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EXISTENCE OF RENORMALIZED SOLUTIONS FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS WITH GENERALIZED GROWTH IN ORLICZ SPACES

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Abstract. This paper deals with an existence result of renormalized solutions for nonlinear parabolic equations of the type

$$\frac{\partial b(x,u)}{\partial t} - \operatorname{div} a(x,t,u,\nabla u) - \operatorname{div} \Phi(x,t,u) = f \quad \text{in } Q_T = \Omega \times (0,T),$$

where $b(x,\cdot)$ is a strictly increasing C^1-function for every $x \in \Omega$ with $b(x,0) = 0$, the lower order term Φ satisfies a natural growth condition described by the appropriate Orlicz function M, and f is an element of $L^1(Q_T)$. We do not assume any growth restrictions neither on M nor on its conjugate $\overline{M}$.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ satisfying the segment property, $N \geq 2$, let $Q_T = \Omega \times (0,T)$, where T is a positive real number, and let M be an Orlicz function. Let $\mathrm{A}(u) := -\operatorname{div} a(x,t,u,\nabla u)$ be a so-called Leray–Lions type operator whose prototype is the p-Laplacian operator and let $b : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $b(x,\cdot)$ is a strictly increasing C^1-function for any fixed $x \in \Omega$ with $b(x,0) = 0$.

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In this paper, we prove the existence of renormalized solutions in the setting of Orlicz spaces to the following Cauchy–Dirichlet boundary value problem

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t} + \mathrm{A}(u) - \operatorname{div} \Phi(x,t,u) = f & \text{in } Q_T, \\ b(x,u)(t=0) = b(x,u_0) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where $u_0 \in L^1(\Omega)$, $f \in L^1(Q_T)$, and Φ satisfies the following natural growth condition

$$|\Phi(x,t,s)| \leq \gamma(x,t) + \overline{M}^{-1}(M(|s|)) \text{ with } \gamma \in E_{\overline{M}}(Q_T), \tag{1.2}$$

where $E_{\overline{M}}$ denotes the closure, in the Orlicz space defined by the complementary $\overline{M}$, of the set of bounded measurable functions with compact support in $\overline{\Omega}$.

Problem (1.1) has been studied in different particular cases, we give some works in this direction. In the classical Sobolev spaces and in a special case where $\Phi \equiv 0$, b is a maximal monotone graph on $\mathbb{R}$ and $a(x,t,s,\xi)$ is independent of s, existence and uniqueness of a renormalized solution have been proved by Blanchard and Murat [9] and by Blanchard and Porretta [11] in the case where $a(x,t,s,\xi)$ is independent of t. Aberqi, Bennouna, and Redwane [1] investigated problem (1.1) in the case $M(t) = t^p$ for a measure $\mu = f - \operatorname{div}(F)$, with $f \in L^1(Q_T)$, $F \in (L^{p'}(Q))^N$, and Φ satisfies the condition

$$|\Phi(x,t,s)| \leq c(x,t)|s|^{\gamma},$$

with $c(x,t) \in L^{\tau}(Q_T)$ for some $\tau = \dfrac{N+p}{p-1}$ and $\gamma = \dfrac{N+2}{N+p}(p-1)$.

Concerning contributions in Orlicz spaces framework, Azroul, Redwane, and Rhoudaf [6] proved the existence of renormalized solution, where Φ depends only on u (without dependence on x) and $b(x,u) = b(u)$, the same result has been given by Redwane [24], where $b(x,u)$ depends on x and u.

Recently, in the setting of Orlicz spaces, Hadj Nassar, Moussa and Rhoudaf [21] have studied the existence of renormalized solution for problem (1.1) in the case $f \in L^1(Q_T)$ under a **nonnatural growth condition** on Φ prescribed by an N-function P that increases essentially less rapidly than the Orlicz function M defining the framework spaces, namely,

$$|\Phi(x,t,s)| \leq \overline{P}^{-1}(P(|s|)) \text{ with } P \prec\prec M. \tag{1.3}$$

Indeed, there is no growth with respect to the spatial variable (x,t), thus, condition (1.3) does not define a general growth condition. Our approach in this paper is how to deal with the issue: Passing from assuming condition (1.3) to assuming the **weaker** one (1.2). Motivated by the above mentioned works, we establish the existence of renormalized solutions for problem (1.1) in Orlicz spaces, where Φ satisfies condition (1.2), without assuming any restriction on the N-function M neither on its complementary $\overline{M}$. We avoid to use the concept of Orlicz function grows essentially more slowly than another, we use a direct and a concise method unlike as in [21]. Note that, if $\gamma(x,t) = 0$, then condition (1.2) is natural and less restrictive than (1.3). Thus, we find the same result as in [21] under a best condition. Moreover, if $\gamma(x,t) \neq 0$, then we get a complete growth condition on

Φ. Thus, our work has two directions, weakening the growth restriction on Φ and restoring the general natural growth condition.

In dealing with this problem, we have encountered some difficulties, essentially, under the natural growth assumption (1.2), it is difficult to prove the existence of solution for the approximate problem and proving its convergence, which are the basic results in the proof of such solutions. The improvement from condition (1.3) to condition (1.2) follows thanks to an algebraic trick combined with the convexity of M and Young's inequality on a specific positive quantities.

This article is organized as follows: In section 2, we recall some well-known preliminaries, results, and properties of Orlicz–Sobolev spaces and inhomogeneous Orlicz–Sobolev spaces. Section 3 is devoted to basic assumptions, problem setting, and the proof of the main result that is Theorem 3.4.

2. Preliminaries

2.1. **Orlicz–Sobolev spaces.** Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and convex function with

$$M(t) > 0 \text{ for } t > 0, \lim_{t\to 0} \frac{M(t)}{t} = 0 \text{ and } \lim_{t\to +\infty} \frac{M(t)}{t} = +\infty.$$

The function M is said an N-function or an Orlicz function, and the N-function complementary to M is defined as

$$\overline{M}(t) = \sup\Big\{st - M(s), s \geq 0\Big\}.$$

We recall that (see [2])

$$M(t) \leq t\overline{M}^{-1}(M(t)) \leq 2M(t) \quad \text{for all } t \geq 0 \tag{2.1}$$

and the Young's inequality: for all $s, t \geq 0$,

$$st \leq \overline{M}(s) + M(t).$$

We say that M satisfies the Δ_2-condition (or $M \in \Delta_2$) if for some $k > 0$,

$$M(2t) \leq kM(t) \quad \text{for all } t \geq 0, \tag{2.2}$$

and if (2.2) holds only for $t \geq t_0$, then M is said to satisfy the Δ_2-condition near infinity.

Let M_1 and M_2 be two N-functions. The notation $M_1 \prec\prec M_2$ means that M_1 grows essentially less rapidly than M_2, that is,

$$\text{for all } \epsilon > 0, \quad \lim_{t\to\infty} \frac{M_1(t)}{M_2(\epsilon t)} = 0,$$

that is the case if and only if

$$\lim_{t\to\infty} \frac{(M_2)^{-1}(t)}{(M_1)^{-1}(t)} = 0.$$

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence class of) real-valued measurable functions u on Ω such that

$$\int_\Omega M(|u(x)|)dx < \infty \quad (\text{resp.} \int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)dx < \infty \text{ for some } \lambda > 0).$$

Endowed with the norm

$$\|u\|_M = \inf\Big\{\lambda > 0 : \int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)dx \le 1\Big\},$$

$L_M(\Omega)$ is a Banach space and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The Orlicz–Sobolev space $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of functions u such that u and its distributional derivatives up to order 1 are in $L_M(\Omega)$ (resp. $E_M(\Omega)$).

This is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the norm closure of the Schwartz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$.

We say that a sequence $\{u_n\}$ converges to u for the modular convergence in $W^1L_M(\Omega)$ if, for some $\lambda > 0$,

$$\int_\Omega M\Big(\frac{|D^\alpha u_n - D^\alpha u|}{\lambda}\Big)\,dx \to 0 \quad \text{for all } |\alpha| \le 1;$$

this implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

If M satisfies the Δ_2-condition on $\mathbb{R}^+$ (near infinity only if Ω has finite measure), then the modular convergence coincides with norm convergence. Recall that the norm $\|Du\|_M$ defined on $W_0^1L_M(\Omega)$ is equivalent to $\|u\|_{1,M}$ (see [17]).

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω, which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $D(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (see [17]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined. For more details one can see, for example, [2] or [22].

2.2. **Inhomogeneous Orlicz–Sobolev spaces.** Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $T > 0$, and set $Q_T = \Omega \times (0,T)$. For each $\alpha \in (\mathbb{N}^*)^N$, denote by D_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in \Omega$. The inhomogeneous Orlicz–Sobolev spaces are defined as follows:

$$W^{1,x}L_M(Q_T) = \Big\{u \in L_M(Q_T) : D_x^\alpha u \in L_M(Q_T) \quad \text{for all} \quad |\alpha| \le 1\Big\},$$

and

$$W^{1,x}E_M(Q_T) = \Big\{u \in E_M(Q_T) : D_x^\alpha u \in E_M(Q_T) \quad \text{for all} \quad |\alpha| \le 1\Big\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\le 1} \|D_x^\alpha u\|_{M,Q_T}.$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q_T)$, which have as many copies as there are α-order derivatives, $|\alpha| \le 1$. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}}))$. If $u \in W^{1,x}L_M(Q_T)$, then the function $t \mapsto u(t) = u(t,\cdot)$ is defined on $(0,T)$ with values in $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q_T)$, then the concerned function is a $W^1E_M(\Omega)$-valued and is strongly measurable. Furthermore the following embedding holds: $W^{1,x}E_M(Q_T) \subset L^1(0,T;W^1E_M(\Omega))$. The space $W^{1,x}L_M(Q_T)$ is not in general separable, if $u \in W^{1,x}L_M(Q_T)$, we cannot conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto \| u(t) \|_{M,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure in $W^{1,x}E_M(Q_T)$ of $D(Q_T)$. It is proved that when Ω has the segment property, then each element u of the closure of $D(Q_T)$ with respect of the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is a limit, in $W^{1,x}L_M(Q_T)$, of some subsequence $(u_n) \subset D(Q_T)$ for the modular convergence; that is, if, for some $\lambda > 0$, such that for all $|\alpha| \le 1$;

$$\int_{Q_T} M\Big(\frac{|D_x^\alpha u_n - D_x^\alpha u|}{\lambda}\Big) dx\, dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

This implies that (u_n) converges to u in $W^{1,x}L_M(Q_T)$ for the weak topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$. Consequently,

$$\overline{D(Q_T)}^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = \overline{D(Q_T)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}.$$

This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore,

$$W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_M.$$

We have then the following complementary system

$$\Big(W_0^{1,x}L_M(Q_T), F, W_0^{1,x}E_M(Q_T), F_0\Big),$$

where F is the dual space of $W_0^{1,x}E_M(Q_T)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q_T)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q_T)$ and it is shown that,

$$W^{-1,x}L_{\overline{M}}(Q_T) = \Big\{f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{M}}(Q_T)\Big\}.$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\le 1} \|f_\alpha\|_{\overline{M},Q_T},$$

where the infimum is taken on all possible decompositions

$$f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha, f_\alpha \in L_{\overline{M}}(Q_T).$$

The space F_0 is then given by

$$W^{-1,x}L_{\overline{M}}(Q_T) = \Big\{ f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q_T) \Big\}$$

and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q_T)$.

Lemma 2.1. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Also let M, P, and Q be N-functions such that $Q \prec\prec P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$,*

$$|f(x, s)| \le c(x) + k_1 P^{-1} M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskii operator N_f, defined by $N_f(u)(x) = f(x, u(x))$, is strongly continuous from $P(E_M, \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.

Lemma 2.2. *Let $u_k, u \in L_M(\Omega)$. If $u_k \to u$ for the modular convergence, then $u_k \to u$ for $\sigma(L_M, L_{\overline{M}})$.*

Lemma 2.3 ([3, Lemma 1])**.** *If $u_n \to u$ for the modular convergence (with every $\lambda > 0$) in $L_M(Q_T)$, then $u_n \to u$ strongly in $L_M(Q_T)$.*

Lemma 2.4 ([17])**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. Let M be a Orlicz function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then, $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & a.e. \text{ in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & a.e. \text{ in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.5 ([17])**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$ and let M be an Orlicz function. we assume that the set of discontinuity points D of F' is finite. Then the mapping $F : W^1L_M(\Omega) \to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.*

Lemma 2.6 ([16])**.** *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \ge 2$, satisfying the segment property. Then*

$$\Big\{ u \in W_0^{1,x} L_{\overline{M}}(Q_T) : \frac{\partial u}{\partial t} \in W^{-1,x} L_{\overline{M}}(Q_T) + L^1(Q_T) \Big\} \subset C\Big([0, T], L^1(\Omega)\Big).$$

Lemma 2.7 (Integral Poincaré's type inequality in inhomogeneous Orlicz spaces [16])**.** *Let Ω be a bounded open subset of $\mathbb{R}^N$ and let M be an Orlicz function. Then there exist two positive constants $\delta, \lambda > 0$ such that*

$$\int_{Q_T} M(\delta|u(x, t)|)\, dx\, dt \le \int_{Q_T} \lambda M(|\nabla u(x, t)|)\, dx\, dt \quad \text{for all } u \in W_0^{1,x} L_M(Q_T).$$

Lemma 2.8. *If $f_n \subset L^1(\Omega)$ with $f_n \to f \in L^1(\Omega)$ a.e. in Ω, $f_n, f \geq 0$ a. e. in Ω and $\displaystyle\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx$, then $f_n \to f$ in $L^1(\Omega)$.*

Lemma 2.9 ([17])**.** *Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_M(\Omega)$. Then, there exists a sequence $(u_n) \subset D(\Omega)$ such that $u_n \to u$ for the modular convergence in $W_0^1 L_M(\Omega)$. Furthermore, if $u \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$, then*

$$\|u_n\|_\infty \leq \ (N+1)\|u\|_\infty.$$

3. Basic assumptions and main result

Through this paper, Ω is a bounded open subset of $\mathbb{R}^N$ satisfying the segment property, $N \geq 2$, $Q_T = \Omega \times (0,T)$ where T is a positive real number, and M is an Orlicz function. Consider $b : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function such that for every $x \in \Omega$, $b(x,s)$ is a strictly increasing C^1-function with $b(x,0) = 0$ and for any $k > 0$, there exist $\lambda_k > 0$, a function $A_k \in L^\infty(\Omega)$, and a function $\widetilde{A}_k \in L_M(\Omega)$ such that,

$$\lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \quad \text{and} \quad \left|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\right| \leq \widetilde{A}_k(x). \tag{3.1}$$

Let $A : D(A) \subset W_0^{1,x} L_M(Q_T) \to W^{-1,x} L_{\overline{M}}(Q_T)$ be an operator of Leray—Lions type of the form

$$\mathrm{A}u := -\mathrm{div}\, a(x,t,u,\nabla u).$$

Our main goal in this study is to prove the existence of renormalized solutions in the setting of Orlicz spaces for the nonlinear problem

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t} - \mathrm{div}\, a(x,t,u,\nabla u) - \mathrm{div}\, \Phi(x,t,u) = f & \text{in } Q_T, \\ b(x,u)(t=0) = b(x,u_0) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{3.2}$$

where $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function satisfying, for almost every $(x,t) \in Q_T$ and for all $s \in \mathbb{R}, \xi, \eta \in \mathbb{R}^N (\xi \neq \eta)$ the following conditions:

(H$_1$): There exist a function $c(x,t) \in E_{\overline{M}}(Q_T)$, some positive constants k_1 and k_2, and an Orlicz function $P \prec\prec M$ such that

$$|a(x,t,s,\xi)| \leq c(x,t) + \overline{M}^{-1}(P(k_1|s|) + \overline{M}^{-1}(M(k_2|\xi|)).$$

(H$_2$): The vector a is strictly monotone

$$\Big(a(x,t,s,\xi) - a(x,t,s,\eta)\Big) \cdot \Big(\xi - \eta\Big) > 0.$$

(H$_3$): a is coercive, there exists a constant $\alpha > 0$ such that

$$a(x,t,s,\xi) \cdot \xi \geq \alpha M(|\xi|).$$

For the lower order term, we assume that $\Phi : Q_T \times \mathbb{R} \to \mathbb{R}^N$ is a Caratheodory function satisfying the following condition:

(H$_4$): For all $s \in \mathbb{R}$ and for almost every $x \in \Omega$,

$$|\Phi(x,t,s)| \leq \gamma(x,t) + \overline{M}^{-1}(M(|s|)), \ \text{with } \gamma \in E_{\overline{M}}(Q_T).$$

For that concern the right hand, $f \in L^1(Q_T)$. $u_0 \in L^1(\Omega)$,

Lemma 3.1 ([21]). *Under assumptions* (H_1)–(H_3), *let* (Z_n) *be a sequence in* $W_0^{1,x}L_M(Q_T)$ *such that*

$$Z_n \rightharpoonup Z \quad \text{in } W_0^{1,x}L_M(Q_T) \text{ for } \sigma(\Pi L_M(Q_T), \Pi E_{\overline{M}}(Q_T)),$$

$$\Big(a(x,t,Z_n,\nabla Z_n)\Big)_n \quad \text{is bounded in } \Big(L_{\overline{M}}(Q_T)\Big)^N,$$

$$\lim_{n,s\to\infty}\int_{Q_T}\Big(a(x,t,Z_n,\nabla Z_n) - a(x,t,Z_n,\nabla Z\chi_s)\Big)\cdot\Big(\nabla Z_n - \nabla Z\chi_s\Big)dxdt = 0,$$

where χ_s *denotes the characteristic function of the set* $\Omega_s = \Big\{x \in \Omega : |\nabla Z| \le s\Big\}$. *Then,*

$$\nabla Z_n \to \nabla Z \quad \text{a.e. in } Q_T,$$

$$\lim_{n\to\infty}\int_{Q_T} a(x,t,Z_n,\nabla Z_n)\nabla Z_n\, dx = \int_{Q_T} a(x,t,Z,\nabla Z)\nabla Z\, dxdt,$$

$$M(|\nabla Z_n|) \longrightarrow M(|\nabla Z|) \quad \text{in } L^1(Q_T).$$

In what follows, we will use the following real function of a real variable, called the truncation at height $k > 0$,

$$T_k(s) = \max\Big(-k, \min(k,s)\Big) = \begin{cases} s & \text{if } |s| \le k,\\ k\dfrac{s}{|s|} & \text{if } |s| > k.\end{cases}$$

Now, we give the definition of a renormalized solution for problem (3.2).

Definition 3.2. A measurable function u defined on Q_T is said a renormalized solution for problem (3.2), if

$$T_k(u) \in W_0^{1,x}L_M(Q_T) \quad \text{for all } k \ge 0, \quad \text{and} \quad b(x,u) \in L^\infty(0,T,L^1(\Omega)),$$

$$\lim_{m\to\infty}\int_{\{m\le|u(x,t)|\le m+1\}} a(x,t,u,\nabla u)\nabla u\, dxdt = 0,$$

and if, for every function r (renormalization) in $W^{1,\infty}(\mathbb{R})$ with compact support, we have

$$\begin{aligned}&\tfrac{\partial B_r(x,u)}{\partial t} - \operatorname{div}(r(u)a(x,t,u,\nabla u)) + r'(u)a(x,t,u,\nabla u)\nabla u\\ &-\operatorname{div}(r(u)\Phi(x,t,u)) + r'(u)\Phi(x,t,u)\nabla u = fr(u) \quad \text{in} \quad D'(Q_T),\end{aligned} \tag{3.3}$$

where $B_r(x,\tau) = \displaystyle\int_0^\tau \frac{\partial b(x,s)}{\partial s} r'(s)\, ds$ and $B_r(x,u)(t=0) = B_r(x,u_0)$ in Ω.

Remark 3.3. [21,24] For every $r \in W^{2,\infty}(\mathbb{R})$ nondecreasing function with $supp(r') \subset [-k,k]$ and (3.1), we have

$$\lambda_k|r(s_1) - r(s_2)| \le |B_r(x,s_1) - B_r(x,s_2)| \le \|A_k\|_{L^\infty(\Omega)}|r(s_1) - r(s_2)|,$$

for almost every $x \in \Omega$ and for every $s_1, s_2 \in \mathbb{R}$.

The following theorem is our main result.

Theorem 3.4. *Suppose that assumptions* (H_1)-(H_4) *hold true and that* $f \in L^1(Q_T)$. *Then there exists at least a renormalized solution for problem* (3.2).

The proof of the above theorem is divided into four steps.

Step 1: Approximate problems.

Let f_n be a sequence of regular function in $C_0^\infty(Q_T)$, which converges strongly to f in $L^1(Q_T)$ and such that $\|f_n\|_{L^1} \leq \|f\|_{L^1}$ and for each $n \in I\!N^*$. Put

$$b_n(x,s) = T_n(b(x,s)) + \frac{1}{n}s,$$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \text{ a.e } (x,t) \in Q_T, \text{ for all } s \in \mathbb{R}, \text{ for all } \xi \in \mathbb{R}^N,$$

and

$$\Phi_n(x,t,s) = \Phi(x,t,T_n(s)) \text{ a.e } (x,t) \in Q_T, \quad \text{for all } s \in \mathbb{R}.$$

Let $u_{0n} \in C_0^\infty(\Omega)$ such that

$$\| b_n(x,u_{0n}) \|_{L^1} \leq \| b(x,u_0) \|_{L^1} \text{ and } b_n(x,u_{0n}) \longrightarrow b(x,u_0) \text{ in } L^1(\Omega).$$

Consider the following approximate problem:

$$\begin{cases} \dfrac{\partial b_n(x,u_n)}{\partial t} - \operatorname{div} a(x,t,u_n,\nabla u_n) - \operatorname{div} \Phi_n(x,t,u_n) = f_n & \text{in } Q_T, \\ b_n(x,u_n)(t=0) = b_n(x,u_0) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{3.4}$$

Let $z_n(x,t,u_n,\nabla u_n) = a_n(x,t,u_n,\nabla u_n) + \Phi_n(x,t,u_n)$, which satisfies (A_1), (A_2), (A_3), and (A_4) of [20]. Indeed, it remains to prove (A_4), to do this we use Young's inequality as follows:

$$\begin{aligned} |\Phi_n(x,t,u_n)\nabla u_n| &\leq |\gamma(x,t)||\nabla u_n| + \overline{M}^{-1}(M(|T_n(u_n)|))|\nabla u_n| \\ &= \frac{\alpha^2}{\alpha+2}\frac{\alpha+2}{\alpha^2}|\gamma(x,t)||\nabla u_n| \\ &\quad + \frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|T_n(u_n)|))\frac{\alpha}{\alpha+1}|\nabla u_n| \\ &\leq \frac{\alpha^2}{\alpha+2}\Big(\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) + M\Big(|\nabla u_n|\Big)\Big) \\ &\quad + \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|T_n(u_n)|))\Big) + M\Big(\frac{\alpha}{\alpha+1}|\nabla u_n|\Big). \end{aligned}$$

While $\dfrac{\alpha}{\alpha+1} < 1$, using the convexity of M and since $\overline{M}$ and $\overline{M}^{-1} \circ M$ are increasing functions, one has

$$\begin{aligned} |\Phi_n(x,t,u_n)\nabla u_n| &\leq \frac{\alpha^2}{\alpha+2}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) + \frac{\alpha^2}{\alpha+2}M\Big(|\nabla u_n|\Big) \\ &\quad + \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(n))\Big) + \frac{\alpha}{\alpha+1}M\Big(|\nabla u_n|\Big). \end{aligned}$$

Then we get

$$\begin{aligned} \Phi_n(x,t,u_n)\nabla u_n &\geq -\Big(\frac{\alpha^2}{\alpha+2} + \frac{\alpha}{\alpha+1}\Big)M\Big(|\nabla u_n|\Big) - \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(n))\Big) \\ &\quad - \frac{\alpha^2}{\alpha+2}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big). \end{aligned}$$

Using this last inequality and (H_3), we obtain

$$\begin{aligned} z_n(x,t,u_n,\nabla u_n)\nabla u_n &\geq \Big(\alpha - \frac{\alpha^2}{\alpha+2} - \frac{\alpha}{\alpha+1}\Big) M\Big(|\nabla u_n|\Big) \\ &\quad - \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(n))\Big) - \frac{\alpha^2}{\alpha+2}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) \\ &\geq \frac{\alpha^2}{(\alpha+1)(\alpha+2)} M\Big(|\nabla u_n|\Big) - \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(n))\Big) \\ &\quad - \frac{\alpha^2}{\alpha+2}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big). \end{aligned}$$

Since $\gamma \in E_{\overline{M}}(Q_T)$, $\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) \in L^1(Q_T)$. Thus, from [16], the approximate problem (3.4) has at least one weak solution $u_n \in W_0^{1,x}L_M(Q_T)$.

Step 2: A priori estimates.

Proposition 3.5. *Suppose that assumptions* (H_1)-(H_4) *hold true and let* $(u_n)_n$ *be a solution of the approximate problem* (3.4)*. Then, for all* $k > 0$*, there exist two constants* C_k *and* $\widehat{C}_k$ *(not depending on* n*), such that*

$$\| T_k(u_n) \|_{W_0^{1,x}L_M(Q_T)} \leq C_k, \tag{3.5}$$

$$\int_\Omega B_k^n(x,u_n)(\sigma)\,dx \leq \widehat{C}_k + k\Big(\|f\|_{L^1(Q_T)} + \|b(x,u_0\|_{L^1(\Omega)}\Big), \tag{3.6}$$

for almost any $\sigma \in (0,T)$*, where* $B_k^n(x,\tau) = \displaystyle\int_0^\tau T_k(s)\frac{\partial b_n(x,s)}{\partial s}\,ds$*, and*

$$\lim_{k\to\infty} meas\Big\{(x,t) \in Q_T : |u_n| > k\Big\} = 0. \tag{3.7}$$

Proof. Testing the approximate problem (3.4) by $T_k(u_n)\chi_{(0,\sigma)}$, one has for every $\sigma \in (0,T)$

$$\begin{aligned} &\int_\Omega \Big(B_k^n(x,u_n)(\sigma) - B_k^n(x,u_{0n})\Big)\,dx + \int_{Q_\sigma} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)\,dx\,dt \\ &\quad + \int_{Q_\sigma} \Phi_n(x,t,u_n)\nabla T_k(u_n)\,dx\,dt = \int_{Q_\sigma} f_n T_k(u_n)\,dx\,dt. \end{aligned} \tag{3.8}$$

First, let us remark that $\Phi_n(x,t,u_n)\nabla T_k(u_n)$ is different from zero only on the set $\{|u_n| \leq k\}$, where $T_k(u_n) = u_n$. From (H_4) and then Young's inequality for an arbitrary $\alpha > 0$ (the constant of coercivity), we have

$$\begin{aligned}
&\int_{Q_\sigma} \Phi_n(x,t,u_n)\nabla T_k(u_n)\,dx\,dt \\
&\le \int_{Q_\sigma} |\gamma(x,t)||\nabla T_k(u_n)|\,dx\,dt \\
&\quad + \int_{Q_\sigma} \overline{M}^{-1}(M(|T_k(u_n)|))|\nabla T_k(u_n)|\,dx\,dt \\
&= \frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} \frac{\alpha+2}{\alpha^2}|\gamma(x,t)||\nabla T_k(u_n)|\,dx\,dt \\
&\quad + \int_{Q_\sigma} \frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|T_k(u_n)|))\frac{\alpha}{\alpha+1}|\nabla T_k(u_n)|\,dx\,dt \\
&\le \frac{\alpha^2}{\alpha+2}\Big(\int_{Q_\sigma} \overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx + \int_{Q_\sigma} M\Big(|\nabla T_k(u_n)|\Big)\,dx\,dt\Big) \\
&\quad + \int_{Q_\sigma} \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|T_k(u_n)|))\Big)\,dx\,dt \\
&\quad + \int_{Q_\sigma} M\Big(\frac{\alpha}{\alpha+1}|\nabla T_k(u_n)|\Big)\,dx\,dt.
\end{aligned}$$

Since $\gamma \in E_{\overline{M}}(Q_\sigma)$, then $\frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} \overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx\,dt = \gamma_0 < +\infty$ and while $\frac{\alpha}{\alpha+1} < 1$, using the convexity of M and the fact that $\overline{M}$ and $\overline{M}^{-1} \circ M$ are increasing functions, we get

$$\begin{aligned}
&\int_{Q_\sigma} \Phi_n(x,t,u_n)\nabla T_k(u_n)\,dx\,dt \\
&\le \gamma_0 + \frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} M\Big(|\nabla T_k(u_n)|\Big)\,dx\,dt \\
&\quad + \int_{Q_\sigma} \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(k))\Big)\,dx\,dt \\
&\quad + \frac{\alpha}{\alpha+1}\int_{Q_\sigma} M\Big(|\nabla T_k(u_n)|\Big)\,dx\,dt.
\end{aligned}$$

Using (2.1), there exists some constant C_k^α such that

$$\int_{Q_\sigma} \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(k))\Big)\,dx\,dt \le \int_{Q_\sigma} \overline{M}\Big(2\frac{\alpha+1}{\alpha k}M(k)\Big)\,dx\,dt = C_k^\alpha,$$

which gives the estimate

$$\begin{aligned}
&\int_{Q_\sigma} \Phi_n(x,t,u_n)\nabla T_k(u_n)\,dx\,dt \\
&\le \gamma_0 + \frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} M\Big(|\nabla T_k(u_n)|\Big)\,dx\,dt \\
&\quad + C_k^\alpha + \frac{\alpha}{\alpha+1}\int_{Q_\sigma} M\Big(|\nabla T_k(u_n)|\Big)\,dx\,dt.
\end{aligned} \tag{3.9}$$

On the other hand, we have $\|f_n\|_{L^1} \leq \|f\|_{L^1}$, which implies that

$$\int_{Q_T} f_n T_k(u_n)\,dx\,dt \leq k\|f\|_{L^1}. \tag{3.10}$$

Concerning the first integral in (3.8), by the construction of $B_k^n(x,u_n)$, we have

$$\int_\Omega B_k^n(x,u_n)(\sigma)\,dx \geq 0 \tag{3.11}$$

and

$$0 \leq \int_\Omega B_k^n(x,u_{0n})\,dx \leq k\int_\Omega |b_n(x,u_{0n})|\,dx \leq k\|b(x,u_0)\|_{L^1(\Omega)}. \tag{3.12}$$

Combining (3.8), (3.9), (3.10), (3.11) and (3.12) we get

$$\begin{aligned}
&\int_{Q_\sigma} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\,dt\\
&\leq \gamma_0 + k\overline{C} + C_k^\alpha + \frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} M\Big(|\nabla T_k(u_n)|\Big)\,dx\,dt\\
&\quad + \frac{\alpha}{\alpha+1}\int_{Q_\sigma} M\Big(|\nabla T_k(u_n)|\Big)\,dx\,dt,
\end{aligned} \tag{3.13}$$

where $\overline{C} = \|f\|_{L^1(\Omega)} + \|b(x,u_0)\|_{L^1(\Omega)}$. Thanks to ($H_3$), we deduce

$$\int_{Q_\sigma}\Big(\alpha - \frac{\alpha^2}{\alpha+2} - \frac{\alpha}{\alpha+1}\Big)M\Big(|\nabla T_k(u_n)|\Big)\,dx\,dt \leq \gamma_0 + k\overline{C} + C_k^\alpha.$$

Since $\Big(\alpha - \frac{\alpha^2}{\alpha+2} - \frac{\alpha}{\alpha+1}\Big) = \frac{\alpha^2}{(\alpha+1)(\alpha+2)} > 0$, finally we have

$$\int_{Q_T} M\Big(|\nabla T_k(u_n)|\Big)\,dx\,dt \leq (\gamma_0 + k\overline{C} + C_k^\alpha)\frac{(\alpha+1)(\alpha+2)}{\alpha^2} = C_k. \tag{3.14}$$

To prove (3.6), we combine (3.8), (3.9), (3.10), (3.12), (3.13), and (3.14) with $\widehat{C}_k = C_k^\alpha + C_k$. Finally, we prove (3.7), to this end, since $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$, there exist $\lambda > 0$ and a constant C_0 such that

$$\int_{Q_T} M\Big(\frac{|T_k(u_n)|}{\lambda}\Big)\,dx\,dt \leq C_0.$$

By using Young's inequality, we obtain

$$\begin{aligned}
\text{meas}\Big\{|u_n| > k\Big\} &= \frac{1}{k}\int_{\{|u_n|>k\}} k\,dx\,dt \leq \frac{1}{k}\int_{Q_T} |T_k(u_n)|\,dx\,dt\\
&\leq \frac{\lambda}{k}\Big(\int_{Q_T} M\Big(\frac{|T_k(u_n)|}{\lambda}\Big)\,dx\,dt + \int_{Q_T} \overline{M}(1)\,dx\,dt\Big)\\
&\leq \frac{\lambda}{k}\Big(C_0 + \overline{M}(1)|Q_T|\Big) \qquad \text{for all } n, \qquad \text{for all } k > 0,\\
&\longrightarrow 0 \quad \text{as } k \longrightarrow \infty,
\end{aligned} \tag{3.15}$$

which implies (3.7). □

Remark 3.6. Notice that, we can get differently another estimate like (3.15), by using the integral Poincaré's type inequality in inhomogeneous Orlicz spaces with the constants δ and λ. Hence,

$$\begin{aligned} M\Big(\delta k\Big)\text{meas}\Big\{|u_n| > k\Big\} &= \int_{\{|u_n|>k\}} M\Big(\delta|T_k(u_n)|\Big)\,dx\,dt \\ &\le \lambda \int_{Q_T} M\Big(|\nabla T_k(u_n|)\Big)\,dx\,dt. \end{aligned}$$

Then, from (3.14) we get

$$\text{meas}\Big\{|u_n| > k\Big\} \le \frac{\lambda C_k}{M\Big(\delta k\Big)} \quad \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

Lemma 3.7. *Let u_n be a solution of the approximate problem (3.4). Then*

(i) $u_n \longrightarrow u$ *a.e. in* Q_T,
(ii) $b_n(x, u_n) \longrightarrow b(x, u)$ *a.e. in* Q_T,
(iii) $b(x, u) \in L^\infty(0, T; L^1(\Omega))$.

Proof. For (i) and (ii), we argue as in [24, Proposition 5.3], we take a $C^2(\mathbb{R})$ nondecreasing function Γ_k such that $\Gamma_k(s) = \begin{cases} s & \text{for } |s| \le \frac{k}{2}, \\ k & \text{for } |s| \ge k, \end{cases}$ and multiplying the approximate problem (3.4) by $\Gamma'_k(u_n)$, we obtain

$$\begin{aligned} \frac{\partial B^n_\Gamma(x,u_n)}{\partial t} = \;& \text{div}\Big(a(x,t,u_n,\nabla u_n)\Gamma'_k(u_n)\Big) - a(x,t,u_n,\nabla u_n)\Gamma''_k(u_n)\nabla u_n \\ &+\text{div}\Big(\Gamma'_k(u_n)\Phi_n(x,t,u_n)\Big) - \Gamma''_k(u_n)\Phi_n(x,t,u_n)\nabla u_n + f_n\Gamma'_k(u_n), \end{aligned} \tag{3.16}$$

where $B^n_\Gamma(x,\tau) = \displaystyle\int_0^\tau \frac{\partial b^n_k(x,s)}{\partial s}\Gamma'_k(s)\,ds$.

Remarking that $\overline{M}^{-1} \circ M$ is an increasing function, $\gamma \in E_{\overline{M}}(Q_T)$, $supp(\Gamma'_k)$, $supp(\Gamma''_k) \subset [-k, k]$, and using Young's inequality, we get

$$\begin{aligned} &\Big|\int_{Q_T} \Gamma'_k(u_n)\Phi_n(x,t,u_n)\,dx\,dt\Big| \\ &\le \|\Gamma'_k\|_{L^\infty}\Big(\int_{Q_T} |\gamma(x,t)|\,dx\,dt + \int_{Q_T} \overline{M}^{-1}(M(|T_k(u_n)|))\,dx\,dt\Big) \\ &\le \|\Gamma'_k\|_{L^\infty}\Big(\int_{Q_T} \Big(\overline{M}(|\gamma(x,t)|) + M(1)\Big)\,dx\,dt + \int_{Q_T} \overline{M}^{-1}(M(k))\,dx\,dt\Big) \\ &< C_{1,k}, \end{aligned}$$

and (here, we use also estimate (3.14))

$$\begin{aligned}
&\Big|\int_{Q_T}\Gamma_k''(u_n)\Phi_n(x,t,u_n)\nabla u_n\,dx\,dt\Big|\\
&\le \|\Gamma_k''\|_{L^\infty}\Big(\int_{Q_T}|\gamma(x,t)|\,dx\,dt+\int_{Q_T}\overline{M}^{-1}(M(|T_k(u_n)|))|\nabla T_k(u_n)|\,dx\,dt\Big)\\
&\le \|\Gamma_k''\|_{L^\infty}\Big[\int_{Q_T}\Big(\overline{M}(|\gamma(x,t)|)+M(1)\Big)\,dx\,dt+\int_{Q_T}M(k)\,dx\,dt\\
&\qquad\qquad+\int_{Q_T}M(|\nabla T_k(u_n)|)\,dx\,dt\Big]\\
&< C_{2,k},
\end{aligned}$$

where $C_{1,k}$ and $C_{2,k}$ are two positive constants independent of n. Then each term in the right-hand side of (3.16) is bounded either in $L^1(Q_T)$ or in $W^{-1,x}L_{\overline{M}}(Q_T)$, which implies that

$$\frac{\partial B_\Gamma^n(x,u_n)}{\partial t}\ \text{is bounded in}\ L^1(Q_T)+W^{-1,x}L_{\overline{M}}(Q_T).$$

Moreover, due to the properties of Γ_k' and (3.1), we have

$$|\nabla B_\Gamma^n(x,u_n)|\le \|A_k\|_{L^\infty(\Omega)}|\nabla T_k(u_n)|\|\Gamma_k'\|_{L^\infty(\Omega)}+k\|\Gamma_k'\|_{L^\infty(\Omega)}\widetilde{A}_k(x),$$

which implies by (3.5), that

$$B_\Gamma^n(x,u_n)\ \text{is bounded in}\ W_0^{1,x}L_M(Q_T).$$

Arguing as in [9, 10, 24], we get (i) and (ii) of Lemma 3.7.

To prove (iii), using (ii), we pass to the limit inferior in (3.6) as $n\longrightarrow+\infty$, and we get

$$\frac{1}{k}\int_\Omega B_k(x,u)(\sigma)\,dx\le\frac{\widehat{C}_k}{k}+\Big(\|f\|_{L^1(Q_T)}+\|b(x,u_0\|_{L^1(\Omega)}\Big),$$

for almost any $\sigma\in(0,T)$. Tanks to the definition of $B_k(x,s)$ and the convergence of $\frac{1}{k}\int_\Omega B_k(x,u)$ to $b(x,u)$ as k goes to $+\infty$, this gives that $b(x,u)\in L^\infty(0,T;L^1(\Omega))$. □

The next lemma will be used later, we prove it now.

Lemma 3.8. *Let u_n be a solution of the approximate problem (3.4); then*

(i) $\{a(x,t,T_k(u_n),\nabla T_k(u_n))\}_n$ *is bounded in* $(L_{\overline{M}}(Q_T))^N$,

(ii) $$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n\,dx=0.$$

Proof. (i) We will use the Banach–Steinhaus theorem. Let $\phi\in(E_M(Q_T))^N$ be an arbitrary function. From (H_2), we can write

$$\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\phi)\Big)\cdot\Big(\nabla T_k(u_n)-\phi\Big)\ge 0,$$

which gives

$$\begin{aligned}&\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\phi\,dx\\&\leq \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\\&\quad+\int_{Q_T} a(x,t,T_k(u_n),\phi)(\phi-\nabla T_k(u_n))\,dx.\end{aligned}$$

Let us denote by J_1 and J_2 the first and the second integral, respectively, in the right-hand side of (3.16), so that

$$J_1=\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx.$$

Going back to (3.13), we obtain

$$\begin{aligned}J_1\leq\gamma_0+k\overline{C}+C_k^\alpha+\frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} M\Big(|\nabla T_k(u_n)|\Big)\,dx\,dt\\+\frac{\alpha}{\alpha+1}\int_{Q_\sigma} M\Big(|\nabla T_k(u_n)|\Big)\,dx\,dt,\end{aligned}$$

and thanks to (3.5), there exists a positive constant C_{J_1} independent of n such that

$$J_1\leq C_{J_1}.$$

Now we estimate the integral J_2. To this end, remark that

$$\begin{aligned}J_2&=\int_{Q_T} a(x,t,T_k(u_n),\phi)(\phi-\nabla T_k(u_n))\,dx\,dt\\&\leq\int_{Q_T}|a(x,t,T_k(u_n),\phi)||\phi|\,dx\,dt+\int_{Q_T}|a(x,t,T_k(u_n),\phi)||\nabla T_k(u_n)|\,dx\,dt.\end{aligned}$$

On the other hand, let η be large enough. From (H_1) and the convexity of $\overline{M}$, we get

$$\begin{aligned}&\int_{Q_T}\overline{M}\Big(\frac{|a(x,t,T_k(u_n),\phi)|}{\eta}\Big)\,dx\,dt\\&\leq\int_{Q_T}\overline{M}\Big(\frac{c(x,t)+\overline{M}^{-1}(P(k_1|T_k(u_n)|)+\overline{M}^{-1}(M(k_2|\phi|))}{\eta}\Big)\,dx\,dt\\&\leq\frac{1}{\eta}\int_{Q_T}\overline{M}(c(x,t))\,dx\,dt+\frac{1}{\eta}\int_{Q_T}\overline{M}\Big(\overline{M}^{-1}(P(k_1|T_k(u_n)|))\Big)\,dx\,dt\\&\quad+\frac{1}{\eta}\int_{Q_T}\overline{M}\Big(\overline{M}^{-1}(M(k_2|\phi|))\Big)\,dx\,dt\\&\leq\frac{1}{\eta}\int_{Q_T}\overline{M}(c(x,t))\,dx\,dt+\frac{1}{\eta}\int_{Q_T}P(k_1k)\,dx\,dt\\&\quad+\frac{1}{\eta}\int_{Q_T}M(k_2|\phi|)\,dx\,dt.\end{aligned}$$

Since $\phi\in(E_M(Q_T))^N$ and $c(x,t)\in E_{\overline{M}}(Q_T)$, we deduce that $\{a(x,t,T_k(u_n),\phi)\}$ is bounded in $(L_{\overline{M}}(Q_T))^N$ and we have $\{\nabla T_k(u_n)\}$ is bounded in $(L_M(Q_T))^N$.

Consequently, $J_2 \le C_{J_2}$, where C_{J_2} is a positive constant not depending on n. Then we obtain

$$\int_{Q_T} a(x, T_k(u_n), \nabla T_k(u_n))\phi \, dx \, dt \le C_{J_1} + C_{J_2}. \quad \text{for all } \phi \in (E_M(Q_T))^N.$$

Finally, $\{a(x,t,T_k(u_n), \nabla T_k(u_n))\}_n$ is bounded in $(L_{\overline{M}}(Q_T))^N$.

(ii) Testing (3.4) by $\theta_m(u_n) = T_{m+1}(u_n) - T_m(u_n)$, we have

$$\begin{aligned} &\int_\Omega B_m(x,u_n)(T)\, dx + \int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla\theta_m(u_n)\, dx\, dt \\ &\quad + \int_{Q_T} \Phi_n(x,t,u_n)\nabla\theta_m(u_n)\, dx\, dt = \int_\Omega B_m(x,u_{0n})\, dx + \int_{Q_T} f_n\theta_m(u_n)\, dx\, dt, \end{aligned} \tag{3.17}$$

where $B_m(x,\tau) = \displaystyle\int_0^\tau \frac{\partial b(x,s)}{\partial s}\theta_m(s)\, ds$. Since $B_m(x,u_n)(T) \ge 0$, hence from (H_3) and (H_4), it follows

$$\begin{aligned} &\alpha \int_{Q_T} M(|\nabla\theta_m(u_n)|)\, dx\, dt \\ &\le \int_{Q_T} \overline{M}^{-1}(M(|u_n|))|\nabla\theta_m(u_n)|\, dx\, dt + \int_{Q_T} |\gamma(x,t)||\nabla\theta_m(u_n)|\, dx\, dt \\ &\quad + \int_\Omega B_m(x,u_{0n})\, dx + \int_{Q_T} f_n\theta_m(u_n)\, dx\, dt. \end{aligned}$$

That means, knowing that $\nabla\theta_m(u_n) = \nabla u_n \chi_{E_m}$ a.e. in Q_T, where

$$E_m := \Big\{(x,t) \in Q_T : m \le |u_n| \le m+1\Big\},$$

and following the same argument as in the proof of (3.5) of Proposition 3.5, we get

$$\begin{aligned} &\alpha \int_{Q_T} M(|\nabla\theta_m(u_n)|)\, dx\, dt \\ &\le \int_{Q_T} \overline{M}^{-1}(M(|u_n|))|\nabla u_n|\chi_{E_m}\, dx\, dt + \int_{E_m} |\gamma(x,t)||\nabla\theta_m(u_n)|\, dx\, dt \\ &\quad + \int_\Omega B_m(x,u_{0n})\, dx + \int_{Q_T} f_n\theta_m(u_n)\, dx\, dt \\ &\le \int_{Q_T} \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|u_n|)\Big)\chi_{E_m}\, dx\, dt + \int_{Q_T} M\Big(\frac{\alpha}{\alpha+1}|\nabla\theta_m(u_n)|\Big)\, dx\, dt \\ &\quad + \frac{\alpha^2}{\alpha+2}\Big(\int_{E_m} \overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\, dx\, dt + \int_{Q_T} M\Big(|\nabla\theta_m(u_n)|\Big)\, dx\, dt\Big) \\ &\quad + \int_\Omega B_m(x,u_{0n})\, dx + \int_{Q_T} f_n\theta_m(u_n)\, dx\, dt. \end{aligned}$$

let $C^{\alpha}_{max} := \max\Big((\alpha+1), \dfrac{(\alpha+1)(\alpha+2)}{\alpha^2}\Big)$. It follows

$$\begin{aligned}&\int_{Q_T} M(|\nabla\theta_m(u_n)|)\,dx\,dt\\ &\leq C^{\alpha}_{max}\Big[\int_{E_m}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx\,dt+\int_\Omega B_m(x,u_{0n})\,dx\\ &\qquad+\int_{E_m}\overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|u_n|)\Big)\,dx\,dt+\int_{Q_T} f_n\theta_m(u_n)\,dx\,dt\Big].\end{aligned}\tag{3.18}$$

Now, let us concentrate on the convergence as $n\to\infty$ of each integral in (3.18), which can be treated by the same way (Lebesgue's dominated convergence theorem). Take, for example, the first one

$$\int_{\{m\leq|u_n|\leq m+1\}}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx=\int_\Omega\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\chi_{\{m\leq|u_n|\leq m+1\}}\,dx\,dt.$$

Put $g_n=\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\chi_{\{m\leq|u_n|\leq m+1\}}$, since χ is continuous, then

$$g_n\longrightarrow g=\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\chi_{\{m\leq|u|\leq m+1\}}\quad\text{a.e. in } Q_T.$$

We have $|g_n|\leq\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)$, which is integrable on Q_T, since $\gamma\in E_{\overline{M}}(Q_T)$. From Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_{Q_T} g_n\,dx\,dt=\int_{Q_T}\lim_{n\to\infty} g_n\,dx\,dt=\int_{Q_T}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\chi_{\{m\leq|u|\leq m+1\}}\,dx\,dt.$$

Passing to the limit as $n\to\infty$ in (3.18), we get

$$\begin{aligned}&\lim_{n\to\infty}\int_{Q_T} M(|\nabla\theta_m(u_n)|)\,dx\,dt\\ &\leq C^{\alpha}_{max}\Big[\int_{\{m\leq|u|\leq m+1\}}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x)|\Big)\,dx\,dt+\int_\Omega B_m(x,u_0)\,dx\\ &\qquad+\int_{\{m\leq|u|\leq m+1\}}\overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|u|)\Big)\,dx\,dt\\ &\qquad+\int_{Q_T} f\theta_m(u)\,dx\,dt\Big].\end{aligned}\tag{3.19}$$

Now, we will pass to the limit as $m\to\infty$, by Lebesgue's theorem each integral in (3.19) goes to zero as m goes to ∞, which gives

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{Q_T} M(|\nabla\theta_m(u_n)|)\,dx\,dt=0.\tag{3.20}$$

Our aim here is to prove that $\lim_{m\to\infty}\lim_{n\to\infty}\int_{Q_T}\Phi_n(x,t,u_n)\nabla\theta_m(u_n)\,dx\,dt = 0$, to this end, Young's inequality allows us to get

$$\int_{Q_T}\Phi_n(x,t,u_n)\nabla\theta_m(u_n)\,dx\,dt \le \int_{Q_T} M(|\nabla\theta_m(u_n)|)\,dx\,dt + \int_{E_m}\overline{M}(\Phi_n(x,t,u_n))\,dx\,dt. \tag{3.21}$$

We have already proved that the first integral in the right-hand side of (3.21) goes to zero as m and n go to ∞. It remains to show that the second one goes to zero again. Indeed, note that, for $n \ge m+1 \ge |u_n|$, we have $T_n(u_n) = T_{m+1}(u_n) = u_n$. Then, from (H_4) and the convexity of $\overline{M}$ we obtain

$$\begin{aligned}
&\int_{\{m\le|u_n|\le m+1\}}\overline{M}(\Phi_n(x,t,u_n))\,dx\,dt\\
&= \int_{\{m\le|u_n|\le m+1\}}\overline{M}(|\Phi(x,t,T_{m+1}(u_n))|)\,dx\,dt\\
&\le \int_{\{m\le|u_n|\le m+1\}}\overline{M}(\overline{M}^{-1}(M(|T_{m+1}(u_n)|))\,dx\,dt\\
&\le \int_{\{m\le|u_n|\le m+1\}} M(|T_{m+1}(u_n)|)\,dx\,dt\\
&\le \int_{Q_T} M(m+1)\,dx\,dt.
\end{aligned}$$

We deduce that

$$\begin{aligned}
&\int_{\{m\le|u_n|\le m+1\}}\overline{M}(|\Phi(x,t,T_{m+1}(u_n))|)\,dx\,dt\\
&= \int_{Q_T}\overline{M}(|\Phi(x,t,T_{m+1}(u_n))|)\,\chi_{\{m\le|u_n|\le m+1\}}dx\,dt \le C_{0,m}.
\end{aligned} \tag{3.22}$$

Let us denote $G_n^m = \overline{M}(|\Phi(x,t,T_{m+1}(u_n))|)\chi_{\{m\le|u_n|\le m+1\}} \longrightarrow G^m$ a.e. in Ω, where

$$G^m = \overline{M}(|\Phi(x,t,T_{m+1}(u)|)\,\chi_{\{m\le|u|\le m+1\}},$$

since $\overline{M}$ is continuous and Φ is a Carathéodory function. From (3.22), G_n^m is bounded independently of n. Using Lebesgue's theorem, it follows, as $n \longrightarrow \infty$

$$\int_{\{m\le|u_n|\le m+1\}}\overline{M}(|\Phi_n(x,t,u_n)|)\,dx\,dt \longrightarrow \int_{\{m\le|u|\le m+1\}}\overline{M}(|\Phi(x,t,u)|)\,dx\,dt.$$

Then

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}}\overline{M}(|\Phi_n(x,t,u_n)|)\,dx\,dt = 0. \tag{3.23}$$

Combining (3.20), (3.21), and (3.23), we get

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{Q_T}\Phi_n(x,t,u_n)\nabla\theta_m(u_n)\,dx\,dt = 0.$$

At the end, let $m, n \longrightarrow \infty$ in (3.17). Then we find

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{3.24}$$

□

Step 3: Almost everywhere convergence of the gradients.
In this step, most parts of the proof of the following proposition are the same argument as in [21].

Proposition 3.9. *Let u_n be a solution of the approximate problem (3.4). Then, for all $k \ge 0$, we have (for a subsequence still denoted by u_n) as $n \to +\infty$,*

(i) $\nabla u_n \to \nabla u$ *a.e. in* Q_T,

(ii) $a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u))$ *weakly in* $(L_{\overline{M}}(Q_T))^N$,

(iii) $M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|)$ *strongly in* $L^1(Q_T)$.

Proof. Let $\theta_j \in D(Q_T)$ be a sequence such that $\theta_j \longrightarrow u$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence and let $\psi_i \in D(\Omega)$ be a sequence that converges strongly to u_0 in $L^1(\Omega)$.

Put $Z_{i,j}^{\mu} = T_k(\theta_j)_\mu + \mathrm{e}^{-\mu t}T_k(\psi_i)$, where $T_k(\theta_j)_\mu$ is the mollification with respect to the time of $T_k(\theta_j)$. Notice that $Z_{\mu,j}^{i}$ is a smooth function having the following properties:

$$\frac{\partial Z_{i,j}^{\mu}}{dt} = \mu(T_k(\theta_j) - Z_{i,j}^{\mu}), \quad Z_{i,j}^{\mu}(0) = T_k(\psi_i) \quad \text{and } |Z_{i,j}^{\mu}| \le k,$$

$$Z_{i,j}^{\mu} \longrightarrow T_k(u)_\mu + \mathrm{e}^{-\mu t}T_k(\psi_i), \quad \text{in } W_0^{1,x}L_M(Q_T) \quad \text{modularly as } j \longrightarrow \infty,$$

$$T_k(u)_\mu + \mathrm{e}^{-\mu t}T_k(\psi_i) \longrightarrow T_k(u), \quad \text{in } W_0^{1,x}L_M(Q_T) \quad \text{modularly as } \mu \longrightarrow \infty.$$

Let now the function h_m be defined on $\mathbb{R}$ for any $m \ge k$ by

$$h_m(r) = \begin{cases} 1 & \text{if } |r| \le m, \\ -|r| + m + 1 & \text{if } m \le |r| \le m+1, \\ 0 & \text{if } |r| \ge m+1. \end{cases}$$

Put $E_m = \Big\{(x,t) \in Q_T : m \le |u_n| \le m+1\Big\}$. Testing the approximate problem (3.4) by the test function $\varphi_{n,j,m}^{\mu,i} = (T_k(u_n) - Z_{i,j}^{\mu})h_m(u_n)$, we get

$$\begin{aligned}
&\Big\langle \frac{\partial b_n(x,u_n)}{dt}, \varphi_{n,j,m}^{\mu,i} \Big\rangle + \int_{Q_T} a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla Z_{i,j}^{\mu})h_m(u_n)\,dx\,dt \\
&\quad + \int_{Q_T} a(x,t,u_n,\nabla u_n)(T_k(u_n) - Z_{i,j}^{\mu})\nabla u_n h_m'(u_n)\,dx\,dt \\
&\quad + \int_{E_m} \Phi_n(x,t,u_n)\nabla u_n h_m'(u_n)(T_k(u_n) - Z_{i,j}^{\mu})\,dx\,dt \\
&\quad + \int_{Q_T} \Phi_n(x,t,u_n)h_m(u_n)(\nabla T_k(u_n) - \nabla Z_{i,j}^{\mu})\,dx\,dt \\
&\quad = \int_{Q_T} f_n\varphi_{n,j,m}^{\mu,i}\,dx\,dt.
\end{aligned}$$

We simply denote by $\epsilon(n,j,\mu,i)$ and $\epsilon(n,j,\mu)$ any quantities such that

$$\lim_{i\to+\infty}\lim_{\mu\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon(n,j,\mu,i)=0,$$

$$\lim_{\mu\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon(n,j,\mu)=0.$$

□

We have the following lemma, which can be found in [21, 24].

Lemma 3.10 (see [21, 24]). *Let $\varphi^{\mu,i}_{n,j,m}=(T_k(u_n)-Z^{\mu}_{i,j})h_m(u_n)$. Then for any $k\geq 0$ we have*

$$\Big\langle\frac{\partial b_n(x,u_n)}{dt},\varphi^{\mu,i}_{n,j,m}\Big\rangle\geq\epsilon(n,j,\mu,i),$$

where $<,>$ denotes the duality pairing between $L^1(Q_T)+W^{-1,x}L_{\overline{M}}(Q_T)$ and $L^\infty(Q_T)\cap W^{1,x}_0L_M(Q_T)$.

To complete the proof of Proposition 3.9, we establish the results below. For any fixed $k\geq 0$, we have

(r_1) $\displaystyle\int_{Q_T}f_n\varphi^{\mu,i}_{n,j,m}\,dx\,dt=\epsilon(n,j,\mu).$

(r_2) $\displaystyle\int_{Q_T}\Phi_n(x,t,u_n)h_m(u_n)(\nabla T_k(u_n)-\nabla Z^{\mu}_{i,j})\,dx\,dt=\epsilon(n,j,\mu).$

(r_3) $\displaystyle\int_{E_m}\Phi_n(x,t,u_n)\nabla u_n h'_m(u_n)(T_k(u_n)-Z^{\mu}_{i,j})\,dx\,dt=\epsilon(n,j,\mu).$

(r_4) $\displaystyle\int_{Q_T}a(x,t,u_n,\nabla u_n)(T_k(u_n)-Z^{\mu}_{i,j})\nabla u_n h'_m(u_n)\,dx\,dt\leq\epsilon(n,j,\mu,m).$

(r_5) $\displaystyle\int_{Q_T}[a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)]$
$\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx\,dt\leq\epsilon(n,j,\mu,m,s).$

The proofs of (r_1), (r_3), (r_4), and (r_5) are the same as in [21, 24].

To prove (r_2), to this end, for $n\geq m+1$, we have

$$\Phi_n(x,t,u_n)h_m(u_n)=\Phi(x,t,T_{m+1}(u_n))h_m(T_{m+1}(u_n))\text{ a.e in }Q_T.$$

Put $P_n=\overline{M}\Big(\dfrac{|\Phi(x,t,T_{m+1}(u_n))-\Phi(x,t,T_{m+1}(u))|}{\eta}\Big)$. Since Φ is continuous with respect to its third argument and $u_n\longrightarrow u$ a.e in Q_T, then $\Phi(x,t,T_{m+1}(u_n))\to\Phi(x,t,T_{m+1}(u))$ a.e in Ω as n goes to infinity. Besides $\overline{M}(0)=0$, it follows

$$P_n\longrightarrow 0,\quad\text{a.e in }\Omega\text{ as }n\to\infty.$$

Using now the convexity of $\overline{M}$ and (H_4), we have for every $\eta > 0$ and $n \geq m+1$,

$$\begin{aligned} P_n &= \overline{M}\Big(\frac{|\Phi(x,t,T_{m+1}(u_n)) - \Phi(x,t,T_{m+1}(u))|}{\eta}\Big) \\ &\leq \overline{M}\Big(\frac{|\Phi(x,t,T_{m+1}(u_n))| + |\Phi(x,t,T_{m+1}(u))|}{\eta}\Big) \\ &\leq \overline{M}\Big(\frac{2}{\eta}|\gamma(x,t)| + \frac{2}{\eta}\overline{M}^{-1}(M(m+1))\Big) \\ &= \overline{M}\Big(\frac{1}{2}\frac{4}{\eta}|\gamma(x,t)| + \frac{1}{2}\frac{4}{\eta}\overline{M}^{-1}(M(m+1))\Big) \\ &\leq \frac{1}{2}\overline{M}(\frac{4}{\eta}|\gamma(x,t)|) + \frac{1}{2}\overline{M}(\frac{4}{\eta}\overline{M}^{-1}(M(m+1))). \end{aligned}$$

We put $C_m^\eta(x,t) = \frac{1}{2}\overline{M}(\frac{4}{\eta}|\gamma(x,t)|) + \frac{1}{2}\overline{M}(\frac{4}{\eta}\overline{M}^{-1}(M(m+1)))$. Since $\gamma \in E_{\overline{M}}(Q_T)$, we have $C_m^\eta \in L^1(Q_T)$, Then by Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_{Q_T} P_n\,dx\,dt = \int_{Q_T}\lim_{n\to\infty} P_n\,dx\,dt = 0.$$

This implies that $\{\Phi(x,t,T_{m+1}(u_n))\}$ converges modularly to $\Phi(x,t,T_{m+1}(u))$ as $n \to \infty$ in $(L_{\overline{M}}(Q_T))^N$. Moreover, $\Phi(x,t,T_{m+1}(u_n))$ and $\Phi(x,t,T_{m+1}(u))$ lie in $(E_{\overline{M}}(Q_T))^N$, indeed, from ($\mathrm{H}_4$), we have for every $\eta > 0$

$$\begin{aligned} &\int_{Q_T}\overline{M}\Big(\frac{|\Phi(x,t,T_{m+1}(u_n))|}{\eta}\Big)\,dx\,dt \\ &\leq \int_{Q_T}\overline{M}\Big(\frac{1}{\eta}|\gamma(x,t)| + \frac{1}{\eta}\overline{M}^{-1}(M(|T_{m+1}(u_n)|))\Big)\,dx\,dt \\ &\leq \int_{Q_T}\overline{M}\Big(\frac{1}{2}\frac{2}{\eta}|\gamma(x,t)| + \frac{1}{2}\frac{2}{\eta}\overline{M}^{-1}(M(m+1))\Big)\,dx\,dt \\ &\leq \int_{Q_T}\frac{1}{2}\overline{M}(\frac{2}{\eta}|\gamma(x,t)|)\,dx\,dt + \int_{Q_T}\frac{1}{2}\overline{M}\Big(\frac{2}{\eta}\overline{M}^{-1}(M(m+1))\Big)\,dx\,dt \\ &< \infty \text{ since } \gamma \in E_{\overline{M}}(Q_T) \text{ and } \Omega \text{ is bounded,} \end{aligned}$$

the same for $\Phi(x,t,T_{m+1}(u))$. Thanks to Lemma 2.3, we deduce that

$$\Phi(x,t,T_{m+1}(u_n)) \longrightarrow \Phi(x,t,T_{m+1}(u)) \text{ strongly in } (E_{\overline{M}}(Q_T))^N.$$

On the other hand, $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_M(Q_T))^N$ as n goes to infinity. It follows that

$$\begin{aligned} &\lim_{n\to\infty}\int_{Q_T}\Phi(x,t,u_n)h_m(u_n)[\nabla T_k(u_n) - \nabla Z_{i,j}^\mu]\,dx\,dt \\ &= \int_{Q_T}\Phi(x,t,u)h_m(u)[\nabla T_k(u) - \nabla Z_{i,j}^\mu]\,dx\,dt. \end{aligned}$$

Using the modular convergence of $Z_{i,j}^\mu$ as $j \longrightarrow \infty$ and then $\mu \longrightarrow \infty$, we get (r_2). As a consequence of Lemma 3.1, the results of Proposition 3.9 follow.

Step 4: Passing to the limit.
The limit u of the approximate solution u_n of (3.4) satisfies

$$\lim_{m\to\infty}\int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt = 0.$$

Proof. Fix $m > 0$, and we can write

$$\begin{aligned}&\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\ &= \Big(\int_{Q_T} a(x,t,u_n,\nabla u_n)(\nabla T_{m+1}(u_n)-\nabla T_m(u_n))\,dx\,dt\Big)\\ &= \Big(\int_{Q_T} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)\,dx\,dt\\ &\quad -\int_{Q_T} a(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n))\,dx\,dt\Big).\end{aligned}$$

Using Proposition 3.9(ii)-(iii) and passing to the limit as n goes to infinity for fixed m, we get

$$\begin{aligned}&\lim_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\\ &= \int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\nabla u\,dx.\end{aligned}$$

Finally, we pass to the limit as m goes to infinity and then we use (3.24). It follows

$$\begin{aligned}&\lim_{m\to\infty}\lim_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\ &= \lim_{m\to\infty}\int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt = 0,\end{aligned}$$

which gives the desired result.

Now, we will pass to the limit. Testing the approximate problem (3.4) by $r(u_n)$ with $r \in W^{1,\infty}(\mathbb{R})$ having a compact support such that for $k > 0$, $supp(r) \subset [-k, k]$, we get

$$\begin{aligned}&\frac{\partial B_r^n(x,u_n)}{\partial t} - \text{div}\,(r(u_n)a(x,t,u_n,\nabla u_n)) + r'(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n\\ &-\text{div}\,(r(u_n)\Phi(x,t,u_n)) + r'(u_n)\Phi(x,t,u_n)\nabla u_n = fr(u_n) \text{ in } D'(Q_T),\end{aligned}\tag{3.25}$$

where $B_r^n(x,\tau) = \displaystyle\int_0^\tau \frac{\partial b_n(x,s)}{\partial s} r'(s)\,ds.$

Our aim here is to pass to the limit in each term in the previous equality. Let us start by the terms of the left-hand side.

Limit of the first term $\dfrac{\partial B_r^n(x,u_n)}{\partial t}$, since r is bounded and $B_r^n(x,u_n) \longrightarrow B_r(x,u)$ a.e in Q_T and in $L^\infty(Q_T)$ weak*, then

$$\frac{\partial B_r^n(x,u_n)}{\partial t} \longrightarrow \frac{\partial B_r(x,u)}{\partial t} \quad \text{in} \quad D'(Q_T) \quad \text{as} \quad n\to\infty.$$

Remark that, since r and r' have a compact support in $\mathbb{R}$, there exists $k > 0$ such that $supp(r), supp(r') \subset [-k, k]$. For n large enough, we have

$$r(u_n)a(x,t,u_n,\nabla u_n) = r(u_n)a(x,t,T_k(u_n),\nabla T_k(u_n)) \quad \text{a.e. in } Q_T,$$

$$r'(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n = r'(u_n)a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \quad \text{a.e. in } Q_T,$$

$$r(u_n)\Phi_n(x,t,u_n) = r(T_k(u_n))\Phi_n(x,t,T_k(u_n)),$$

$$r'(u_n)\Phi_n(x,t,u_n)\nabla u_n = r'(T_k(u_n))\Phi_n(x,t,T_k(u_n))\nabla T_k(u_n).$$

For the second term of (3.25), since $r(u_n) \to r(u)$ a.e in Q_T as $n \to \infty$, r is bounded and by Proposition 3.9(ii)-(iii), we have

$$r(u_n)a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup r(u)a(x,t,T_k(u),\nabla T_k(u))$$

weakly in $(L_{\overline{M}}(Q_T))^N$ for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$. Then

$$r(u_n)a(x,t,u_n,\nabla u_n) \rightharpoonup r(u)a(x,t,u,\nabla u) \text{ weakly in } (L_{\overline{M}}(Q_T))^N.$$

Concerning the third term of (3.25), since $r'(u_n) \to r'(u)$ a.e in Q_T as $n \to \infty$, r' is bounded, and using Proposition 3.9(ii)-(iii) we obtain, as $n \to \infty$

$$r'(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n \rightharpoonup r'(u)a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) \text{ weakly in } L^1(Q_T).$$

Then

$$r'(u)a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) = r'(u)a(x,t,u,\nabla u)\nabla u \quad \text{a.e. in } Q_T.$$

Arguing similarly, we get the limit of the fourth term of (3.25),

$$r(u_n)\Phi_n(x,t,u_n) \to r(u)\Phi(x,t,u) \text{ strongly in } (E_M(Q_T))^N.$$

For the remaining term of the left-hand side, we have $r'(u_n)$ converges to $r'(u)$ and $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_M(Q_T))^N$ as $n \to +\infty$, while $\Phi_n(x, T_k(u_n))$ is uniformly bounded with respect to n and converges a.e. in Q_T to $\Phi(x, T_k(u))$ as n tends to $+\infty$. Therefore

$$r'(u_n)\Phi_n(x,t,u_n)\nabla u_n \rightharpoonup r'(u)\Phi(x,t,u)\nabla u \text{ weakly in } L_M(Q_T).$$

Concerning the right-hand side of (3.25), due to Lemma 3.7(i) and the fact that f_n converges strongly to f in $L^1(Q_T)$, we have

$$f_n r(u_n) \longrightarrow f r(u) \quad \text{strongly in } L^1(Q_T) \text{ as } n \to \infty.$$

Now, we are ready to pass to the limit as $n \to \infty$ in each term of (3.25) to conclude that u satisfies (3.3). It remains to show that $B_r(x,u)$ satisfies the initial condition of (3.4). To do this, recall that, r' has a compact support, we have $B_r^n(x,u_n)$ is bounded in $L^\infty(Q_T)$. Moreover, (3.25) and the above considerations on the behavior of the terms of this equation show that $\dfrac{\partial B_r^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q_T)+W^{-1,x}L_{\overline{M}}(Q_T)$. As a consequence, an Aubin's type lemma (see [25, Corollary 4]) and Lemma 2.6 imply that $B_r^n(x,u_n)$ is in a compact set of $C^0([0,T]; L^1(\Omega))$. It follows that $B_r^n(x,u_n)(t=0)$ converges to $B_r(x,u)(t=0)$ strongly in $L^1(\Omega)$. Due to Remark 3.3 and the fact that $b_n(x,u_{0n}) \longrightarrow b(x,u_0)$ in $L^1(\Omega)$, we conclude that $B_r^n(x,u_n)(t=0) = B_r^n(x,u_{0n})$ converges to $B_r(x,u)(t=0)$ strongly in $L^1(\Omega)$. Then we conclude that $B_r(x,u)(t=0) = B_r(x,u_0)$ in Ω.

That is the full proof of the main result. □

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