Khayyam J. Math. 7 (2021), no. 2, 211-218
DOI: 10.22034/KJM.2021.202782.1576

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

MULTI-DIMENSIONAL WAVELETS ON SOBOLEV SPACES

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Communicated by J.M. Aldaz

ABSTRACT. For admissible and integrable function ψ in $L^2(\mathbb{R}^n)$, the multi-dimensional continuous wavelet transform on Sobolev spaces is defined. The inversion formula for this transform on Sobolev spaces is established, and as a result, it is concluded that there is an isometry of Sobolev spaces $H_s(\mathbb{R}^n)$ into $H_{0,s}(\mathbb{R}^n \times \mathbb{R}_0^+ \times S^{n-1})$, for arbitrary real s. Also, among other things, it is shown that the range of this transform is a reproducing kernel Hilbert space and the reproducing kernel is found.

1. Introduction and preliminaries

Wavelet analysis is a particular time-scale or space-scale representation of signals, which has become popular in physics, mathematics, and engineering in the last decade. The transformed signal composed of the inner product with shifted and scaled versions of a fixed function is called analyzing or basic wavelets. In the literature, one often defines wavelet transform via an irreducible unitary representation of group of affine linear transformations of the real axis ($'ax+b'$-group). For a detailed description of these group theoretical aspects, we refer to [5–7].

Multi-dimensional wavelets may be derived from the similitude group of $\mathbb{R}^n (n > 1)$ denoted by $SIM(n) = \mathbb{R}^n \times (\mathbb{R}_0^+ \times So(n))$, consisting of dilations, rotations, and translations. This group has the following natural action on an n-dimensional signal

$$f_{b,a,R}(x) = [\pi(b,a,R)f](x) = a^{-n/2} f(a^{-1}R^{-1}(x-b)), \tag{1.1}$$

for all $(b,a,R) \in SIM(n)$. In [1, Theorem 14.2.1], it has been shown that the operator defined in (1.1) is a unitary irreducible representation of $SIM(n)$ in $L^2(\mathbb{R}^n)$. Also, this representation is square integrable. A vector $0 \neq \psi \in L^2(\mathbb{R}^n)$

Date: Received: 24 September 2019; Revised: 6 January 2021; Accepted: 11 January 2021.
1991 *Mathematics Subject Classification.* Primary 43A15; Secondary 43A85.
Key words and phrases. Sobolev space, similitude group, multi-dimensional wavelet, reproducing kernel.

is called admissible if $\prec \psi_{b,a,R}, \psi \succ_{L^2(\mathbb{R}^n)}$ is in $L^2(SIM(n))$. Moreover, one can check a vector $\psi \in L^2(\mathbb{R}^n)$ is admissible if and only if it satisfies

$$c_\psi = (2\pi)^n A_{n-1} \int_{\mathbb{R}^n} |\widehat{\psi}(k)|^2 \frac{dk}{|k|^n} < \infty, \tag{1.2}$$

where $A_{n-1} = \prod_{k=2}^{n-1} \frac{2\pi^{k/2}}{\Gamma(k/2)}$ is the volume of $So(n-1)$. The admissible vector ψ is called an admissible wavelet if $\|\psi\| = 1$. The continuous wavelet transform corresponding to the wavelet $\psi \in L^2(\mathbb{R}^n)$ is defined as

$$W_\psi f(b,a,R) = c_\psi^{-1/2} \prec \psi_{b,a,R}, f \succ, \tag{1.3}$$

for all $f \in L^2(\mathbb{R}^n)$. Note that if the wavelet ψ is axially symmetric, that is, $So(n-1)-$invariant, then one can replace everywhere $So(n)$ by $\frac{So(n)}{So(n-1)} \simeq S^{n-1}$, the unit sphere in $\mathbb{R}^n$. The rotation R becomes $R \equiv R(\varpi), \varpi \in So(n-1)$, and continuous wavelet transform leads to $W_\psi f(b,a,\varpi) \in L^2(X, d\upsilon)$, in which

$$X = \frac{SIM(n)}{So(n-1)} = \mathbb{R}^n \times (\mathbb{R}_0^+ \times S^{n-1}),$$

and $d\upsilon = \frac{da}{a^{n+1}} d\varpi db$ is an $SIM(n)-$invariant measure for X (for more details see [1]).

In [3], the discrete wavelet transform has been extended to the Sobolev spaces by Daubechies, and the continuous wavelet transform on affine group has been studied in [7]. In this article, we verify some known results for the extension of multi-dimensional wavelet transform to Sobolev spaces. In particular, we show that this continuous wavelet transform on Sobolev spaces is an isometry. Among other things, we show that the range of multi-dimensional continuous wavelet transform on Sobolev space is a reproducing kernel Hilbert space.

2. Main results

In this section, it is shown that the multi-dimensional wavelet transform of Sobolev space $H_s(\mathbb{R}^n)$ into $H_{0,s}(\mathbb{R}^n \times \mathbb{R}_0^+ \times S^{n-1})$ is an isometry, for arbitrary real s. Also, we investigate that the range of this continuous wavelet is a reproducing kernel Hilbert space and determine it. For the reader's convenience, we review the definition of Sobolev spaces (for more details one may see [4, 8]).

Suppose that $k \in \mathbb{N}$, and let H_k be the space of all $f \in L^2(\mathbb{R}^n)$ whose distribution derivatives $\partial^\alpha f$ are L^2-functions, for multi-index α with $|\alpha| \le k$. It has been shown that H_k is a Hilbert space with the inner product

$$(f,g)_k = \sum_{|\alpha| \le k} \int_{\mathbb{R}^n} (\partial^\alpha f)(x)(\overline{\partial^\alpha g})(x) dx,$$

for $f, g \in H_k$. It is more convenient to use an equivalent inner product defined in terms of the Fourier transform. One can check that $f \in H_k$ if and only if $(1+|\xi|^2)^{k/2} \widehat{f} \in L^2(\mathbb{R}^n)$ and that the norms $f \mapsto (\sum_{|\alpha| \le k} \|\partial^\alpha f\|_2^2)^{1/2}$ and $f \mapsto$

$\|(1+|\xi|^2)^{k/2}\widehat{f}\|_2$ are equivalent. Furthermore, the Sobolev space H_s for any $s \in \mathbb{R}$ is defined as

$$H_s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n); \int (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi < \infty\},$$

in which $\mathcal{S}'(\mathbb{R}^n)$ is the tempered distribution space. Moreover, the inner product and norm on $H_s(\mathbb{R}^n)$ are given by

$$(f,g)_s = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi$$

and

$$\|f\|_s = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi.$$

In what follows, we assume that $\psi \in L^2(\mathbb{R}^n)$ is an admissible wavelet and integrable. The multi-dimensional continuous wavelet transform is defined on the similitude group $SIM(n)$ as

$$W_\psi f(b,a,\varpi) = c_\psi^{-1/2} \prec \psi_{b,a,\varpi}, f \succ, \tag{2.1}$$

where $f \in L^2(\mathbb{R}^n)$ and $\psi_{b,a,\varpi}, c_\psi$ are as in (1.1) and (1.2), respectively. We now prove that for $f \in H_s(\mathbb{R}^n)$, $W_\psi f$ is in $L^2((\mathbb{R}_0^+ \times S^{n-1}, \frac{da}{a^{n+1}}d\varpi), H_s(\mathbb{R}^n))$ in which

$$\begin{aligned} &L^2((\mathbb{R}_0^+ \times S^{n-1}, \frac{da}{a^{n+1}}d\varpi), H_s(\mathbb{R}^n)) \\ &= \{f \in H_s(\mathbb{R}^n), \int_{\mathbb{R}_0^+ \times S^{n-1}} \|f(\cdot,a,\varpi)\|_s^2 \frac{da}{a^{n+1}} d\varpi < \infty\}. \end{aligned}$$

To this end, we need two auxiliary lemmas (Lemmas 2.1 and 2.3) as follows.

Lemma 2.1. *For an admissible wavelet $\psi \in L^2(\mathbb{R}^n)$, we have*

$$(W_\psi f(\cdot,a,\varpi))^\wedge(\xi) = c_\psi^{-1/2} a^{n/2} \widehat{\psi}(-a\varpi\xi)\widehat{\overline{f}}(\xi), \tag{2.2}$$

in which $f \in H_s(\mathbb{R}^n)$.

Proof. Let $\psi \in L^2(\mathbb{R}^n)$ be an admissible wavelet. For $f \in H_s(\mathbb{R}^n)$, we have

$$\begin{aligned} W_\psi f(b,a,\varpi) &= c_\psi^{-1/2} \prec \psi_{b,a,\varpi}, f \succ \\ &= c_\psi^{-1/2} \int_{\mathbb{R}^n} \psi_{b,a,\varpi}(x)\overline{f}(x)dx \\ &= c_\psi^{-1/2} \int_{\mathbb{R}^n} a^{-n/2}\psi(a^{-1}\varpi^{-1}(x-b))\overline{f}(x)dx \\ &= c_\psi^{-1/2} \int_{\mathbb{R}^n} D_{-a}L_\varpi\psi(b-x)\overline{f}(x)dx \\ &= c_\psi^{-1/2}(D_{-a}L_\varpi\psi * \overline{f})(b). \end{aligned}$$

Then

$$(W_\psi f(\cdot,a,\varpi))^\wedge(\xi) = c_\psi^{-1/2}(D_{-a}L_\varpi\psi * \overline{f})^\wedge(\xi)$$

$$= c_\psi^{-1/2}(D_{-a}L_\varpi\psi)\widehat{\ }(\xi)\overline{\widehat{\widehat{f}}}(\xi)$$

$$= c_\psi^{-1/2}a^{n/2}\widehat{\psi}(-a\varpi\xi)\overline{\widehat{f}}(\xi),$$

in which the operators $D_a\psi(x) := \frac{1}{\sqrt{a^n}}\psi(a^{-1}x)$ and $L_\varpi\psi(x) := \psi(\varpi^{-1}x)$ are dilation and rotation operators, respectively.

□

For the next lemma, we need to recall some notations and [4, Theorem 2.49], which we point out them here.

For $x \in \mathbb{R}^n \setminus \{0\}$, the polar coordinate of x is $r = |x|, x' = \frac{x}{|x|} \in S^{n-1}$. The map $\phi(x) = (r, x')$ is continuous bijection from $\mathbb{R}^n \setminus \{0\}$ to $(0,\infty) \times S^{n-1}$ whose continuous inverse is $\phi^{-1}(r, x') = rx'$. The Borel measure on $(0,\infty)\times S^{n-1}$, which is induced by ϕ from Lebesgue measure on $\mathbb{R}^n$, is denoted by m_* . Moreover, the measure ρ on $(0,\infty)$ is defined by $\rho(E) = \int_E r^{n-1}dr$ (see [4]).

Theorem 2.2 ([4, Theorem 2.49])**.** *There is a unique Borel measure σ on S^{n-1} such that $m_* = \rho \times \sigma$. If f is Borel measurable on $\mathbb{R}^n$ and $f \geq 0$ or $f \in L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} f(x)dx = \int_{(0,\infty)}\int_{S^{n-1}} f(rx')r^{n-1}d\sigma(x')dr. \tag{2.3}$$

Lemma 2.3. *With the above notations, let $\psi \in L^2(\mathbb{R}^n)$ be admissible and integrable. Then*

$$\int_{\mathbb{R}_0^+\times S^{n-1}} |\widehat{\psi}(a\varpi.\xi)|^2\frac{da}{a}d\varpi = \int_{\mathbb{R}^n}\frac{|\widehat{\psi}(\eta)|^2}{|\eta|^n}d\eta. \tag{2.4}$$

Proof. Set $\eta = a|\xi|\frac{\varpi\xi}{|\xi|}$, in which $a|\xi| = r = |\eta|$ and $\frac{\varpi\xi}{|\xi|} = x'$ are introduced as above. We have $da = \frac{dr}{|\xi|}$ and $|\xi| = \frac{|\eta|}{a}$. So by substituting in (2.3), we get

$$\int_{\mathbb{R}_0^+\times S^{n-1}} |\widehat{\psi}(a\varpi\xi)|^2\frac{da}{a}d\varpi = \int_{\mathbb{R}^n}\frac{|\widehat{\psi}(\eta)|^2}{|\eta|^n}d\eta.$$

□

Proposition 2.4. *Let $\psi \in L^2(\mathbb{R}^n)$ be admissible and integrable. For $f \in H_s(\mathbb{R}^n)$, the continuous wavelet transform $W_\psi f$ is in $L^2((\mathbb{R}_0^+ \times S^{n-1}, \frac{da}{a^{n+1}}d\varpi), H_s(\mathbb{R}^n))$.*

Proof. Fix $(a,\varpi) \in \mathbb{R}_0^+ \times S^{n-1}$, and let $f \in H_s(\mathbb{R}^n)$. First of all, we investigate that $W_\psi f(\cdot, a, \varpi)$ is in $H_s(\mathbb{R}^n)$. By using Lemma 2.1 and the fact

$$|\widehat{\psi}(\xi)| \leq \|\psi\|_1,$$

we get

$$\begin{aligned}\|W_\psi f(\cdot,a,\varpi)\|_s^2 &= \int_{\mathbb{R}^n}(1+|\xi|^2)^s|(W_\psi f(\cdot,a,\varpi))\widehat{\ }(\xi)|^2d\xi \\ &= c_\psi^{-1}\int_{\mathbb{R}^n}(1+|\xi|^2)^s|D_a(L_\varpi\psi)\widehat{\ }(\xi)|^2|\widehat{f}(\xi)|^2d\xi\end{aligned}$$

$$\leq c_\psi^{-1}\|L_\varpi\psi\|_1^2\|f\|_s^2$$

$$\leq c_\psi^{-1}\|\psi\|_1^2\|f\|_s^2.$$

Also,

$$\int_{\mathbb{R}_0^+\times S^{n-1}}\|W_\psi f(\cdot,a,\varpi)\|_s^2\frac{da}{a^{n+1}}d\varpi<\infty.$$

Now by using Lemma 2.3, we have

$$\int_{\mathbb{R}_0^+\times S^{n-1}}\|W_\psi f(\cdot,a,\varpi)\|_s^2\frac{da}{a^{n+1}}d\varpi$$

$$=c_\psi^{-1}\int_{\mathbb{R}_0^+\times S^{n-1}}\int_{\mathbb{R}^n}a^n(1+|\xi|^2)^s|\widehat{f}(\xi)|^2|\widehat{\psi}(a\xi\varpi)|^2\frac{da}{a^{n+1}}d\varpi d\xi$$

$$=c_\psi^{-1}(\int_{\mathbb{R}^n}\frac{|\widehat{\psi}(\xi)|^2}{|\xi|^n}d\xi)(\int_{\mathbb{R}^n}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2)$$

$$=\|f\|_s^2.$$

□

It is worthwhile to note that the inner product and the norm on $L^2((\mathbb{R}_0^+\times S^{n-1},\frac{da}{a^{n+1}}d\varpi),H_s(\mathbb{R}^n))$ are

$$\prec\prec\varphi,\psi\succ\succ=\int_{\mathbb{R}_0^+\times S^{n-1}}(\varphi(\cdot,a,\varpi),\psi(\cdot,a,\varpi))_s\frac{da}{a^{n+1}}d\varpi$$

and

$$|||\varphi|||=\int_{\mathbb{R}_0^+\times S^{n-1}}\|\varphi(\cdot,a,\varpi)\|_s^2\frac{da}{a^{n+1}}d\varpi,$$

for φ,ψ in $L^2((\mathbb{R}_0^+\times S^{n-1},\frac{da}{a^{n+1}}d\varpi),H^s(\mathbb{R}^n))$.
The inversion formula for the wavelet transform on the Sobolev spaces is given in the next theorem.

Theorem 2.5 (Inversion formula)**.** *Let $\psi\in L^2(\mathbb{R}^n)$ be admissible and integrable. Then for $f,g\in H_s(\mathbb{R}^n)$, it follows that*

$$(f,g)_s=\prec\prec W_\psi f,W_\psi g\succ\succ.$$

Proof. By Lemmas 2.1 and 2.3, for $f,g\in H_s(\mathbb{R}^n)$, we have

$$\begin{aligned}\prec\prec W_\psi f,W_\psi g\succ\succ &= \frac{1}{c_\psi}\int_{SIM(n)}(1+|\xi|^2)^s\widehat{f}(\xi)\overline{\widehat{g}(\xi)}|\widehat{\psi}(a\xi\varpi)|^2\frac{da}{a^{n+1}}d\varpi)d\xi\\ &= \frac{1}{c_\psi}\int_{\mathbb{R}^n}\int_{\mathbb{R}_0^+\times S^{n-1}}|\widehat{\psi}(a\varpi\xi)|^2(1+|\xi|^2)^s\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\frac{da}{a^{n+1}}d\varpi)d\xi\\ &= \frac{1}{c_\psi}(\int_{\mathbb{R}^n}\frac{|\widehat{\psi}(\xi)|^2}{|\xi|^n}d\xi)(f,g)_s\\ &= (f,g)_s.\end{aligned}$$

□

As an important consequence of the inversion formula is that, the continuous multi-dimensional wavelet transform is an isometry from $H_s(\mathbb{R}^n)$ into $H_{0,s}(\mathbb{R}^n\times\mathbb{R}_0^+\times S^{n-1})$.

Corollary 2.6. *The multi-dimensional continuous wavelet transform is an isometry from $H_s(\mathbb{R}^n)$ into $H_{0,s}(\mathbb{R}^n\times\mathbb{R}_0^+\times S^{n-1})$. In particular, $|||W_\psi f|||=\|f\|_s$.*

Proof. By using [2, Theorems 12.6.1 and 12.7.2], we have

$$\begin{aligned}L^2((\mathbb{R}_0^+\times S^{n-1},\frac{da}{a^{n+1}}d\varpi),H_s(\mathbb{R}^n)) &\simeq L^2(\mathbb{R}_0^+\times S^{n-1})\otimes H_s(\mathbb{R}^n)\\ &\simeq H_{0,s}(\mathbb{R}^n\times\mathbb{R}_0^+\times S^{n-1}).\end{aligned}$$

□

Proposition 2.7. *Let $\varphi,\psi\in L^2(\mathbb{R}^n)$ be two admissible wavelets. For $f,g\in H_s(\mathbb{R}^n)$, we have*

$$\prec\prec W_\varphi f,W_\psi g\succ\succ=\frac{c_{\varphi,\psi}}{\sqrt{c_\varphi c_\psi}}(f,g)_s,$$

in which $c_{\varphi,\psi}=\int_{\mathbb{R}^n}|\widehat{\varphi}(\xi)||\widehat{\psi}(\xi)|\frac{d\xi}{|\xi|^n}$.

Proof. Let $f,g\in H_s(\mathbb{R}^n)$. Then by Lemma 2.3, we have

$$\begin{aligned}&\prec\prec W_\varphi f,W_\psi g\succ\succ\\ &=\frac{1}{\sqrt{c_\psi c_\varphi}}\int_{SIM(n)}(1+|\xi|^2)^s\widehat{f}(\xi)\overline{\widehat{g}(\xi)}|\widehat{\varphi}(a\varpi\xi)||\widehat{\psi}(a\varpi\xi)|\frac{da}{a^{n+1}}d\varpi)d\xi\\ &=\frac{1}{\sqrt{c_\psi c_\varphi}}\int_{\mathbb{R}^n}\int_{\mathbb{R}_0^+\times S^{n-1}}|\widehat{\varphi}(a\varpi^{-1}.\xi)||\widehat{\psi}(a\varpi^{-1}.\xi)|(1+|\xi|^2)^s\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\frac{da}{a^{n+1}}d\varpi)d\xi\\ &=\frac{1}{\sqrt{c_\psi c_\varphi}}(f,g)_s\int_{\mathbb{R}^n}\frac{|\widehat{\varphi}(\xi)||\widehat{\psi}(\xi)|}{|\xi|^n}d\xi\\ &=\frac{c_{\varphi,\psi}}{\sqrt{c_\varphi c_\psi}}(f,g)_s.\end{aligned}$$

□

In the next theorem, we obtain an explicit expression for W_ψ^* defined from $H_{0,s}(\mathbb{R}^n\times\mathbb{R}_0^+\times S^{n-1})$ into $H_s(\mathbb{R}^n)$.

Theorem 2.8. *The range of continuous wavelet transform defined as (2.1) on Sobolev space is a reproducing kernel Hilbert space with the reproducing kernel*

$$K((b_2,a_2,\varpi_2),(b_1,a_1,\varpi_1))=\frac{1}{\sqrt{c_\psi}}W_\psi\psi(a_2^{-1}\varpi_2^{-1}(b_2-b_1),a_1^{-1}a_2,\varpi_1^{-1}\varpi_2).\quad(2.5)$$

Proof. Let $f\in\mathcal{S}(\mathbb{R}^n)$ and let $g(b,a,\varpi)=g_1(b)g_2(a,\varpi)$, in which $g_1\in\mathcal{S}(\mathbb{R}^n)$, $g_2\in C_0^\infty(\mathbb{R}_0^+\times S^{n-1})$. We have

$$\prec\prec W_\psi f,g\succ\succ$$

$$= \int_{\mathbb{R}_0^+ \times S^{n-1}} (W_\psi f(\cdot, a, \varpi), g(\cdot, a, \varpi))_s \frac{da}{a^{n+1}} d\varpi$$

$$= \int_{\mathbb{R}_0^+ \times S^{n-1}} \int_{\mathbb{R}^n} (1+|\xi|^2)^s (W_\psi f(\cdot, a, \varpi))\widehat{\ }(\xi) \widehat{g}(\cdot, a, \varpi)(\xi) \frac{da}{a^{n+1}} d\varpi d\xi$$

$$\leq \int_{\mathbb{R}_0^+ \times S^{n-1}} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |(W_\psi f(\cdot, a, \varpi))\widehat{\ }(\xi)|^2 d\xi \right)^{1/2}$$

$$\cdot \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{g}(\cdot, a, \varpi)(\xi)|^2 d\xi \right)^{1/2} \frac{da}{a^{n+1}} d\varpi$$

$$\leq \int_{\mathbb{R}_0^+ \times S^{n-1}} \|W_\psi f(\cdot, a, \varpi)\|_s \|g(\cdot, a, \varpi)\|_s \frac{da}{a^{n+1}} d\varpi$$

$$\leq |||W_\psi f|||.|||g|||,$$

which allows one to change the order of integration, so

$$\prec\prec W_\psi f, g \succ\succ$$

$$= c_\psi^{-1/2} \int_{\mathbb{R}^n} (1+|\xi|^2)^s \widehat{f}(\xi) \int_{\mathbb{R}_0^+ \times S^{n-1}} (D_{-a} L_\varpi \psi)\widehat{\ }(\xi) \widehat{g}(\cdot, a, \varpi)(\xi) \frac{da}{a^{n+1}} d\varpi d\xi.$$

Set

$$Tg(\xi) = c_\psi^{-1/2} \int_{\mathbb{R}_0^+ \times S^{n-1}} (D_{-a} L_\varpi \psi)\widehat{\ }(\xi) \widehat{g}(\cdot, a, \varpi)(\xi) \frac{da}{a^{n+1}} d\varpi.$$

Then $Tg \in L^2(\mathbb{R}^n)$. In fact,

$$\begin{aligned} |Tg(\xi)|^2 &\leq c_\psi^{-1} \int_{\mathbb{R}_0^+ \times S^{n-1}} |(D_{-a} L_\varpi \psi)\widehat{\ }(\xi)|^2 |\widehat{g}(\cdot, a, \varpi)(\xi)|^2 \frac{da}{a^{n+1}} d\varpi \\ &\leq c_\psi^{-1} \|D_{-a} L_\varpi \psi\|_1^2 \int_{\mathbb{R}_0^+ \times S^{n-1}} |\widehat{g}(\cdot, a, \varpi)(\xi)|^2 \frac{da}{a^{n+1}} d\varpi, \end{aligned}$$

and using the Cauchy Schwarz inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |Tg(\xi)|^2 d\xi &\leq c_\psi^{-1} \|D^{-a} L_\varpi \psi\|_1^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}_0^+ \times S^{n-1}} |\widehat{g}(\cdot, a, \varpi)(\xi)|^2 \frac{da}{a^{n+1}} d\varpi \\ &\leq c_\psi^{-1} \|D^{-a} L_\varpi \psi\|_1^2 |||g|||_{s=0}. \end{aligned}$$

Therefore, there exists $W_\psi^* g \in L^2(\mathbb{R}^n)$ such that $(W_\psi^* g)\widehat{\ }(\xi) = Tg(\xi)$. Then we get

$$\prec\prec W_\psi f, g \succ\succ = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \widehat{f}(\xi) \widehat{Tg}(\xi) d\xi = (f, W_\psi^* g)_s.$$

Moreover,

$$W_\psi^* g(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{Tg}(\xi) e^{i\xi . x} d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \frac{1}{\sqrt{c_\psi}} \int_{\mathbb{R}_0^+ \times S^{n-1}} \int_{\mathbb{R}^n} (D_{-a} L_\varpi \psi * g(\cdot, a, \varpi))\widehat{\ }(\xi) e^{i\xi x} d\xi \frac{da}{a^{n+1}} d\varpi$$

$$= \frac{1}{\sqrt{c_\psi}} \int_{\mathbb{R}_0^+ \times S^{n-1}} D_{-a} L_\varpi \psi * g(x, a, \varpi) \frac{da}{a^{n+1}} d\varpi.$$

Now, we show that the range of W_ψ is a reproducing kernel Hilbert space with reproducing kernel:

$$K((b_2, a_2, \varpi_2), (b_1, a_1, \varpi_1)) = \frac{1}{\sqrt{c_\psi}} W_\psi \psi(a_2^{-1} \varpi_2^{-1}(b_2 - b_1), a_1^{-1} a_2, \varpi_1^{-1} \varpi_2).$$

Indeed, $g \in rangeW_\psi$ if and only if $W_\psi W_\psi^* g = g$. So,

$$\begin{aligned} g(b_2, a_2, \varpi_2) &= W_\psi W_\psi^* g(b_2, a_2, \varpi_2) \\ &= \frac{1}{\sqrt{c_\psi}} \prec \psi_{b_2, a_2, \varpi_2}, W_\psi^* g \succ \\ &= \frac{1}{\sqrt{c_\psi}} \prec W_\psi \psi_{b_2, a_2, \varpi_2}, g \succ \\ &= \frac{1}{\sqrt{c_\psi}} \int_{\mathbb{R}_0^+ \times S^{n-1}} \int_{\mathbb{R}^n} W_\psi \psi_{b_2, a_2, \varpi_2}(b_1, a_1, \varpi_1) \overline{g}(b_1, a_1, \varpi_1) db_1 \frac{da_1}{a_1^{n+1}} d\varpi_1 \\ &= \frac{1}{c_\psi} \int_{\mathbb{R}_0^+ \times S^{n-1}} \int_{\mathbb{R}^n} \prec \psi_{b_1, a_1, \varpi_1}, \psi_{b_2, a_2, \varpi_2} \succ \overline{g}(b_1, a_1, \varpi_1) db_1 \frac{da_1}{a_1^{n+1}} d\varpi_1. \end{aligned}$$

□

Acknowledgement. The author would like to thank Professor R.A. Kamyabi Gol for his helpful consult and also reading this paper. The author would also like to thank referees for their valuable comments and remarks.

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