



MODULATION INVARIANT SPACES ON LOCALLY COMPACT ABELIAN GROUPS

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ABSTRACT. We define and investigate modulation invariant spaces on a locally compact abelian group G with respect to a closed subgroup of the dual group \widehat{G} . Using a range function approach, we establish a characterization of modulation invariant spaces. Finally, we define a metric on the collection of all modulation invariant spaces and study some topological properties of the metric space.

1. INTRODUCTION AND PRELIMINARIES

For a locally compact abelian (LCA) group G , a translation invariant space is defined to be a closed subspace of $L^2(G)$ that is invariant under translations by elements of a closed subgroup Γ of G . Following the main idea in [8, 9], translation invariant spaces in case of Γ closed, discrete, and cocompact, called shift invariant spaces, have been studied in [3, 4, 11–13, 16], and extended to the case of Γ closed and cocompact (but not necessarily discrete) in [2]. Recently, translation invariant spaces have been generalized in [1] to the case when Γ is closed (not necessarily discrete or cocompact); see also [10].

The two useful tools in the Gabor theory are translation invariant spaces and modulation invariant spaces [6]. Because of the significance of the Gabor theory in mathematical analysis, it is important to study modulation invariant spaces as well as translation invariant spaces. Our main result in this article is a characterization of modulation invariant spaces in terms of range functions. The basic idea is the fact that the image of a modulation invariant subspace of $L^2(G)$ under the Fourier transform is a translation invariant subspace of $L^2(\widehat{G})$. We use this

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fact to transform modulation invariant spaces to translation invariant spaces, and then we follow the ideas in [1, 2]. Indeed, we achieve our goal by transforming $L^2(G)$ into a vector valued space, in such a way that modulations by a closed subgroup of \widehat{G} become multiplications by a nice family of functions, and then using the ideas of [2], we characterize modulation invariant spaces in terms of range functions. The other goal of the article is to consider the collection of all modulation invariant spaces as a metric space. Using the range functions, we define a metric on the collection of all modulation invariant spaces and study some topological properties of this metric space. For example, we show that it is a complete, noncompact, and disconnected metric space. The manuscript is organized as follows. The rest of this section is devoted to stating some required facts about translation invariant spaces from [1, 2]. Section 2 contains our main results related to modulation invariant spaces. We investigate modulation invariant spaces using a range function approach. Following the ideas of Section 2, we then define and investigate a topology (in fact a metric) on the collection of all modulation invariant spaces on G in Section 3.

Let (Ω, m) be a σ -finite measure space and let \mathcal{H} be a separable Hilbert space. A range function is a mapping $J : \Omega \rightarrow \{ \text{closed subspaces of } \mathcal{H} \}$. We write $P_J(\omega)$ for the orthogonal projection of \mathcal{H} onto $J(\omega)$. A range function J is measurable if the mapping $\omega \mapsto \langle P_J(\omega)(a), b \rangle$ is measurable for all $a, b \in \mathcal{H}$. Consider the space $L^2(\Omega, \mathcal{H})$ of all measurable functions ϕ from Ω to \mathcal{H} such that $\|\phi\|_2^2 = \int_{\Omega} \|\phi(\omega)\|_{\mathcal{H}}^2 dm(\omega) < \infty$ with the inner product $\langle \phi, \psi \rangle = \int_{\Omega} \langle \phi(\omega), \psi(\omega) \rangle_{\mathcal{H}} dm(\omega)$. A subset \mathcal{D} of $L^\infty(\Omega)$ is said to be a determining set for $L^1(\Omega)$, if for all $f \in L^1(\Omega)$, $\int_{\Omega} fg dm = 0$ for all $g \in \mathcal{D}$ implies that $f = 0$. A closed subspace V of $L^2(\Omega, \mathcal{H})$ is called a multiplicatively invariant subspace with respect to a determining set \mathcal{D} , if for each $\phi \in V$ and $g \in \mathcal{D}$ one has $g\phi \in V$. Bownik and Ross in [2, Theorem 2.4] showed that there is a correspondence between multiplicatively invariant spaces and measurable range functions as follows.

Proposition 1.1. *Suppose that $L^2(\Omega)$ is separable such that $L^2(\Omega, \mathcal{H})$ is also separable. Then for a closed subspace V of $L^2(\Omega, \mathcal{H})$ and a determining set \mathcal{D} for $L^1(\Omega)$, the following statements are equivalent:*

- (1) V is multiplicatively invariant with respect to \mathcal{D} .
- (2) V is multiplicatively invariant with respect to $L^\infty(\Omega)$.
- (3) There exists a measurable range function J such that

$$V = \{ \phi \in L^2(\Omega, \mathcal{H}) : \phi(\omega) \in J(\omega) , \quad a.e. \omega \in \Omega \}.$$

Identifying range functions that are equivalent almost everywhere, the correspondence between \mathcal{D} -multiplicatively invariant spaces and measurable range functions is one to one and onto. Moreover, there is a countable subset \mathcal{A} of $L^2(\Omega, \mathcal{H})$ such that V is the smallest closed \mathcal{D} -multiplicatively invariant space containing \mathcal{A} . For any such \mathcal{A} , the measurable range function associated with V satisfies

$$J(\omega) = \overline{\text{span}}\{ \phi(\omega) : \phi \in \mathcal{A} \} \quad a.e. \omega \in \Omega.$$

Now assume that G is a second countable LCA group and that Γ is a closed subgroup of G . Let Γ^* be the annihilator of Γ in \widehat{G} . Also suppose that Ω is a

measurable section for the quotient \widehat{G}/Γ^* and that C is a measurable section for the quotient G/Γ . For $\gamma \in \Gamma$, we denote by X_γ the associated character on \widehat{G} , that is, $X_\gamma(\chi) = \chi(\gamma)$ for $\chi \in \widehat{G}$. One can see that the set $\{X_\gamma|_\Omega : \gamma \in \Gamma\}$ is a determining set for $L^1(\Omega)$. A closed subspace $V \subseteq L^2(G)$ is called a Γ -translation invariant space, if $T_\gamma V \subseteq V$ for all $\gamma \in \Gamma$. We say that V is generated by a countable subset \mathcal{A} of $L^2(G)$, when $V = S^\Gamma(\mathcal{A}) = \overline{\text{span}}\{T_\gamma f : f \in \mathcal{A}, \gamma \in \Gamma\}$. In [1, Proposition 6.4], it was shown that there exists an isometric isomorphism between $L^2(G)$ and $L^2(\Omega, L^2(C))$, namely, $Z : L^2(G) \rightarrow L^2(\Omega, L^2(C))$ satisfying

$$Z(T_\gamma \phi) = X_\gamma|_\Omega Z(\phi). \tag{1.1}$$

Let Z be as in (1.1). The forthcoming proposition, which is [1, Theorem 6.5], states that Z turns Γ -translation invariant spaces in $L^2(G)$ into multiplicatively invariant spaces in $L^2(\Omega, L^2(C))$ with respect to the determining set $\mathcal{D} = \{X_\gamma|_\Omega : \gamma \in \Gamma\}$ and vice versa. It also establishes a characterization of Γ -translation invariant spaces in terms of range functions.

Proposition 1.2. *Let $V \subseteq L^2(G)$ be a closed subspace and let Z be as in (1.1). Then the following statements are equivalent:*

- (1) V is a Γ -translation invariant space.
- (2) $Z(V)$ is a multiplicatively invariant subspace of $L^2(\Omega, L^2(C))$ with respect to the determining set $\mathcal{D} = \{X_\gamma|_\Omega : \gamma \in \Gamma\}$.
- (3) There exists a measurable range function

$$J : \Omega \rightarrow \{\text{closed subspaces of } L^2(C)\}$$

such that

$$V = \{f \in L^2(G) : Z(f)(\omega) \in J(\omega), \text{ for a.e. } \omega \in \Omega\}.$$

Identifying range functions that are equivalent almost everywhere, the correspondence between Γ -translation invariant spaces and measurable range functions is one to one and onto. Moreover if $V = S^\Gamma(\mathcal{A})$ for some countable subset \mathcal{A} of $L^2(G)$, the measurable range function J associated with V is given by

$$J(\omega) = \overline{\text{span}}\{Z(\phi)(\omega) : \phi \in \mathcal{A}\} \text{ a.e. } \omega \in \Omega.$$

2. MODULATION INVARIANT SPACES

Our goal in this section is a characterization of modulation invariant spaces in terms of range functions. We transfer modulation invariant spaces to translation invariant spaces, and then we use Proposition 1.2 to characterize modulation invariant spaces in terms of range functions. Let G be an LCA group and let Λ be a closed subgroup of \widehat{G} , which is not necessarily discrete or cocompact. A closed subspace $W \subseteq L^2(G)$ is called Λ -modulation invariant, if $M_\lambda W \subseteq W$ for all $\lambda \in \Lambda$, where M_λ is the modulation operator defined as $M_\lambda : L^2(G) \rightarrow L^2(G)$, $M_\lambda f(x) = \lambda(x)f(x)$. Similar to the translation case, we say that W is generated by a countable subset \mathcal{A} of $L^2(G)$, when $W = M^\Lambda(\mathcal{A}) = \overline{\text{span}}\{M_\lambda f : f \in \mathcal{A}, \lambda \in \Lambda\}$. When $\mathcal{A} = \{\varphi\}$, $M^\Lambda(\varphi)$ is called a principal modulation invariant space. Assume that Λ^* is the annihilator of Λ in G . In addition, suppose that Π is a measurable section for the quotient G/Λ^* and that D is a measurable section for

the quotient \widehat{G}/Λ . For $\lambda \in \Lambda$, we denote by X_λ the associated character on G . One can see that the set $\mathcal{D} = \{X_\lambda|_\Pi : \lambda \in \Lambda\}$ is a determining set for $L^1(\Pi)$. Let $Z : L^2(\widehat{G}) \rightarrow L^2(\Pi, L^2(D))$ be similar to (1.1) and let \mathcal{F} be the Fourier transform on $L^2(G)$. We define an isometric isomorphism as

$$\tilde{Z} : L^2(G) \rightarrow L^2(\Pi, L^2(D)), \quad \tilde{Z} := Z \circ \mathcal{F}. \tag{2.1}$$

In the next theorem, we show that \tilde{Z} turns Λ -modulation invariant spaces in $L^2(G)$ into multiplicatively invariant spaces in $L^2(\Pi, L^2(D))$ and vice versa. Furthermore, we establish a characterization of Λ -modulation invariant spaces in terms of range functions. The main idea of the proof is that the Fourier transform maps Λ -modulation invariant subspaces of $L^2(G)$ to Λ -translation invariant subspaces of $L^2(\widehat{G})$.

Theorem 2.1. *Let $W \subseteq L^2(G)$ be a closed subspace and let \tilde{Z} be as in (2.1). Then the following statements are equivalent:*

- (1) W is a Λ -modulation invariant space.
- (2) $\tilde{Z}(W)$ is a multiplicatively invariant subspace of $L^2(\Pi, L^2(D))$ with respect to the determining set $\mathcal{D} = \{X_\lambda|_\Pi : \lambda \in \Lambda\}$.
- (3) There exists a measurable range function

$$J : \Pi \rightarrow \{\text{closed subspaces of } L^2(D)\}$$

such that

$$W = \{f \in L^2(G) : \tilde{Z}(f)(x) \in J(x), \text{ for a.e. } x \in \Pi\}. \tag{2.2}$$

Identifying range functions that are equivalent almost everywhere, the correspondence between Λ -modulation invariant spaces and measurable range functions is one to one and onto. Moreover if $W = M^\Lambda(\mathcal{A})$, for some countable subset \mathcal{A} of $L^2(G)$, the measurable range function J associated with W is given by

$$J(x) = \overline{\text{span}}\{\tilde{Z}(\phi)(x) : \phi \in \mathcal{A}\} \text{ a.e. } x \in \Pi.$$

Proof. For (1) \Rightarrow (2) assume that W is a Λ -modulation invariant space and that \mathcal{F} is the Fourier transform on $L^2(G)$. Then $\mathcal{F}(W)$ is clearly a Λ -translation invariant subspace of $L^2(\widehat{G})$. By [1, Theorem 6.5], $\tilde{Z}(W) = Z(\mathcal{F}(W))$ is a multiplicatively invariant subspace of $L^2(\Pi, L^2(D))$ with respect to the determining set $\mathcal{D} = \{X_\lambda|_\Pi : \lambda \in \Lambda\}$. For (2) \Rightarrow (3), suppose that $\tilde{Z}(W)$ is multiplicatively invariant. By Proposition 1.1,

$$\tilde{Z}(W) = \{\phi \in L^2(\Pi, L^2(D)) : \phi(x) \in J(x), \text{ a.e. } x \in \Pi\},$$

for some measurable range function J . Applying \tilde{Z}^{-1} , we get

$$W = \{f \in L^2(G) : \tilde{Z}(f)(x) \in J(x), \text{ for a.e. } x \in \Pi\}.$$

Finally, for (3) \Rightarrow (1), assume that (2.2) holds for some measurable range function J . For $f \in W$ and $\lambda \in \Lambda$, using (1.1), we have

$$\begin{aligned} \tilde{Z}(M_\lambda f)(x) &= Z(\mathcal{F}(M_\lambda f))(x) \\ &= Z(T_\lambda \hat{f})(x) \end{aligned}$$

$$\begin{aligned}
&= X_\lambda|_\Pi(x)Z(\hat{f})(x) \\
&= X_\lambda|_\Pi(x)\tilde{Z}(f)(x) \in J(x) \text{ a.e. } x \in \Pi,
\end{aligned}$$

which proves (1). Now let $W = M^\Lambda(\mathcal{A})$ for some countable subset \mathcal{A} of $L^2(G)$. Put $\mathcal{B} = \tilde{Z}(\mathcal{A})$; then \mathcal{B} is a countable subset of $L^2(\Pi, L^2(D))$. By (1.1)

$$\tilde{Z}(W) = \overline{\text{span}}\{X_\lambda(x)\phi(x) : \lambda \in \Lambda, \phi \in \mathcal{B}\}.$$

Now by Proposition 1.1, the range function associated with $\tilde{Z}(W)$ is

$$J(x) = \overline{\text{span}}\{\Phi(x) : \Phi \in \mathcal{B}\} = \overline{\text{span}}\{\tilde{Z}\phi(x) : \phi \in \mathcal{A}\}.$$

The proof of one to one correspondence between Λ -modulation invariant spaces and measurable range functions is similar to [1, Theorem 6.5] and is omitted. \square

Corollary 2.2. *For an LCA group G , a closed subspace W of $L^2(G)$ that is invariant under all modulations (i.e., W is \widehat{G} -modulation invariant) can be written as*

$$W = \{f \in L^2(G) : \text{supp } f \subseteq F\}, \quad (2.3)$$

for some measurable subset $F \subseteq G$.

Proof. In this case, $\mathcal{F}(W)$ is a \widehat{G} -translation invariant subspace of $L^2(\widehat{G})$. By [1, Remark 6], $\mathcal{F}(W) = \{f \in L^2(\widehat{G}) : \text{supp } \hat{f} \subseteq E\}$, for some measurable subset $E \subseteq G$ and (2.3) follows. \square

For a countable subset $\mathcal{A} \subseteq L^2(G)$, we can give characterizations of frames and Riesz bases generated by \mathcal{A} in terms of the operator \tilde{Z} defined in (2.1).

Theorem 2.3. *Let $\mathcal{A} \subseteq L^2(G)$ be a countable subset and let J be the measurable range function associated with $W = M^\Lambda(\mathcal{A})$. Assume that $E^\Lambda(\mathcal{A}) := \{M_\lambda\phi : \phi \in \mathcal{A}\}$. The following conditions are equivalent.*

- (1) $E^\Lambda(\mathcal{A})$ is a continuous frame (continuous Riesz basis) for W with bounds $0 < A \leq B < \infty$.
- (2) The set $\{\tilde{Z}\phi(x) : \phi \in \mathcal{A}\}$ is a frame (Riesz basis) with bounds A and B , for almost every $x \in \Pi$.

Proof. Using the fact that unitary operators preserve frames and Riesz bases [5, Section 5.3], we know that $E^\Lambda(\mathcal{A})$ is a frame for $M^\Lambda(\mathcal{A})$ if and only if

$$\mathcal{F}(E^\Lambda(\mathcal{A})) = \{T_\lambda\hat{\phi} : \phi \in \mathcal{A}\}$$

is a frame (Riesz basis) for

$$\mathcal{F}(M^\Lambda(\mathcal{A})) = \overline{\text{span}}\{T_\lambda\hat{\phi} : \phi \in \mathcal{A}\}.$$

Equivalently, for almost every $x \in \Pi$, the set $\{Z\hat{\phi}(x) : \phi \in \mathcal{A}\}$ is a frame (Riesz basis) for $J(x)$, where Z is as in (1.1) (see [1, Theorem 6.10]). Now by (2.1)

$$\{Z\hat{\phi}(x) : \phi \in \mathcal{A}\} = \{\tilde{Z}\phi(x) : \phi \in \mathcal{A}\}.$$

\square

The following proposition states that every Λ -modulation invariant space can be decomposed to mutually orthogonal Λ -modulation invariant spaces each of which is generated by a single function in $L^2(G)$. The proof is similar to [2, Theorem 5.3], and so it is omitted.

Proposition 2.4. *Let W be a Λ -modulation invariant subspace of $L^2(G)$. Then there exist functions $\phi_n \in W$, $n \in \mathbb{N}$ with the following properties:*

- (1) *The set $\{M_\lambda \phi_n : \lambda \in \Lambda\}$ is a Parseval frame for $M^\Lambda(\phi_n)$.*
- (2) *The space W can be decomposed as an orthogonal sum*

$$W = \bigoplus_{n \in \mathbb{N}} M^\Lambda(\phi_n).$$

Example 2.5. For a fixed prime number p , the field of p -adic numbers \mathbb{Q}_p is the completion of rational numbers $x = \sum_{j=m}^\infty c_j p^j$ for $m \in \mathbb{Z}$ and $c_j \in \{0, 1, \dots, p-1\}$ under the p -adic norm $|\cdot|_p$ defined as follows (see [7, Chapter 2]). Every nonzero rational x can be uniquely written as $x = \frac{r}{s} p^n$, where $r, s, n \in \mathbb{Z}$ and p does not divide r or s . We define the p -adic norm of x by $|x|_p = p^{-n}$, in addition $|0|_p = 0$. Then \mathbb{Q}_p is an additive LCA group and $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\} = \left\{ \sum_{j=0}^\infty c_j p^j : c_j \in \{0, 1, \dots, p-1\} \right\}$ is a closed, compact, and open subgroup of \mathbb{Q}_p . A fundamental domain of \mathbb{Z}_p in \mathbb{Q}_p is

$$\Omega = \left\{ \sum_{j=-1}^{-1} c_j p^j : c_j \in \{0, 1, \dots, p-1\} \right\}.$$

By Theorem 2.1, every \mathbb{Z}_p -modulation invariant space of $L^2(\mathbb{Q}_p)$ is of the form

$$W = \{f \in L^2(\mathbb{Q}_p) : \tilde{Z}f(x) \in J(x) \text{ a.e. } x \in \Omega\},$$

for some measurable range function $J : \Omega \rightarrow \{\text{closed subspaces of } L^2(\Omega)\}$, where $\tilde{Z} : L^2(\mathbb{Q}_p) \rightarrow L^2(\Omega, L^2(\Omega))$ is the Zak transform defined as (2.1).

3. MODULATION METRIC

In [14], we defined a translation metric on the collection of all translation invariant spaces and studied some topological properties of the metric space. In this section, we define a similar metric for the modulation case. Indeed, we introduce and investigate topological properties of a modulation metric θ , a metric on the collection of all modulation invariant subspaces of $L^2(G)$. Let $MI(G)$ denote the collection of all modulation invariant subspaces of $L^2(G)$. For each V and W in $MI(G)$ define

$$\theta(V, W) = \inf\{\alpha > 0 : m(\{x \in \Pi : \|P_{J_V}(x) - P_{J_W}(x)\| > \alpha\}) = 0\},$$

where J_V and J_W are the measurable range functions associated with V and W , $P_{J_V}(x)$ and $P_{J_W}(x)$, $x \in \Pi$, are the orthogonal projections onto $J_V(x)$ and $J_W(x)$ respectively, $\|\cdot\|$ denotes the operator norm, and m is the Haar measure of G . In the forthcoming proposition, we show that θ is a metric on $MI(G)$, which is called modulation metric. Note that if V and W are modulation invariant spaces, then $\theta(V, W) \leq \epsilon$ if and only if $\|P_{J_V}(x) - P_{J_W}(x)\| \leq \epsilon$, for a.e. $x \in \Pi$.

Proposition 3.1. *With the notation as above, θ is a metric on $MI(G)$.*

Proof. The positivity of θ follows from its definition. For V and W in $MI(G)$, if $\theta(V, W) = 0$, one can find a sequence (α_n) of positive numbers converging to 0 and a set E of measure zero such that $\|P_{J_V}(x) - P_{J_W}(x)\| \leq \alpha_n$, for all $n \in \mathbb{N}$ and for $x \in \Pi$. It follows that $\|P_{J_V}(x) - P_{J_W}(x)\| = 0$ for a.e. $x \in \Pi$, so the projections onto $J_W(x)$ and $J_V(x)$ are the same a.e. Hence $V = W$, in the sense of the usual convention that two modulation invariant spaces are equal if the corresponding range functions are equal a.e. On the other hand, $V = W$ implies that $J_V(x) = J_W(x)$ for a.e. $x \in \Pi$, which in turn implies that $\|P_{J_V}(x) - P_{J_W}(x)\| > 0$ only on a set of measure 0. Hence $\theta(V, W) = 0$. For the triangle inequality, if U, V , and W are modulation invariant spaces and $\epsilon > 0$, one can get $M_1, M_2 > 0$ such that $M_1 < \theta(V, U) + \frac{\epsilon}{2}$, $M_2 < \theta(U, W) + \frac{\epsilon}{2}$, $m(\{x \in \Pi : \|P_{J_V}(x) - P_{J_U}(x)\| > M_1\}) = 0$, and $m(\{x \in \Pi : \|P_{J_U}(x) - P_{J_W}(x)\| > M_2\}) = 0$. Applying the triangle inequality for the norm gives

$$\|P_{J_V}(x) - P_{J_W}(x)\| \leq M_1 + M_2 \quad \text{a.e. } x \in \Pi.$$

It follows that $\theta(V, W) \leq \theta(V, U) + \theta(U, W)$. Finally, it is obvious that $\theta(V, U) = \theta(U, V)$. \square

In what follows, we show that $MI(G)$ is a complete, noncompact, and disconnected metric space. First of all, we need the following lemma.

Lemma 3.2. *Let (J_n) be a sequence of measurable range functions and let $(P_n(x))$ be the corresponding sequence of orthogonal projections onto J_n 's. Suppose that $(P_n(x))$ converges to the orthogonal projection $P(x)$ in the operator norm for $x \in \Pi$. If $J(x)$ is the range of $P(x)$, then J is a measurable range function.*

Proof. Let $f \in L^2(D)$. Setting $F_n(\xi) = P_n(\xi)f$ and $F(\xi) = P(\xi)f$, we have

$$\|F_n(\xi) - F(\xi)\| \leq \|P_n(\xi) - P(\xi)\| \|f\|.$$

It now follows that $F(\xi) = \lim F_n(\xi)$. Thus F is the limit of a sequence (F_n) of vector valued measurable functions and hence is measurable. That is J is measurable. \square

Theorem 3.3. *The space $MI(G)$ is complete in the modulation metric.*

Proof. Suppose that (W_n) is Cauchy in $MI(G)$. Then $(P_{J_{W_n}}(x))$ is Cauchy in the Banach space $BL(L^2(D))$, the space of all bounded linear operators on $L^2(D)$. Hence it converges to an orthogonal projection $P(x)$ for a.e. $x \in \Pi$. Let $J(x)$ be the closed subspace of $L^2(D)$ associated with the orthogonal projection $P(x)$. Consider the modulation invariant space $W := \{\varphi \in L^2(G) : \tilde{Z}\varphi(x) \in J(x) \text{ a.e. } x \in \Pi\}$. We have $J_W(x) = J(x)$ for a.e. $x \in \Pi$, and hence $P_{J_W}(x) = P(x)$ for a.e. $x \in \Pi$. Consequently, (W_n) converges to W in the modulation metric. \square

As a consequence of Theorem 3.3, we have the following corollary. Let $PMI(G)$ denote the collection of all principal modulation invariant subspaces of $L^2(G)$.

Corollary 3.4. *The space $PMI(G)$ is complete in the modulation metric.*

Proof. Suppose that (W_n) is a Cauchy sequence in $PMI(G)$. By Theorem 3.3, (W_n) converges to some $W \in MI(G)$. We need only to show that W has a single generator. For $0 < \epsilon < 1$, choose $p \in \mathbb{N}$ such that $\theta(W_n, W) < \epsilon$ for all $n \geq p$. This implies that $\|P_{J_{W_n}}(x) - P_{J_W}(x)\| < \epsilon$ for a.e. x whenever $n \geq p$. Hence $\dim J_W(x) = \dim J_{W_n}(x) = 1$ for a.e. x (see [17, Theorem 4.35]). This proves that W can be generated by a single function, and hence $W \in PMI(G)$. \square

Let $FMI(G)$ be the collection of all modulation invariant spaces generated by a fixed number of elements of $L^2(G)$. With the same proof as Corollary 2.4, one can see the following corollary.

Corollary 3.5. *The collection $FMI(G)$ is complete in the modulation metric.*

Now we show that $MI(G)$ is not a compact metric space.

Proposition 3.6. *The space $MI(G)$ is not compact in the modulation metric topology.*

Proof. Using [15, Theorem 45.1], it is enough to show that $MI(G)$ is not totally bounded in the modulation metric. First choose a countable basis $\{\varphi_1, \varphi_2, \dots\}$ for $L^2(G)$. Set $W_m = M^\Lambda(\mathcal{A}_m)$, where $\mathcal{A}_m = \{\varphi_1, \varphi_2, \dots, \varphi_m\}$. Then $W_m \subset W_{m+1}$ for any m , and hence $\|P_{J_{W_m}}(x) - P_{J_{W_{m+1}}}(x)\| = 1$ for all $x \in \Pi$ (see [17, Theorem 4.30]). That is, $\theta(W_m, W_{m+1}) = 1$ for all m . Hence for $\epsilon = \frac{1}{2}$, no finite collection of ϵ -balls can contain all W_m 's. \square

In the next theorem, we show that the metric space $MI(G)$ is disconnected.

Theorem 3.7. *The space $MI(G)$ is disconnected in the modulation metric.*

Proof. It is enough to show that $MI(G)$ has an open and closed proper subset. That $PMI(G)$ is closed follows from Corollary 2.4. Now we show that it is open. Let $W \in PMI(G)$; put $r = \frac{1}{2}$. We show that $B_r(V) \subseteq PMI(G)$, where $B_r(W)$ is an open ball with center W and radius r . Let $V \in B_r(V)$; then $\theta(W, V) < \frac{1}{2}$. Hence $\dim J_W(x) = \dim J_V(x)$ for a.e. $x \in \Pi$ (see [17, Theorem 4.35]). Hence $V \in PMI(G)$. Since W is arbitrary, then $PMI(G)$ is an open subspace of $MI(G)$. That is $MI(G)$ is disconnected. \square

Remark 3.8. In the case that the closed subgroup Λ of \widehat{G} is also cocompact, our results in this article can be phrased in terms of the so called fiberization map $\mathcal{T} : L^2(\widehat{G}) \rightarrow L^2(\Pi, l^2(\Lambda^*))$, $\mathcal{T}f(x) = \{\hat{f}(x+k)\}_{k \in \Lambda^*}$, instead of the Zak transform Z . In this case, Theorem 2.1 says that a closed subspace W of $L^2(G)$ is Λ -modulation invariant if and only if $W = \{f \in L^2(G) : \tilde{\mathcal{T}}(f)(x) \in J(x), \text{ for a.e. } x \in \Pi\}$, for some measurable range function

$$J : \Pi \rightarrow \{\text{closed subspaces of } L^2(D)\},$$

in which $\tilde{\mathcal{T}} : L^2(G) \rightarrow L^2(\Pi, l^2(\Lambda^*))$, $\tilde{\mathcal{T}} := \mathcal{T} \circ \mathcal{F}$, and \mathcal{F} is the Fourier transform on $L^2(G)$. Moreover if $W = M^\Lambda(\mathcal{A})$, for some countable subset \mathcal{A} of $L^2(G)$, then the measurable range function J is given by

$$J(x) = \overline{\text{span}}\{\tilde{\mathcal{T}}(\phi)(x) : \phi \in \mathcal{A}\} \text{ a.e. } x \in \Pi$$

(compare with [2, Theorem 3.8]).

Example 3.9. We give an example of a sequence of \mathbb{Z} -modulation invariant subspaces of $L^2(\mathbb{R})$ converging in the modulation metric. Define $\phi := \mathbf{1}_{(0,1)} \in L^2(\mathbb{R})$, and suppose that (ϕ_n) is the sequence defined by $\phi_n(x) = \frac{n+1}{n} \mathbf{1}_{(0,1)}(x)$. Let $W = M^{\mathbb{Z}}(\phi)$ and let $W_n = M^{\mathbb{Z}}(\phi_n)$. A direct calculation from [2, Theorem 3.8] shows that $J_V(\xi) = \text{span}\{e_0\}$ and $J_{V_n}(\xi) = \text{span}\{\frac{n+1}{n}e_0\}$, where (e_k) is the standard basis of $l^2(\Lambda^*)$. So the projections onto $J_{W_n}(\xi)$'s and $J_W(\xi)$ are the same, and we can conclude that (W_n) converges to W in the modulation metric.

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