and $n_1 \in \mathbb{N}$ such that

$$d(f^{n_1}(x), f^{n_1}(y)) > \delta.$$

By (3.1), we have $n_1 > m$. By replacing n_1 instead of m, there exists $0 < \delta_1 < \delta'$ such that $d(x,y) < \delta_1$ implies that $d(f^i(x), f^i(y)) < \delta$ for each $1 \leq i \leq n_1$ and any $x, y \in X$. Now, we consider $U_1 \subset U$ as its diameter less than δ_1. Since f is sensitive, there exists $n_2 \in S_f(U_1, \delta)$ and so $n_2 > n_1$. Since that $U_1 \subset U$, so $S_f(U_1,\delta) \subset S_f(U,\delta)$ and $n_2 \in S_f(U,\delta)$. By maintaining this process and replacing n_k instead of n_{k-1} for $k > 1$, there exists $0 < \delta_k < \delta'$ such that $d(x,y) < \delta_k$ implies that $d(f^i(x), f^i(y)) < \delta$ for each $1 \leq i \leq n_k$ and any $x, y \in X$. Hence, we can choose the neighborhood $U_k \subset U_{k-1} \subset \cdots \subset U_1 \subset U$ such that its diameter is less than δ_k that tends to zero when $k \to \infty$. Again, by sensitivity of f, there is $n_{k+1} \in S_f(U_k,\delta)$ such that $n_{k+1} > n_k$ and $n_{k+1} \in S_f(U,\delta)$. Therefore, the number of n_k, $n_k > m$ that $n_k \in S_f(U,\delta)$ is infinite. By Lemma 3.7, for any $n \in S_f(U,\delta)$, we have $[n-m, n] \subset S_f(U,\delta')$ and the number of such intervals is infinite, then f is almost thick sensitive. □

Question: Is Proposition 3.8 true for an NDS? The following proposition gives a positive answer to the above question, if NDS is constructed by a finite number of continuous maps, but we cannot judge in general case. First, we state and prove the following lemma.

Lemma 3.9. *Let $f_{1,\infty}$ be an NDS on a compact metric space X constructed by continuous maps $g_1, \ldots, g_k$. If $f_{1,\infty}$ is sensitive with sensitivity constant δ and $m \in \mathbb{N}$ is fixed ($m \neq 1$), then for any open neighborhood U of $x \in X$, there exists $0 < \delta_1 < \delta$ such that for any $n \in S_{f_{1,\infty}}(U,\delta)$, we have $[n-m+1, n] \subset S_{f_{1,\infty}}(U,\delta_1)$.*

Proof. Since g_j is continuous for $j = 1, \ldots, k$, then for the above $\delta > 0$, there exist $0 < \delta_1 < \delta_2 < \cdots < \delta_m = \delta$ such that

$$d(x,y) < \delta_i \Longrightarrow d(g_j(x), g_j(y)) < \delta_{i+1},$$

for each $1 \leq i \leq m-1$ and $1 \leq j \leq k$ and any $x, y \in X$. Since $f_{1,\infty}$ is constructed by $g_1, \ldots, g_k$, we have

$$d(x,y) < \delta_1 \Longrightarrow d(f_t^l(x), f_t^l(y)) < \delta_m \tag{3.2}$$

for each $0 \leq l \leq m-1$ and any $x, y \in X$ and $t > 0$. If there exists $j \in [n-m+1, n]$ such that $j \notin S_{1,\infty}(U,\delta_1)$, then $d(f_1^j(x), f_1^j(y)) < \delta_1$. By (3.2), we have

$$d(f_{j+1}^{i+j}(f_1^j(x)), f_{j+1}^{i+j}(f_1^j(y))) = d(f_1^{i+j}(x), f_1^{i+j}(y)) < \delta,$$

for any $1 \leq i \leq m-1$. On the other hand, $n-m+2 \leq i+j \leq n+m-1$ and $n \in [n-m+2, n+m-1]$, then it contradicts with $n \in S_{f_{1,\infty}}(U,\delta)$ and hence $[n-m+1, n] \subset S_{f_{1,\infty}}(U,\delta_1)$. □

Proposition 3.10. *Let $f_{1,\infty}$ be an NDS on a compact metric space X constructed by continuous maps $g_1, \ldots, g_k$. If $f_{1,\infty}$ is sensitive, then $f_{1,\infty}$ is almost thick sensitive.*

Proof. Suppose that δ is the sensitivity constant of $f_{1,\infty}$, and let $m \in \mathbb{N}$ be fixed ($m \neq 1$). By Lemma 3.9, for any open neighborhood U of $x \in X$, there exists $0 < \delta_1 < \delta$ such that for any $n \in S_{f_{1,\infty}}(U,\delta)$, we have $[n-m+1,n] \subset S_{f_{1,\infty}}(U,\delta_1)$. It is sufficient to prove that for any open neighborhood U of $x \in X$, $S_{f_{1,\infty}}(U,\delta)$ is infinite. By sensitivity of $f_{1,\infty}$, we have for any $x \in X$ and open neighborhood U of x with diameter less than δ_1, there exist $y \in U$ and $n_1 \in \mathbb{N}$ such that

$$d(f_1^{n_1}(x), f_1^{n_1}(y)) > \delta_m = \delta.$$

By using (3.2), we have $n_1 > m-1$. By replacing n_1 instead of $m-1$, there exists $0 < \delta_1' < \delta_1$ such that $d(x,y) < \delta_1'$ implies that $d(f_1^i(x), f_1^i(y)) < \delta$ for each $1 \leq i \leq n_1$ and any $x, y \in X$. Now, we consider $U_1 \subset U$ as its diameter is less than δ_1'. Since $f_{1,\infty}$ is sensitive, there exists $n_2 \in S_{f_{1,\infty}}(U_1,\delta)$ and so $n_2 > n_1$. Since $U_1 \subset U$, so $S_{f_{1,\infty}}(U_1,\delta) \subset S_{f_{1,\infty}}(U,\delta)$ and $n_2 \in S_{f_{1,\infty}}(U,\delta)$. By continuing this process and replacing n_k instead of n_{k-1} for $k > 1$, there exists $0 < \delta_k' < \delta_1$ such that $d(x,y) < \delta_k'$ implies that $d(f_1^i(x), f_1^i(y)) < \delta$ for each $1 \leq i \leq n_k$ and any $x, y \in X$. Hence we can choose the neighborhood $U_k \subset U_{k-1} \subset \cdots \subset U_1 \subset U$ such that its diameter is less than δ_k' that tends to zero when $k \to \infty$. Again, by sensitivity of $f_{1,\infty}$, there is $n_{k+1} \in S_{f_{1,\infty}}(U_k,\delta)$ such that $n_{k+1} > n_k$ and $n_{k+1} \in S_{f_{1,\infty}}(U,\delta)$. Therefore, the number of n_k, $n_k > m-1$, that $n_k \in S_{f_{1,\infty}}(U,\delta)$ is infinite. By Lemma 3.9, for any $n \in S_{f_{1,\infty}}(U,\delta)$, we have $[n-m+1,n] \subset S_{f_{1,\infty}}(U,\delta_1)$ and the number of such intervals is infinite, then $f_{1,\infty}$ is almost thick sensitive. □

Remark 3.11. Assume that f is topologically transitive and that f^2 is not topologically transitive. Then there are nonempty open sets U and V such that $f^n(U) \cap V = \emptyset$ for every even number n. Hence $N_f(U,V)$ contains no intervals and so f is not almost thick transitive. Therefore in spite of sensitivity, Proposition 3.8 is not true for transitivity.

Proposition 3.12. *Let $f_{1,\infty}$ be an NDS on a compact metric space X. If $f_{1,\infty}$ is thick transitive and syndetic, then $f_{1,\infty}$ has sensitive dependence on initial conditions.*

Proof. Let $0 < \delta < \frac{\text{diam}(X)}{6}$. Then for every point $x \in X$, there exists $y \in X$ such that $d(x,y) > 3\delta$. Hence for every nonempty open set V with $\text{diam}(V) < \delta$, there exists a nonempty open set U such that $d(U,V) > \delta$, where $d(U,V) = \inf_{u\in U, v\in V} d(u,v)$. Since $f_{1,\infty}$ is syndetic, there exists $m \in \mathbb{N}$ such that $[n,n+m] \cap N_{f_{1,\infty}}(V,V) \neq \emptyset$ for all $n \in \mathbb{Z}_+$. Since $f_{1,\infty}$ is thick transitive, there exists $K \in \mathbb{Z}_+$ such that $[K,K+m] \subset N_{f_{1,\infty}}(V,U)$. Choose $n_0 \in [K,K+m] \cap N_{f_{1,\infty}}(V,V)$. Hence $f_1^{n_0}(V) \cap U \neq \emptyset$ and $f_1^{n_0}(V) \cap V \neq \emptyset$. This shows that $n_0 \in S_{f_{1,\infty}}(V,\delta)$. Hence $f_{1,\infty}$ has sensitive dependence on initial conditions. □

As an application of the above proposition, Examples 3.3 and 3.4 have sensitive dependence on initial conditions. Indeed, since that examples are topologically mixing, so we can see that they are thick transitive and syndetic and by the above proposition, they have sensitive dependence on initial conditions.

Remark 3.13. Resemble the proof of Proposition 3.12, if f is thick transitive and syndetic, then f is sensitive. On the other hand, by Proposition 3.8, f is almost thick sensitive and hence almost thick chaotic. Furthermore, since the topologically mixing function f is thick transitive and syndetic, by Proposition 3.12, if f is topologically mixing, then it is almost thick chaotic.

Let $C(X)$ denote the collection of continuous maps on X. For any $f, g \in C(X)$, the supremum metric is defined by

$$D(f,g) = \sup_{x\in X} d(f(x), g(x)).$$

It can be seen that a sequence (f_n) in $C(X)$ converges to f in $(C(X), D)$ if and only if f_n converges to f uniformly on X and hence the topology generated by the supremum metric is called the topology of uniform convergence; see [22]. Sharma and Raghav [22] studied relations between topological transitivity, weak mixing, topological mixing, and sensitive dependence on initial conditions of $(X, f_{1,\infty})$ generated by a sequence (f_n) of continuous maps uniformly converging to f with the autonomous system (X, f).

Vasisht and Das [25] proved some stronger forms of transitivity and sensitivity in an NDS $(X, f_{1,\infty})$ generated by a sequence of continuous maps converging uniformly to a map f. A system $(X, f_{1,\infty})$ is said to be *feeble open* if for any nonempty open set U in X, $int(f_n(U))$ is nonempty for each $n \in \mathbb{N}$; see [25].

Let $(X, f_{1,\infty})$ be an NDS generated by a family of feeble open maps commuting with a continuous map f on X. We want to evaluate necessary and sufficient condition for NDS $(X, f_{1,\infty})$ to be almost thick transitive. In order to prove, we need to state the following proposition and corollary for the system $(X, f_{1,\infty})$.

Proposition 3.14 ([22])**.** *Let $(X, f_{1,\infty})$ be an NDS generated by a family $f_{1,\infty}$ and let f be any continuous map on X. If the family $f_{1,\infty}$ commutes with f, then $d(f_1^k(x), f^k(x)) \leq \sum_{i=1}^{k} D(f_i, f)$ for any $x \in X$ and any $k \in \mathbb{N}$.*

Corollary 3.15 ([22])**.** *Let $(X, f_{1,\infty})$ be an NDS generated by a family $f_{1,\infty}$ and let f be any continuous map on X. If the family $f_{1,\infty}$ commutes with f, then for any $x \in X$ and any $k \in \mathbb{N}$, $d(f_1^{n+k}(x), f^k(f_1^n(x))) \leq \sum_{i=1}^{k} D(f_{i+n}, f)$.*

Theorem 3.16. *Let $(X, f_{1,\infty})$ be an NDS generated by a family $f_{1,\infty}$ of feeble open maps commuting with f such that $\sum_{i=1}^{\infty} D(f_i, f) < \infty$. Then (X, f) is almost thick transitive if and only if $(X, f_{1,\infty})$ is almost thick transitive.*

Proof. Let (X, f) be almost thick transitive. Assume that $x, y \in X$ and $\epsilon > 0$ are given. Let U and V be neighborhoods of x and y with radius ϵ, respectively. Since $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, there exists t such that $\sum_{i=t}^{\infty} D(f_i, f) < \frac{\epsilon}{2}$. As the family $f_{1,\infty}$ consists of feeble open maps, then $f_1^t(U)$ has nonempty interior. Let $U' = int(f_1^t(U))$ and let V' be a neighborhood of y with radius $\frac{\epsilon}{2}$ that are nonempty open sets in X. Since (X, f) is almost thick transitive, there exists $k \in \mathbb{Z}_+$ such that $[k, k+n] \subset N_f(U', V')$. Let $m \in N_f(U', V')$; then $f^m(U') \cap V' \neq \emptyset$. Since $U' = int(f_1^t(U))$, there exists $u \in U$ such that $f^m(f_1^t(u)) \in V'$. By Corollary 3.15, we have

$$d(f_1^{m+t}(U), f^m(f_1^t(U))) \leq \sum_{i=1}^{m} D(f_{i+t}, f) < \frac{\epsilon}{2}.$$

Then by the triangle inequality, we get

$$d(y, f_1^{m+t}(U)) \leq d(y, f^m(f_1^t(U))) + d(f^m(f_1^t(U)), f_1^{m+t}(U)) < \epsilon,$$

which implies that $f_1^{m+t}(U) \cap V \neq \emptyset$; hence $m + t \in N_{f_{1,\infty}}(U, V)$ and $N_f(U', V') + t \subseteq N_{f_{1,\infty}}(U, V)$. Thus $(X, f_{1,\infty})$ is almost thick transitive.

On the contrary, let $\epsilon > 0$ be given and let $B(x, \epsilon)$ and $B(y, \epsilon)$ be two nonempty open sets in X.
Since $\sum_{i=1}^{\infty} D(f_i, f) < \infty$, choose $r \in \mathbb{N}$ such that $\sum_{i=r}^{\infty} D(f_i, f) < \frac{\epsilon}{2}$. Applying almost thick transitivity of $(X, f_{1,\infty})$ to open sets $U = (f_1^r)^{-1}B(x, \epsilon)$ and $V = B(y, \frac{\epsilon}{2})$, there exists $k \in \mathbb{Z}_+$ such that $[k, k+m] \in N_{f_{1,\infty}}(U, V)$. Choose m such that $f_1^{r+m}(U) \cap V \neq \emptyset$. There exists $u \in U$ such that $d(f_1^{r+m}(u), y) < \frac{\epsilon}{2}$. By Corollary 3.15, we have

$$d(f_1^{r+m}(u), f^m(f_1^r(u))) < \sum_{i=1}^{m} D(f_{r+i}, f) < \frac{\epsilon}{2},$$

and the triangle inequality implies

$$d(y, f^m(f_1^r(u))) < \epsilon.$$

Since $f_1^r(u) \in B(x, \epsilon)$, we have $f^m(B(x, \epsilon)) \cap B(y, \epsilon) \neq \emptyset$, which indicates $N_{f_{1,\infty}}(U, V) - r \subseteq N_f(B(x, \epsilon), B(y, \epsilon))$. Thus (X, f) is almost thick transitive. □

Remark 3.17. Similar to the proof of Theorem 3.16, we can show that $(X, f_{1,\infty})$ is thick transitive if and only if (X, f) is thick transitive.

Also [25, Example 3.3] shows that Theorem 3.16 fails, if the condition of taking $(f_n, n \in \mathbb{N})$ to be feeble open maps is omitted from the hypothesis.

By [25, Theorem 4.6], we have the following theorem.

Theorem 3.18. *Let $(X, f_{1,\infty})$ be an NDS generated by a family $f_{1,\infty}$ of feeble open maps commuting with f such that $\sum_{i=1}^{\infty} D(f_i, f) < \infty$. Then (X, f) is almost thick sensitive if and only if $(X, f_{1,\infty})$ is almost thick transitive.*

By Theorems 3.16 and 3.18, we have the following corollary.

Corollary 3.19. *Let $(X, f_{1,\infty})$ be an NDS generated by a family $f_{1,\infty}$ of feeble open maps commuting with f such that $\sum_{i=1}^{\infty} D(f_i, f) < \infty$. Then $(X, f_{1,\infty})$ is almost thick chaotic if and only if (X, f) is almost thick chaotic.*

4. Continuously almost thick transitive NDS

Here, we introduce a new notion that is stronger than almost thick transitivity, and we obtain strongly topological ergodicity of NDS.

Definition 4.1. A nonautonomous dynamical system $f_{1,\infty}$ is called *continuously almost thick transitive*, if for any pair of open subsets $U, V \subset X$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\text{card}(I) \geq \frac{1}{\epsilon}$, for every interval $I \subset N_{f_{1,\infty}}(U, V) \cap [N, +\infty)$.

In the following, we give an example of continuously almost thick transitive NDS.

Example 4.2. Let g_0 be a topologically transitive continuous map and let $g_1 = Id$ be the identity map of a compact metric space X. Then there exists an NDS $f_{1,\infty}$ constructed by $\{g_0\,, g_1\}$ such that $f_{1,\infty}$ is continuously almost thick transitive. Similar to Remark 3.5, suppose that $\{L_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ is a sequence of positive integers such that $\lim_{i\to+\infty} L_i = +\infty$. If we replace m by L_i in Example 3.2, we have the following:

$$f_1^{m_{j_1}+\sum_{i=1}^{j_1} L_i}(x) := \underbrace{g_1 \circ \cdots \circ g_1}_{L_{j_1}} \circ \underbrace{g_0 \circ \cdots \circ g_0}_{m_{j_1}-m_{(j_1-1)}} \circ \cdots \circ \underbrace{g_1 \circ \cdots \circ g_1}_{L_2} \circ \underbrace{g_0 \circ \cdots \circ g_0}_{m_2-m_1} \circ \underbrace{g_1 \circ \cdots \circ g_1}_{L_1} \circ \underbrace{g_0 \circ \cdots \circ g_0}_{m_1}(x).$$

Indeed, there exist $U_{i_{j_1}}, U_{i_{j_1}} \in \{U_i\}_{i=1}^{\infty}$ such that

$$f_1^{m_{j_1}+\sum_{i=1}^{j_1} L_i}(U_{i_{j_1}}) \cap (U_{j_{j_1}}) = (g_0)^{m_{j_1}}(U_{i_{j_1}}) \cap (U_{j_{j_1}}) \neq \emptyset.$$

Hence $[m_{j_1} + \sum_{i=1}^{j_1-1} L_i, m_{j_1} + \sum_{i=1}^{j_1} L_i] \subset N_{f_{1,\infty}}(U_{i_{j_1}}) \cap (U_{j_{j_1}})$.

In fact, $f_{1,\infty}$ is constructed as below:

$$f_i = \begin{cases} g_0, & if \quad 1 \le i \le m_1, \\ g_1, & if \quad m_1 + 1 \le i \le m_1 + L_1, \\ g_0, & if \quad m_1 + L_1 + 1 \le i \le m_2 + L_1, \\ g_1, & if \quad m_2 + L_1 + 1 \le i \le m_2 + (L_1 + L_2), \\ \vdots & \\ g_0, & if \quad m_{j_1-1} + \sum_{i=1}^{j_1-1} L_i + 1 \le i \le m_{j_1} + \sum_{i=1}^{j_1-1} L_i, \\ g_1, & if \quad m_{j_1} + \sum_{i=1}^{j_1-1} L_i + 1 \le i \le m_{j_1} + \sum_{i=1}^{j_1} L_i, \\ g_0, & if \quad m_{j_1} + \sum_{i=1}^{j_1} L_i + 1 \le i \le m_{j_1+1} + \sum_{i=1}^{j_1} L_i, \\ g_1, & if \quad m_{j_1+1} + \sum_{i=1}^{j_1} L_i + 1 \le i \le m_{j_1+1} + \sum_{i=1}^{j_1+1} L_i. \end{cases}$$

Keeping on this process for any n, we get

$$f_i = \begin{cases} g_0, & if \quad m_{j_{n-1}} + \sum_{i=1}^{j_{n-1}+(n-1)} L_i + 1 \le i \le m_{j_{n-1}+1} + \sum_{i=1}^{j_{n-1}+(n-1)} L_i, \\ g_1, & if \quad m_{j_{n-1}+1} + \sum_{i=1}^{j_{n-1}+(n-1)} L_i + 1 \le i \le m_{j_{n-1}+1} + \sum_{i=1}^{j_{n-1}+n} L_i, \\ \vdots & \\ g_0, & if \quad m_{j_n} + \sum_{i=1}^{j_n+(n-1)} L_i + 1 \le i \le m_{j_n+1} + \sum_{i=1}^{j_{n-1}+(n-1)} L_i, \\ g_1, & if \quad m_{j_n+1} + \sum_{i=1}^{j_n+(n-1)} L_i + 1 \le i \le m_{j_n+1} + \sum_{i=1}^{j_n+n} L_i. \end{cases}$$

Suppose that U and V are two nonempty open subsets of X and that $\epsilon > 0$ is given. By the above process and Example 3.2, there exists a positive integer N such that $N_{f_{1,\infty}}(U, V) \cap [N, +\infty) \neq \emptyset$. Since $\lim_{i\to+\infty} L_i = +\infty$, there exists some positive integer $k \ge N$ such that $L_k > \frac{1}{\epsilon}$. By the construction of maximal interval $I \subset N_{f_{1,\infty}}(U, V) \cap [N, +\infty)$, $\mathrm{card}(I) > L_k > \frac{1}{\epsilon}$ for every $k \ge N$. Hence, there exists an NDS $f_{1,\infty}$ constructed by $\{g_0\,, g_1\}$ such that $f_{1,\infty}$ is continuously almost thick transitive.

Theorem 4.3. *If $f_{1,\infty}$ is continuously almost thick transitive and syndetic, then $f_{1,\infty}$ is strongly topologically ergodic.*

Proof. Suppose that U and V are two nonempty open subsets of X. Since $f_{1,\infty}$ is continuously almost thick transitive, then for any $j \in \mathbb{N}$, there exists N_j such that $\operatorname{card}(I) \geq j$ for every $I \subset N_{f_{1,\infty}}(U, V) \cap [N_j, +\infty)$. On the other hand, for $m \in \mathbb{N}$ in the definition of syndetic, there exists $j \in \mathbb{N}$ such that

$$\frac{j}{j+m} \geq 1 - \epsilon^2,$$

for given $\epsilon > 0$. Moreover, there is $n \in \mathbb{N}$ such that

$$\frac{N_j}{n(m+j)} < \epsilon.$$

Therefore

$$\begin{aligned}
&\frac{\operatorname{card}\{\{1, \dots, N_j + n(m+j))\} \cap N_{f_{1,\infty}}(U, V)\}}{N_j + n(m+j)} \\
&= \frac{\operatorname{card}\{\{1, \dots, N_j\} \cap N_{f_{1,\infty}}(U, V)\}}{N_j + n(m+j)} \\
&\quad + \frac{\operatorname{card}\{\{N_j + 1, \dots, N_j + n(m+j))\} \cap N_{f_{1,\infty}}(U, V)\}}{N_j + n(m+j)} \\
&\geq \frac{nj}{N_j + nm + nj} = \frac{j}{(m+j)(\frac{N_j}{n(m+j)} + 1)} \\
&\geq \frac{j}{(j+m)(1+\epsilon)} \geq \frac{1-\epsilon^2}{1+\epsilon} = 1 - \epsilon.
\end{aligned}$$

This implies that

$$\limsup_{n \to \infty} \frac{\operatorname{card}\{\{1, \dots, n-1\} \cap N_{f_{1,\infty}}(U, V)\}}{n} = 1,$$

and hence $f_{1,\infty}$ is topologically strongly ergodic. □

Assume that (X, f) is a dynamical system. Let $\beta(X)$ be the sigma-algebra of Borel subsets of X and let μ be a probability measure on the measurable space $(X, \beta(X))$ such that $\mu(U) > 0$ for every nonempty open set U in X.

Definition 4.4. A continuous function $\phi : X \to \mathbb{R}$ is said to satisfy *the large deviations theorem* for (f, μ), if for every $\epsilon > 0$ there is $h(\epsilon) > 0$ such that

$$\mu(\{x \in X : |\frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) - \int \phi d\mu| > \epsilon\}) \leq e^{-nh(\epsilon)},$$

for all $n \in \mathbb{N}$ sufficiently large. A pair (f, μ) is said to satisfy the large deviations theorem, if every continuous function $\phi : X \to \mathbb{R}$ satisfies the large deviations theorem for (f, μ); see [29].

Gu in [12, Theorem 4.1] showed that if the pair (f, μ) satisfies the large deviations theorem and the map f is topologically strongly ergodic, then f is sensitive dependence on initial conditions. Therefore we have the following corollary.

Corollary 4.5. *If f is syndetic, continuously almost thick transitive and satisfies the large deviations theorem, then f is almost thick chaotic.*

Proof. By Theorem 4.3, f is strongly topologically ergodic. Hence by [12, Theorem 4.1], f is sensitive. Furthermore, according to Proposition 3.8, f is almost thick sensitive. Hence f is almost thick chaotic. □

Wu and Chen [26] proved that a dynamical system (X, f) satisfying the large deviations theorem is ergodic and this system is chaotic in the sense of Li–Yorke and Devaney provided that its periodic points are dense. Wu, Wang, and Chen [28] achieved remarkable results about the concept of the large deviations theorem and its relation with ergodic properties and chaotic behaviors of a dynamical system (X, f). By the above argument, naturally we have the following questions.
Is it possible to generalize the concept of the large deviations theorem for NDS $f_{1,\infty}$?
Are the above discussions satisfied for an NDS $f_{1,\infty}$?
The above questions can be as a research subject in the future.

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