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PERMANENCE AND STABILITY OF MULTI-SPECIES NONAUTONOMOUS LOTKA–VOLTERRA COMPETITIVE SYSTEMS WITH DELAYS AND FEEDBACK CONTROLS ON TIME SCALES

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Abstract. We consider a multi-species Lotka–Volterra type competitive system with delays and feedback controls on time scales. A general criteria on the permanence is established, and then by constructing suitable Lyapunov functionals, sufficient conditions are derived for the existence and uniform asymptotic stability of unique positive almost periodic solution of the system.

1. Introduction

The mathematical models predator-prey, competition, and mutualism are most suitable for real world situations in population dynamics for multiple species, which can be expressed as a set of parameterized differential or difference equations, or dynamical systems. The Lotka–Volterra (LV) model is the most appropriated and frequently used in competition models. The n-dimensional nonautonomous competitive LV model is described by the following system:

$$x_i'(t) = x_i(t)[a_i(t) - \sum_{j=1}^{n} b_{ij}x_j(t)],\ i = 1, 2, \dots, n, \tag{1.1}$$

where $x_i(t)$ is the density of the ith species at time t, $a_i(t)$ represents the intrinsic growth rate of species i at time t, and $b_{ij}(t)$ are the competing coefficients between species j and i at time t. In mathematical ecology, (1.1) describes an n-species

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dynamical system, in which each individual competes with others of the system for the unique resources.

Moreover, the ecosystems in the real world are continuously distributed by unpredictable forces, which can result in changes in the biological parameters such as survival rates. In ecology, a question of practical interest is whether or not an ecosystem can withstand those unpredictable disturbances that persist for a finite period of time. In the language of control theory, we call the disturbance functions as control variables. In 1993, Gopalsamy and Weng [3] introduced a feedback control variable into the delayed logistic model and discussed the asymptotic behavior of solutions in logistic models with feedback controls, in which the control variables satisfy a certain differential equation. In the last decade, much work has been done on the ecosystem with feedback controls (see [3, 6, 8, 9, 14–17] and the references therein). In particular, Li and Liu [8], Lalli and Yu [6], Li and Wang [9], Shi and chen [14], Shi et al. [15] have studied delay equations with feedback controls.

The study of dynamical systems on time scales is now an active area of research. This study reveals that the existence of positive almost periodic solutions of population models is not worthwhile to establish results for differential equations and again for difference equations separately. One can unify such problems in the frame of dynamic equations on time scales [2, 10, 11, 13]. Recently, Prasad and Khuddush [12] studied the 3-species Lotka–Volterra competition model on time scales,

$$\begin{aligned}
x_1^{\Delta}(t) &= r_1(t) - \exp\{x_1(t)\} - \alpha\exp\{x_2(t)\} - \beta\exp\{x_3(t)\},\\
x_2^{\Delta}(t) &= r_2(t) - \beta\exp\{x_1(t)\} - \exp\{x_2(t)\} - \alpha\exp\{x_3(t)\},\\
x_3^{\Delta}(t) &= r_3(t) - \alpha\exp\{x_1(t)\} - \beta\exp\{x_2(t)\} - \exp\{x_3(t)\},
\end{aligned}$$

and derived sufficient conditions for the existence and uniform asymptotic stability of unique positive almost periodic solution of system. Motivated by aforementioned works in this paper, we study the permanence and positive almost periodic solutions of the following n-species Lotka–Volterra system on time scales:

$$\left.\begin{aligned}
&x_i^{\Delta}(t) = x_i(t)\Big[b_i(t) - a_i(t)x_i^{\sigma}(t) - \sum_{j=1}^{n} a_{ij}(t)x_j\big(t-\tau_{ij}(t)\big) - c_i(t)u_i\big(t-\delta_i(t)\big)\Big],\\
&u_i^{\Delta}(t) = r_i(t) - d_i(t)u_i(t) + e_i(t)x_i\big(t-\eta_i(t)\big), \quad i = 1, 2, \ldots, n,
\end{aligned}\right\} \tag{1.2}$$

where $x_i(t)$ represents the population density of the ith species at time $t \in \mathbb{T}^+$, $u_i(t)$ denotes indirect control variable [5, 7], of the ith species at time $t \in \mathbb{T}^+$ $\big(\mathbb{T}^+$ is a nonempty closed subset of $\mathbb{R}^+ = (0,\infty)\big)$, $x_i^{\sigma}(t) = x_i(\sigma(t))$, $\sigma(t)$ is the forward jump operator, and the functions a_i, b_i, c_i, d_i, e_i, δ_i, η_i, a_{ij}, τ_{ij}, $(i, j = 1, 2, \ldots, n)$ are bounded positive almost periodic functions.

2. Preliminaries

For a function $g(t)$ defined on $\mathbb{T}^+$, we set

$$g^L = \inf\big\{g(t) : t \in \mathbb{T}^+\big\},\ g^U = \sup\big\{g(t) : t \in \mathbb{T}^+\big\}.$$

Definition 2.1 ([1]). A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. Also $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ defined, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}, \ \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}, \ \textit{and} \ \mu(t) = \sigma(t) - t,$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, and right-scattered if $\rho(t) = t, \rho(t) < t, \sigma(t) = t$, and $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

Definition 2.2 ([1]). A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t) f(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions (a function $g : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$) $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} : 1 + \mu(t) f(t) > 0, \text{for all } t \in \mathbb{T}\}$.

Lemma 2.3 ([4]). *Assume that $a > 0$, that $b > 0$, and that $-a \in \mathcal{R}^+$. Then*

$$y^{\Delta}(t) \geq (\leq) b - a y(t), \ y(t) > 0, \ t \in [t_0, \infty)_{\mathbb{T}}$$

implies

$$y(t) \geq (\leq) \frac{b}{a} \Big[1 + \Big(\frac{a y(t_0)}{b} - 1 \Big) e_{(-a)}(t, t_0) \Big], \ t \in [t_0, \infty)_{\mathbb{T}}.$$

Lemma 2.4 ([4]). *Assume that $a > 0$ and that $b > 0$. Then*

$$y^{\Delta}(t) \leq (\geq) y(t) \big(b - a y(\sigma(t)) \big), \ y(t) > 0, \ t \in [t_0, \infty)_{\mathbb{T}}$$

implies

$$y(t) \leq (\geq) \frac{b}{a} \Big[1 + \Big(\frac{b}{a y(t_0)} - 1 \Big) e_{(-b)}(t, t_0) \Big], \ t \in [t_0, \infty)_{\mathbb{T}}.$$

Definition 2.5 ([9]). A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\prod = \{\tau \in \mathbb{R} : t + \tau \in \mathbb{T}, \text{for all } \ t \in \mathbb{T}\} \neq \{0\}.$$

Definition 2.6 ([9]). Let $\mathbb{T}$ be an almost periodic time scale. A function $x \in C(\mathbb{T}, \mathbb{R}^n)$ is called an almost periodic function if the ε-translation set of x, that is,

$$E\{\varepsilon, x\} = \Big\{ \tau \in \prod : |x(t + \tau) - x(t)| < \varepsilon, \text{for all } \ t \in \mathbb{T} \Big\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$, that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon) > 0$ contains $\tau(\varepsilon) \in E\{\varepsilon, x\}$ such that $|x(t+\tau) - x(t)| < \varepsilon$, for all $t \in \mathbb{T}$. Moreover, τ is called the ε-translation number of $x(t)$, and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, x\}$.

Definition 2.7 ([9])**.** Let $\mathbb{D}$ be an open set in $\mathbb{R}^n$ and let $\mathbb{T}$ be a positive almost periodic time scale. A function $f \in C(\mathbb{T} \times \mathbb{D}, \mathbb{R}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in \mathbb{D}$ if the ε-translation set of f

$$E\{\varepsilon, f, \mathbb{S}\} = \Big\{\tau \in \prod : |f(t+\tau) - f(t)| < \varepsilon, \text{for all } (t,x) \in \mathbb{T} \times \mathbb{S}\Big\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$ and for each compact subset $\mathbb{S}$ of $\mathbb{D}$, that is, for any given $\varepsilon > 0$ and each compact subset $\mathbb{S}$ of $\mathbb{D}$, there exists a constant $l(\varepsilon, \mathbb{S}) > 0$ such that each interval of length $l(\varepsilon, \mathbb{S})$ contains $\tau(\varepsilon, \mathbb{S}) \in E\{\varepsilon, f, \mathbb{S}\}$ such that

$$|f(t+\tau, x) - f(t,x)| < \varepsilon, \text{ for all } (t,x) \in \mathbb{T} \times \mathbb{S}.$$

Consider the following system:

$$x^{\Delta}(t) = h(t,x), \tag{2.1}$$

and its associate product system

$$x^{\Delta}(t) = h(t,x), \; z^{\Delta}(t) = h(t,z),$$

where $h : \mathbb{T}^+ \times \mathbb{S}_B \to \mathbb{R}^n$, $\mathbb{S}_B = \{x \in \mathbb{R}^n : \|x\| < B\}$ and $h(t,x)$ is almost periodic in t uniformly for $x \in \mathbb{S}_B$ and is continuous in x.

Lemma 2.8 ([18])**.** *Suppose that there exists a Lyapunov function $V(t,x,z)$ defined on $\mathbb{T}^+ \times \mathbb{S}_B \times \mathbb{S}_B$ satisfying the following conditions:*

(i) $a(\|x-z\|) \le V(t,x,z) \le b(\|x-z\|)$, *where* $a, b \in \mathbb{K}$,
$\mathbb{K} = \{\alpha \in C(\mathbb{R}^+, \mathbb{R}^+) : \alpha(0) = 0$ *and* α *is increasing*$\}$;
(ii) $|V(t,x,z) - V(t,x_1,z_1)| \le L(\|x-x_1\| + \|z-z_1\|)$, *where* $L > 0$ *is a constant,*
(iii) $D^+V^{\Delta}(t,x,z) \le -cV(t,x,z)$, *where* $c > 0, -c \in \mathcal{R}^+$.

Furthermore, if there exists a solution $x(t) \in \mathbb{S}$ of system (2.1) for $t \in \mathbb{T}^+$, where $\mathbb{S} \cup \mathbb{S}_B$ is a compact set, then there exist a unique almost periodic solution $f(t) \in \mathbb{S}$ of system (2.1), which is uniformly asymptotically stable.

Definition 2.9. System (1.2) is said to be permanent if there exist positive constants m, M such that $m \le \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) \le M$, $i = 1, 2, \dots, n$, and $m \le \liminf_{t\to\infty} u_i(t) \le \limsup_{t\to\infty} u_i(t) \le M$, $i = 1, 2, \dots, n$, for any solution $\big(x_1(t), x_2(t), \dots, x_n(t), u_1(t), u_2(t), \dots, u_n(t)\big)$ of (1.2).

For system (1.2), we introduce the following assumption. Let $t_0 \in \mathbb{T}$ be a fixed positive initial time.

(H_1) $a_i(t), b_i(t), c_i(t), d_i(t), e_i(t), \delta_i(t), \eta_i(t), r_i(t), \tau_{ij}(t), a_{ij}(t) (i,j = 1, 2, \dots, n)$ are bounded almost periodic functions and satisfy $0 < a^L \le a_i(t) \le a_i^U$, $0 < b_i^L \le b_i(t) \le b_i^U$, $0 < c_i^L \le c_i(t) \le c_i^U$, $0 < d_i^L \le d_i(t) \le d_i^U$, $0 < e_i^L \le e_i(t) \le e_i^U$, $0 < r_i^L \le r_i(t) \le r_i^U$, $0 < \delta^L \le \delta_i(t) \le \delta_i^U$, $0 < \eta^L \le \eta_i(t) \le \eta_i^U$, $0 < \tau_{ij}^L \le \tau_{ij}(t) \le \tau_{ij}^U$, $0 < a_{ij}^L \le a_{ij}(t) \le a_{ij}^U$ for $i, j = 1, 2, 3, \dots, n$.

Define

$$C^n_{rd^+}[-\tau,0] = \Big\{\phi = (\phi_1,\phi_2,\dots,\phi_n) \in C^n_{rd}[-\tau,0] : \phi_i(s) \geq 0$$
$$\text{and } \phi_i(0) > 0 \text{ for all } s \in [-\tau,0] \text{ and } i = 1,2,\dots,n\Big\},$$

where $\tau = \sup\big\{\delta_i(t), \eta_i(t), \tau_{ij}(t) : t \geq t_0, i = j = 1,2,\dots,n\big\}$. Then $C^n_{rd}[-\tau,0]_{\mathbb{T}}$ is the Banach space of bounded rd-continuous functions $\phi : [-\tau,0]_{\mathbb{T}} \to \mathbb{R}^+$ with the supremum norm defined by $\|\phi\|_c = \sup_{-\tau\leq s\leq 0}|\phi(s)|$, where $\phi = (\phi_1,\phi_2,\dots,\phi_n)$ and $|\phi(s)| = \sum_{i=1}^n |\phi_i(s)|$. We know, for any $(\phi,\psi) \in C^n_{rd}[-\tau,0]\times C^n_{rd}[-\tau,0]$, that system (1.2) has a unique solution

$$Z(t,\phi,\psi) = \big(x_1(t,\phi), x_2(t,\phi),\dots,x_n(t,\phi), u_1(t,\psi), u_2(t,\psi),\dots,u_n(t,\psi)\big)$$

satisfying the initial condition $Z_{t_0}(\cdot,\phi,\psi) = (\phi,\psi)$.

Due to the biological background of system (1.2), positive solutions are only meaningful. So, we restrict our attention to positive solutions of equation (1.2). It is easy to see that the solution $Z(t,\phi,\psi)$ of system (1.2) is positive, if the initial function (ϕ,ψ) is in $C^n_{rd^+}[-\tau,0]\times C^n_{rd^+}[-\tau,0]$.

3. Permanence of solutions

In this section, we derive the sufficient conditions for system (1.2) to be permanent.

Lemma 3.1. *Suppose that assumption (H_1) holds. Then for any positive solution $Z(t) = \big(x(t),u(t)\big)$ of system (1.2), there exist positive constants M and T such that $x_i(t) < M$ and $u_i(t) < M (i = 1,2,\dots,n)$ for $t > T$.*

Proof. Let $Z(t) = \big(x(t),u(t)\big)$ be any positive solution of system (1.2). From the ith equation of system (1.2), we have $x_i^\Delta(t) \leq x_i(t)\Big[b_i^U - a_i^L x_i\big(\sigma(t)\big)\Big]$. Hence, by Lemma 2.4, there exist positive constants M_i^* and T_i^* such that for any positive solution $\big(x(t),u(t)\big)$ of system (1.2), we have $x_i(t) \leq b_i^U/a_i^L := M_i^*$ for all $t \geq T_i^*$. Let $M^* = \max_{1\leq i\leq n}\{M_i^*\}$ and let $T^* = \max_{1\leq i\leq n}\{T_i^*\}$. Then $x_i(t) \leq M^*$ for all $t \geq T^*,\ i = 1,2,\dots,n$. Furthermore, from the $(n+i)$th equation of system (1.2), we have $u_i^\Delta(t) \leq (r_i^U + e_i^U M^*) - d_i^L u_i(t)$ for all $t \geq T^* + \tau$. Hence, by Lemma 2.3, there exist positive constants $M_i* > 0$ and $T_i* > T^* + \tau$ such that for any positive solution $\big(x(t),u(t)\big)$ of system (1.2), we have $u_i(t) \leq (r_i^U + e_i^U M^*)/d_i^L := M_i*$ for all $t \geq T_i*$. Now let $M = \max\{M^*, M_i* : i = 1,2,\dots,n\}$ and let $T = \max\{T^*, T_i* : i = 1,2,\dots,n\}$. Then $x_i(t) \leq M$ and $u_i(t) \leq M$ for all $t \geq T,\ i = 1,2,\dots,n$. ☐

Lemma 3.2. *Suppose that (H_1) holds. Furthermore, for each $i = 1,2,\dots,n$, if $b_i^L > \big[\sum_{j=1}^n a_{ij}^U + c_i^U\big]M$, then system (1.2) is permanent.*

Proof. Let $Z(t) = \big(x(t),u(t)\big)$ be any positive solution of system (1.2). From Lemma 3.1, there are positive constants $M > 0$ and $T \geq t_0$ such that $0 < x_i(t) < M$ and $0 < u_i(t) < M$ for all $t \geq T$, $i = 1,2,\dots,n$. From the ith equation of system (1.2), we have

$$x_i^\Delta(t) \geq x_i(t)\Big[b_i^L - \sum_{j=1}^n a_{ij}^U M - c_i^U M - a_i^U x_i\big(\sigma(t)\big)\Big]$$

for all $t \geq T + \tau$. Hence, from the hypothesis and Lemma 2.4, we can get a constant $m_1^* > 0$ such that for any positive solution $\big(x(t), u(t)\big)$ of system (1.2), there is $\hat{T}_i > T + \tau$ such that

$$x_i(t) \geq \frac{b_i^L - \big(\sum_{j=1}^n a_{ij}^U + c_i^U\big)M}{a_i^U} := m_i^*$$

for all $t \geq \hat{T}_i$. Let $m^* = \min_{1\leq i\leq n}\{m_i^*\}$ and let $\hat{T}^* = \max_{1\leq i\leq n}\{\hat{T}_i\}$. Then $x_i(t) \geq m^*$ for all $t \geq \hat{T}^*, i = 1, 2, \dots, n$. Furthermore, from the above, assumption (H_1), and the $(n+i)$th equation of system (1.2), we have $u_i^{\Delta}(t) \geq (r_i^L + e_i^L m^*) - d_i^U u_i(t)$ for all $t \geq \hat{T}^* + \tau$. Hence, by Lemma 2.3, we can obtain a constant $m_i* > 0$ such that for any positive solution $\big(x(t), u(t)\big)$ of system (1.2), there is $\hat{T}_i* > \hat{T}^* + \tau$ such that

$$u_i(t) \geq \frac{r_i^L + e_i^L m^*}{d_i^U} := m_i* \text{ for all } t \geq \hat{T}_i*.$$

Finally, let $m = \min\{m^*, m_i* : i = 1, 2, \dots, n\}$ and let $\hat{T} = \max\{\hat{T}^*, \hat{T}_i* : i = 1, 2, \dots, n\}$. Then $x_i(t) \geq m$ and $u_i(t) \geq m$ for all $t \geq \hat{T},\ i = 1, 2, \dots, n$. □

Define

$$\Omega = \Big\{Z(t) = \big(x(t), u(t)\big) : \big(x(t), u(t)\big) = \big(x_1(t), \dots, x_n(t), u_1(t), \dots, u_n(t)\big) \text{be a solution of (1.2) and } 0 < x_* \leq x_i \leq x^*,\ 0 < u_* \leq u_i \leq u^*,\ i = 1, \dots, n\Big\}.$$

It is clear that Ω is an invariant set of system (1.2) and by Lemma 3.2, we have $\Omega \neq \emptyset$.

4. Uniform asymptotic stability

In this section, we establish sufficient conditions for the existence and uniform asymptotic stability of the unique positive almost periodic solution to system (1.2).

Theorem 4.1. *Suppose that the hypothesis of Lemma 3.2 holds and that all delays of the system (1.2) are constants. Furthermore, the following holds:*
(H_2) *For $i = 1, 2, \dots, n$,*

$$\alpha_i = (x_i^U)^2\Big[b_i^L - c_i^U u_i^U - \sum_{j=1}^n a_{ij}^U x_j^U\Big] \geq 0,\ \ \alpha_i^* = \frac{d_i^L}{(u_i^L)^2} - \frac{c_i^U \sigma^U (x_i^U)^2}{x_i^L} \geq 0,$$

$$\beta_i = (x_i^U)^2\Bigg[\sum_{j=1}^n \frac{a_{ij}^U (x_i^U)^2}{x_i^L} + \frac{(1 + d_i^U \mu^U) e_i^U}{(u_i^L)^2}\Bigg] \geq 0,\ \ \beta_i^* = \frac{(d_i^U)^2 \mu^U}{(u_i^L)^2} \geq 0,$$

and $B < A$ with $-A, -B \in \mathcal{R}^+$, where $A = \min\{\Gamma_1, \Gamma_1^*\}$ and $B = \max\{\Gamma_2, \Gamma_2^*\}$ in which $\Gamma_1 = \min\limits_{1\leq i\leq n} \alpha_i$, $\Gamma_1^* = \min\limits_{1\leq i\leq n} \alpha_i^*$, $\Gamma_2 = \max\limits_{1\leq i\leq n} \beta_i$, and $\Gamma_2^* = \max\limits_{1\leq i\leq n} \beta_i^*$. Then the dynamic system (1.2) has a unique almost periodic solution $Z(t) = \big(x(t), u(t)\big) \in \Omega$ and is uniformly asymptotically stable.

Proof. According to Lemma 3.2, every solution $Z(t) = \big(x(t), u(t)\big)$ of system (1.2) satisfies that $x_i^L \le x_i \le x_i^U$ and that $u_i^L \le u_i \le u_i^U$. Hence, $|x_i(t)| \le A_i$ and $|u_i(t)| \le B_i$, where $A_i = \max\{|x_{i^*}|, |x_i^*|\}$ and $B_i = \max\{|u_{i^*}|, |u_i^*|\}$ $i = 1, 2, \ldots, n$. Denote

$$\|Z\| = \|(x, u)\| = \sup_{t\in\mathbb{T}^+} \sum_{i=1}^{n} |x_i(t)| + \sup_{t\in\mathbb{T}^+} \sum_{i=1}^{n} |u_i(t)|.$$

Suppose that $Z = \big(x(t), u(t)\big)$ and $\hat{Z} = \big(\hat{x}(t), \hat{u}(t)\big)$ are two positive solutions of (1.2). Then $\|Z\| \le C$ and $\|\hat{Z}\| \le C$, where $C = \sum_{i=1}^{n}(A_i + B_i)$. In view of (1.2), we have

$$x_i^{\Delta}(t) = x_i(t)\Big[b_i(t) - a_i(t)x_i^{\sigma}(t) - \sum_{j=1}^{n} a_{ij}(t)x_j\big(t - \tau_{ij}\big) - c_i(t)u_i\big(t - \delta_i\big)\Big],$$

$$\begin{aligned} u_i^{\Delta}(t) &= r_i(t) - d_i(t)u_i(t) + e_i(t)x_i\big(t - \eta_i\big),\\ \hat{x}_i^{\Delta}(t) &= \hat{x}_i(t)\Big[b_i(t) - a_i(t)\hat{x}_i^{\sigma}(t) - \sum_{j=1}^{n} a_{ij}(t)\hat{x}_j\big(t - \tau_{ij}\big) - c_i(t)\hat{u}_i\big(t - \delta_i\big)\Big],\\ \hat{u}_i^{\Delta}(t) &= r_i(t) - d_i(t)\hat{u}_i(t) + e_i(t)\hat{x}_i\big(t - \eta_i\big), \quad i = 1, 2, \ldots, n. \end{aligned}$$

Define the Lyapunov function $V(t, Z, \hat{Z})$ on $\mathbb{T}^+ \times \Omega \times \Omega$ as

$$V(t, Z, \hat{Z}) = \sum_{i=1}^{n} \Big| x_i(t) - \hat{x}_i(t) \Big| + \sum_{i=1}^{n} \left| \frac{1}{u_i(t)} - \frac{1}{\hat{u}_i(t)} \right|.$$

Then the following two norms are equivalent:

$$\|Z(t) - \hat{Z}(t)\| = \sup_{t\in\mathbb{T}^+} \sum_{i=1}^{n} \Big| x_i(t) - \hat{x}_i(t) \Big| + \sup_{t\in\mathbb{T}^+} \sum_{i=1}^{n} \left| \frac{1}{u_i(t)} - \frac{1}{\hat{u}_i(t)} \right|.$$

$$\|Z(t) - \hat{Z}(t)\|_* = \sup_{t\in\mathbb{T}^+} \left[\sum_{i=1}^{n} (x_i(t) - \hat{x}_i(t))^2 + \sum_{i=1}^{n} \left(\frac{1}{u_i(t)} - \frac{1}{\hat{u}_i(t)} \right)^2 \right]^{1/2}.$$

That is, there exist two constants $\eta_1, eta_2 > 0$ such that

$$\eta_1\|Z(t) - \hat{Z}(t)\| \le \|Z(t) - \hat{Z}(t)\|_* \le \eta_2\|Z(t) - \hat{Z}(t)\|.$$

Hence,

$$(\eta_1\|Z(t) - \hat{Z}(t)\|)^2 \le V(t, Z, \hat{Z}) \le (\eta_2\|Z(t) - \hat{Z}(t)\|)^2.$$

Let $a, b \in C(\mathbb{R}^+, \mathbb{R}^+)$, let $a(x) = \eta_1^2 x^2$, and let $b(x) = \eta_2^2 x^2$. Then the assumption (i) of Lemma 2.8 is satisfied. On the other hand, we have

$$\begin{aligned}
&\Big|V\big(t, Z(t), \hat{Z}(t)\big) - V\big(t, Z^*(t), \hat{Z}^*(t)\big)\Big| \\
&= \Bigg| \sum_{i=1}^{n} \Big|x_i(t) - \hat{x}_i(t)\Big| + \sum_{i=1}^{n} \Big|\frac{1}{u_i(t)} - \frac{1}{\hat{u}_i(t)}\Big| - \sum_{i=1}^{n} \Big|x_i^*(t) - \hat{x}_i^*(t)\Big| \\
&\qquad - \sum_{i=1}^{n} \Big|\frac{1}{u_i^*(t)} - \frac{1}{\hat{u}_i^*(t)}\Big| \Bigg| \\
&\le \sum_{i=1}^{n} \Big|x_i(t) - x_i^*(t)\Big| + \sum_{i=1}^{n} \Big|\frac{1}{u_i(t)} - \frac{1}{u_i^*(t)}\Big| + \sum_{i=1}^{n} \Big|\hat{x}_i(t) - \hat{x}_i^*(t)\Big| \\
&\qquad + \sum_{i=1}^{n} \Big|\frac{1}{\hat{u}_i(t)} - \frac{1}{\hat{u}_i^*(t)}\Big| \\
&= L\Big(\|Z - Z^*(t)\| + \|\hat{Z}(t) - \hat{Z}^*(t)\|\Big),
\end{aligned}$$

where $L = 1$, so condition (ii) of Lemma 2.8 is satisfied. Now consider a function $\hat{V}(t) = \sum_{i=1}^{n} V_i(t)$, $V_i(t) = V_{i1}(t) + V_{i2}(t) + V_{i3}(t) + V_{i4}(t)$ for $i = 1, 2, \ldots, n$, and

$$\begin{aligned}
V_{i1}(t) &= (x_i^U)^2 \Big|\frac{1}{x_i(t)} - \frac{1}{\hat{x}_i(t)}\Big|, \ V_{i2}(t) = \frac{1}{(u_i^L)^2}\Big|u_i(t) - \hat{u}_i(t)\Big|, \\
V_{i3}(t) &= \sum_{j=1}^{n} \frac{a_{ij}^U (x_i^U)^2}{x_i^L} \int_{t-\tau_{ij}}^{t} |x_j(s) - \hat{x}_j(s)|\Delta s \\
&\quad + \frac{(1 + d_i^U \mu^U) e_i^U}{(u_i^L)^2} \int_{t-\eta_i}^{t} |x_i(s) - \hat{x}_i(s)|\Delta s, \\
V_{i4}(t) &= \frac{c_i^U (x_i^U)^2}{x_i^L} \int_{t-\delta_i}^{\sigma(t)} |u_i(s) - \hat{u}_i(s)|\Delta s.
\end{aligned}$$

For $i = 1, 2, \ldots, n$,

$$\begin{aligned}
D^+ V_{i1}^{\Delta}(t) &= (x_i^U)^2 \Big|\frac{1}{x_i(t)} - \frac{1}{\hat{x}_i(t)}\Big|^{\Delta} \\
&= (x_i^U)^2 sign\big(\hat{x}_i^\sigma(t) - x_i^\sigma(t)\big)\Big[\frac{1}{x_i(t)} - \frac{1}{\hat{x}_i(t)}\Big]^{\Delta} \\
&= (x_i^U)^2 sign\big(\hat{x}_i^\sigma(t) - x_i^\sigma(t)\big)\Big[-\frac{x_i^\Delta(t)}{x_i(t) x_i^\sigma(t)} + \frac{\hat{x}_i{}^\Delta(t)}{\hat{x}_i(t)\hat{x}_i^\sigma(t)}\Big] \\
&= (x_i^U)^2 sign\big(\hat{x}_i^\sigma(t) - x_i^\sigma(t)\big) \\
&\quad \times \Bigg\{ -\frac{1}{x_i^\sigma}\Big[b_i(t) - a_i(t) x_i^\sigma(t) - \sum_{j=1}^{n} a_{ij}(t) x_j\big(t - \tau_{ij}\big) - c_i(t) u_i\big(t - \delta_i\big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\hat{x}_i^\sigma}\left[b_i(t)-a_i(t)\hat{x}_i^\sigma(t)-\sum_{j=1}^n a_{ij}(t)\hat{x}_j\big(t-\tau_{ij}\big)-c_i(t)\hat{u}_i\big(t-\delta_i\big)\right]\Bigg\}\\
&=(x_i^U)^2\frac{sign\big(\hat{x}_i^\sigma(t)-x_i^\sigma(t)\big)}{x_i^\sigma(t)\hat{x}_i{}^\sigma(t)}\Bigg\{b_i(t)\big[x_i^\sigma(t)-\hat{x}_i{}^\sigma(t)\big]\\
&-\sum_{j=1}^n a_{ij}(t)[x_i^\sigma(t)\hat{x}_j(t-\tau_{ij})-x_j(t-\tau_{ij})\hat{x}_i{}^\sigma(t)]\\
&-c_i(t)[x_i^\sigma(t)\hat{u}_i(t-\delta_i)-\hat{x}_i^\sigma u_i(t-\delta_i)]\Bigg\}\\
&\le\frac{(x_i^U)^2}{x_i^\sigma(t)\hat{x}_i{}^\sigma(t)}\Bigg\{-b_i(t)\big|\hat{x}_i{}^\sigma(t)-x_i^\sigma(t)\big|\\
&+\sum_{j=1}^n a_{ij}(t)x_i^\sigma(t))\big|\hat{x}_j(t-\tau_{ij})-x_j(t-\tau_{ij})\big|\\
&+\sum_{j=1}^n a_{ij}(t)x_j(t-\tau_{ij})\big|\hat{x}_i^\sigma(t)-x_i^\sigma(t)\big|\\
&+c_i(t)x_i^\sigma(t)\big|\hat{u}_i(t-\delta_i)-u_i(t-\delta_i)\big|\\
&+c_i(t)u_i(t-\delta_i)\big|\hat{x}_i{}^\sigma(t)-x_i^\sigma(t)\big|\Bigg\}\\
&\le\frac{(x_i^U)^2}{x_i^\sigma(t)\hat{x}_i{}^\sigma(t)}\Big[c_i(t)\hat{u}_i(t-\delta_i)-b_i(t)+\sum_{j=1}^n a_{ij}(t)\hat{x}_j(t-\tau_{ij})\Big]\\
&\times\Big|x_i^\sigma(t)-\hat{x}_i{}^\sigma(t)\Big|+\frac{1}{\hat{x}_i^\sigma}\Big[\sum_{j=1}^n a_{ij}(t)\big|\hat{x}_j(t-\tau_{ij})-x_j(t-\tau_{ij})\big|\\
&+c_i(t)\big|u_i(t-\delta_i)-\hat{u}_i(t-\delta_i)\big|\Big]\\
&\le(x_i^U)^2\left[c_i^Uu_i^U-b_i^L+\sum_{j=1}^n a_{ij}^Ux_j^U\right]\left|\frac{1}{x_i^\sigma}-\frac{1}{\hat{x}_i^\sigma}\right|\\
&+\frac{(x_i^U)^2}{x_i^L}\sum_{j=1}^n a_{ij}^U\big|x_j(t-\tau_{ij})-\hat{x}_j(t-\tau_{ij})\big|\\
&+\frac{c_i^U(x_i^U)^2}{x_i^L}\big|u_i(t-\delta_i)-\hat{u}_i(t-\delta_i)\big|,
\end{aligned}
$$

Let $v_i(t)=u_i(t)-\hat{u}_i(t)$; then

$$
\begin{aligned}
(u_i^L)^2D^+V_{i2}^\Delta(t)&=\mathrm{sign}\big(v_i^\sigma(t)\big)v_i^\Delta(t)\\
&=\mathrm{sign}\big(v_i^\sigma(t)\big)\Big\{-d_i(t)v_i(t)+e_i(t)\big[x_i(t-\eta_i)-\hat{x}_i(t-\eta_i)\big]\Big\}\\
&\le-d_i(t)sign\big(v_i^\sigma(t)\big)[v_i^\sigma(t)-\mu(t)v_i^\Delta(t)]\\
&\quad+e_i(t)\big[x_i(t-\eta_i)-\hat{x}_i(t-\eta_i)\big]
\end{aligned}
$$

$$\begin{aligned}&\leq -d_i(t)|v_i^\sigma(t)| + d_i(t)\mu(t)|v_i^\Delta(t)| + e_i(t)\big[x_i(t-\eta_i) - \hat{x}_i(t-\eta_i)\big]\\&\leq -d_i(t)|v_i^\sigma(t)| + d_i^2(t)\mu(t)|v_i(t)|\\&\quad + (1+d_i(t)\mu(t))e_i(t)\big[x_i(t-\eta_i) - \hat{x}_i(t-\eta_i)\big].\end{aligned}$$

So,

$$\begin{aligned}D^+V_{i2}^\Delta(t) \leq &\frac{-d_i^L}{(u_i^L)^2}|u_i^\sigma(t) - \hat{u}_i^\sigma(t)| + \frac{(d_i^U)^2\mu^U}{(u_i^L)^2}\big|u_i(t) - \hat{u}_i(t)\big|\\&+ \frac{(1+d_i^U\mu^U)e_i^U}{(u_i^L)^2}\big|x_i(t-\eta_i) - \hat{x}_i(t-\eta_i)\big|,\end{aligned}$$

$$\begin{aligned}D^+V_{i3}^\Delta(t) \leq &\sum_{j=1}^n \frac{a_{ij}^U(x_i^U)^2}{x_i^L}\big[|x_j(t) - \hat{x}_j(t)| - |x_j(t-\tau_{ij}) - \hat{x}_j(t-\tau_{ij})|\big]\\&+ \frac{(1+d_i^U\mu^U)e_i^U}{(u_i^L)^2}\big[|x_i(t) - \hat{x}_i(t)| - |x_i(t-\eta_i) - \hat{x}_i(t-\eta_i)|\big],\\D^+V_{i4}^\Delta(t) \leq &\frac{c_i^U(x_i^U)^2}{x_i^L}\big[\sigma^U|u_i^\sigma(t) - \hat{u}_i^\sigma(t)| - |u_i(t-\delta_i) - \hat{u}_i(t-\delta_i)|\big],\end{aligned}$$

where $\sigma^U = \max_{t\in\mathbb{T}^+}\sigma^\Delta(t)$. Since

$$\Big|x_i(t) - \hat{x}_i(t)\Big| = x_i(t)\hat{x}_i(t)\Big|\frac{1}{x_i(t)} - \frac{1}{\hat{x}_i(t)}\Big| \leq (x_i^U)^2\Big|\frac{1}{x_i(t)} - \frac{1}{\hat{x}_i(t)}\Big|$$

and

$$\Big|\frac{1}{u_i(t)} - \frac{1}{\hat{u}_i(t)}\Big| = \frac{1}{u_i(t)\hat{u}_i(t)}\Big|u_i(t) - \hat{u}_i(t)\Big| \leq \frac{1}{(u_i^L)^2}\Big|u_i(t) - \hat{u}_i(t)\Big|,$$

it follows that

$$\begin{aligned}D^+V^\Delta(t) \leq &D^+\hat{V}^\Delta(t)\\\leq &\sum_{i=1}^n\Bigg\{(x_i^U)^2\Big[c_i^Uu_i^U - b_i^L + \sum_{j=1}^n a_{ij}^Ux_j^U\Big]\Big|\frac{1}{x_i^\sigma(t)} - \frac{1}{\hat{x}_i^\sigma(t)}\Big|\\&+ \Big[\frac{-d_i^L}{(u_i^L)^2} + \frac{c_i^U\sigma^U(x_i^U)^2}{x_i^L}\Big]|u_i^\sigma(t) - \hat{u}_i^\sigma(t)| + \sum_{j=1}^n\frac{a_{ij}^U(x_i^U)^2}{x_i^L}\big|x_j(t) - \hat{x}_j(t)\big|\\&+ \frac{(1+d_i^U\mu^U)e_i^U}{(u_i^L)^2}\big|x_i(t) - \hat{x}_i(t)\big| + \frac{(d_i^U)^2\mu^U}{(u_i^L)^2}|u_i(t) - \hat{u}_i(t)|\Bigg\}\\\leq &-\sum_{i=1}^n(x_i^U)^2\Big[b_i^L - c_i^Uu_i^U - \sum_{j=1}^n a_{ij}^Ux_j^U\Big]\Big|\frac{1}{x_i^\sigma(t)} - \frac{1}{\hat{x}_i^\sigma(t)}\Big|\\&- \sum_{i=1}^n\Big[\frac{d_i^L}{(u_i^L)^2} - \frac{c_i^U\sigma^U(x_i^U)^2}{x_i^L}\Big]|u_i^\sigma(t) - \hat{u}_i^\sigma(t)|\\&+ \sum_{i=1}^n\Big[\sum_{j=1}^n\frac{a_{ij}^U(x_i^U)^2}{x_i^L} + \frac{(1+d_i^U\mu^U)e_i^U}{(u_i^L)^2}\Big]\big|x_i(t) - \hat{x}_i(t)\big|\end{aligned}$$

$$
\begin{aligned}
&+ \frac{(d_i^U)^2\mu^U}{(u_i^L)^2}|u_i(t)-\hat{u}_i(t)| \\
\leq &- \sum_{i=1}^{n}(x_i^U)^2\Big[b_i^L - c_i^U u_i^U - \sum_{j=1}^{n} a_{ij}^U x_j^U\Big]\Big|\frac{1}{x_i^\sigma(t)} - \frac{1}{\hat{x}_i^\sigma(t)}\Big| \\
&- \sum_{i=1}^{n}\Big[\frac{d_i^L}{(u_i^L)^2} - \frac{c_i^U\sigma^U(x_i^U)^2}{x_i^L}\Big]|u_i^\sigma(t)-\hat{u}_i^\sigma(t)| \\
&+ \sum_{i=1}^{n}\Big[\sum_{j=1}^{n}\frac{a_{ij}^U(x_i^U)^2}{x_i^L} + \frac{(1+d_i^U\mu^U)e_i^U}{(u_i^L)^2}\Big]\cdot(x_i^U)^2\Big|\frac{1}{x_i(t)} - \frac{1}{\hat{x}_i(t)}\Big| \\
&+ \frac{(d_i^U)^2\mu^U}{(u_i^L)^2}|u_i(t)-\hat{u}_i(t)| \\
= &- \Gamma_1\sum_{i=1}^{n}\Big|\frac{1}{x_i^\sigma} - \frac{1}{\hat{x}_i^\sigma}\Big| - \Gamma_1^*\sum_{i=1}^{n}|u_i^\sigma(t)-\hat{u}_i^\sigma(t)| \\
&+ \Gamma_2\sum_{i=1}^{n}\Big|\frac{1}{x_i(t)} - \frac{1}{\hat{x}_i(t)}\Big| + \Gamma_2^*\sum_{i=1}^{n}\big|u_i(t)-\hat{u}_i(t)\big| \\
= &- AV(\sigma(t)) + BV(t) = (B-A)V(t) - A\mu(t)D^+V^\Delta(t).
\end{aligned}
$$

It follows that $D^+V^\Delta(t) \leq (B-A)/(1+A\mu^U)V(t) = -\gamma V(t)$. By (H_2), we have $\gamma = (A-B)/(1+A\mu^U) > 0$ and $-\gamma \in \mathcal{R}^+$. Thus, the assumption (iii) of Lemma 2.8 is satisfied, and hence, it follows from Lemma 2.8 that there exists a unique uniformly asymptotically stable almost periodic solution $Z(t) = \big(x(t), u(t)\big)$ of dynamic system (1.2) and that $Z(t) \in \Omega$. □

5. Numerical simulations

Example 5.1. Consider the following system for $\mathbb{T}^+ = \mathbb{Z}^+$:

$$
\left.
\begin{aligned}
x_i(t+1) = x_i(t)\Big[1 + b_i(t) - a_i(t)x_i(t+1) - \sum_{j=1}^{n} a_{ij}(t)x_j\big(t-\tau_{ij}(t)\big) \\
- c_i(t)u_i\big(t-\delta_i(t)\big)\Big], \\
u_i(t+1) = r_i(t) + (1-d_i(t))u_i(t) + e_i(t)x_i\big(t-\eta_i(t)\big), \quad i=1,2,3,
\end{aligned}
\right\}
\tag{5.1}
$$

in which, for $i, j = 1, 2, 3$,

$$
\tau_{ij}(t) = \delta_i(t) = \eta_i(t) = 1 \text{ and } \begin{bmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix},
$$

$$
\begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix}
$$

$$= \begin{bmatrix} 0.03+0.02|\sin(t)| & 0.02+0.01|\cos(\sqrt{3}t)| & 0.03+0.01|\sin(\sqrt{2}t)| \\ 0.02+0.01|\sin(\sqrt{7}t)| & 0.03+0.02|\cos(\sqrt{2}t)| & 0.02+0.01|\cos(\sqrt{5}t)| \\ 0.02+0.01|\sin(\sqrt{2}t)| & 0.02+0.01|\cos(\sqrt{5}t)| & 0.03+0.01|\sin(\sqrt{2}t)| \end{bmatrix},$$

$$\begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{bmatrix} = \begin{bmatrix} 0.003+0.001\cos(\pi t) \\ 0.004+0.002\cos(\pi t) \\ 0.002+0.001\sin(\pi t) \end{bmatrix}, \begin{bmatrix} d_1(t) \\ d_2(t) \\ d_3(t) \end{bmatrix} = \begin{bmatrix} 0.5+0.38\sin(\pi t) \\ 0.6+0.36\sin(\pi t) \\ 0.3+0.1\cos(\pi t) \end{bmatrix},$$

$$\begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{bmatrix} = \begin{bmatrix} 1+0.02|\cos(\sqrt{2}t)| \\ 1+0.01|\sin(\sqrt{3}t)| \\ 2+0.02|\cos(t)| \end{bmatrix}, \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} = \begin{bmatrix} 0.002+0.001\sin(\pi t) \\ 0.003+0.002\sin(\pi t) \\ 0.002+0.001\cos(\pi t) \end{bmatrix}.$$

By a direct calculation, we get $x_1^U = x_2^U = 5$, $x_3^U = 2.5$, $u_1^U = 0.15833$, $u_2^U = 0.12917, u_3^U = 0.0525$, $x_1^L = 1.86275$, $x_2^L = 1.93069$, $x_3^L = 1.73267$, $u_1^L = 0.004389$, $u_2^L = 0.004094, u_3^L = 0.006832$,

$$\frac{b_1^L}{\sum_{j=1}^n a_{1j}^U + c_1^U} = 8.06452, \ \frac{b_2^L}{\sum_{j=1}^n a_{2j}^U + c_2^U} = 8.19672,$$

$$\frac{b_3^L}{\sum_{j=1}^n a_{3j}^U + c_3^U} = 8.06452, \ M = 5 < \frac{b_i^L}{\sum_{j=1}^n a_{ij}^U + c_i^U}, i = 1,2,3.$$

Therefore, system (5.1) is permanent. Furthermore, $\alpha_1 = 4.42083, \alpha_2 = 4.46042$, $\alpha_3 = 4.57375$, $\alpha_1^* = 0.14842$, $\alpha_2^* = 0.01897$, $\alpha_3^* = 0.08857$, $\beta_1 = 1.75153$, $\beta_2 = 1.66936$, $\beta_3 = 0.38696$, $\beta_1^* = 1.88$, $\beta_2^* = 1.96$, $\beta_3^* = 1.4$. It follows that $A = 4.57375 > 1.96 = B$. It is easy to see that the conditions of Theorem 4.1 are verified. Therefore, system (5.1) has a unique positive almost periodic solution, which is uniformly asymptotic stable. Our numerical simulations support our results. From Figure 1–14, it can be seen that for system (5.1), there exists a positive almost periodic solution denoted by $(x_1^*(t), x_2^*(t), x_3^*(t))$.

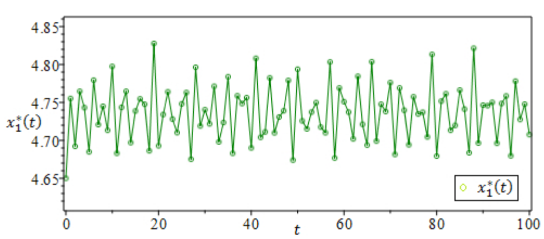

FIGURE 1. Positive almost periodic solution of system (5.1). Time series of $x_1^*(t)$ with initial value $x_1^*(0) = 4.65$ and t over $[0, 100]$.

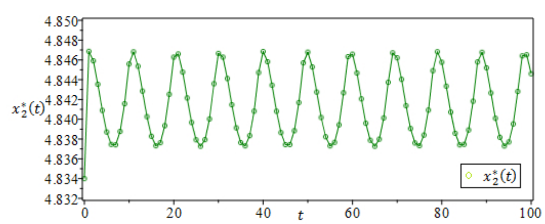

FIGURE 2. Positive almost periodic solution of system (5.1). Time series of $x_2^*(t)$ with initial value $x_1^*(0) = 4.834$ and t over $[0, 100]$.

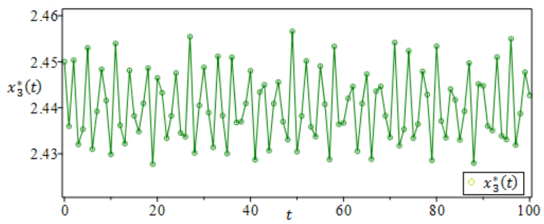

FIGURE 3. Positive almost periodic solution of system (5.1). Time series of $x_3^*(t)$ with initial value $x_1^*(0) = 2.45$ and t over $[0, 100]$.

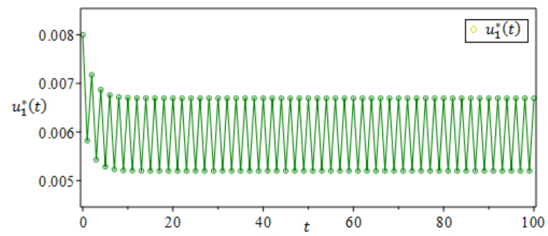

FIGURE 4. Positive almost periodic solution of system (5.1). Time series of $u_1^*(t)$ with initial value $u_1^*(0) = 0.008$ and t over $[0, 100]$.

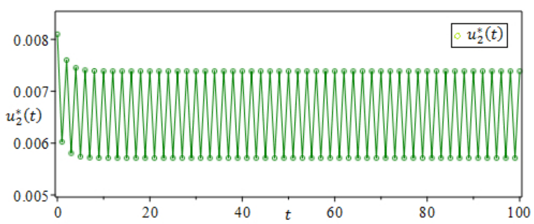

FIGURE 5. Positive almost periodic solution of system (5.1). Time series of $u_2^*(t)$ with initial value $u_2^*(0) = 0.0081$ and t over $[0, 100]$.

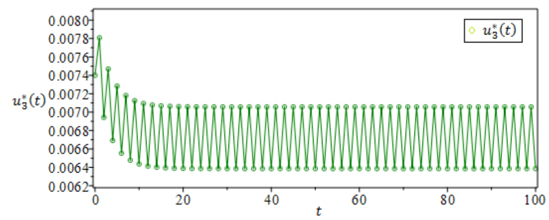

FIGURE 6. Positive almost periodic solution of system (5.1). Time series of $u_3^*(t)$ with initial value $u_3^*(0) = 0.0074$ and t over $[0, 100]$.

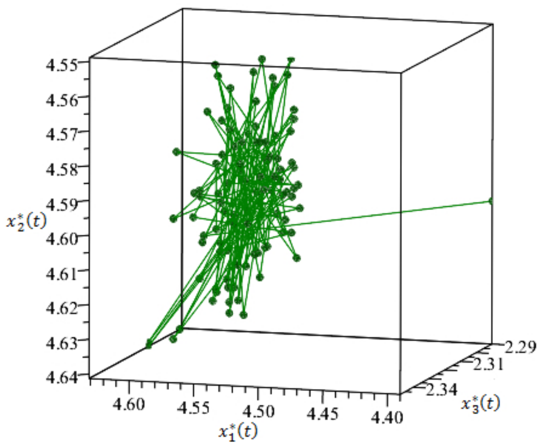

FIGURE 7. Positive almost periodic solution of system (5.1). Time series of $(x_1^*(t), x_2^*(t), x_3^*(t))$ with initial value $(x_1^*(0), x_2^*(0), x_3^*(0)) = (4.39, 4.6, 2.29)$.

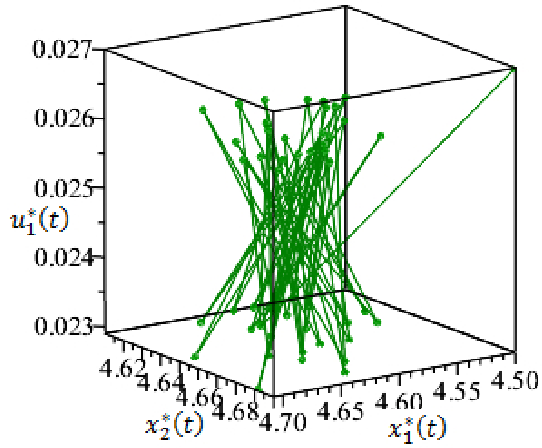

FIGURE 8. Positive almost periodic solution of system (5.1). Time series of $(x_1^*(t), x_2^*(t), u_1^*(t))$ with initial value $(x_1^*(0), x_2^*(0), u_1^*(0)) = (4.5, 4.7, 0.027)$.

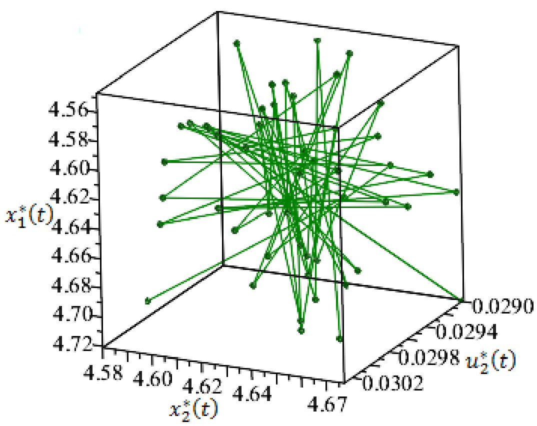

FIGURE 9. Positive almost periodic solution of system (5.1). Time series of $(x_1^*(t), x_2^*(t), u_2^*(t))$ with initial value $(x_1^*(0), x_2^*(0), u_2^*(0)) = (4.71, 4.58, 0.0298)$.

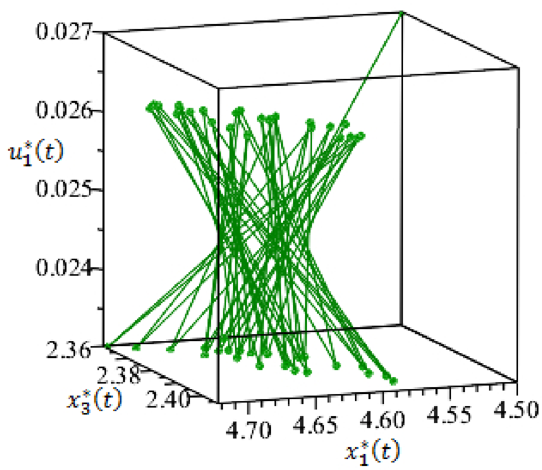

FIGURE 10. Positive almost periodic solution of system (5.1). Time series of $(x_1^*(t), x_3^*(t), u_1^*(t))$ with initial value $(x_1^*(0), x_3^*(0), u_1^*(0)) = (4.5, 2.36, 0.027)$.

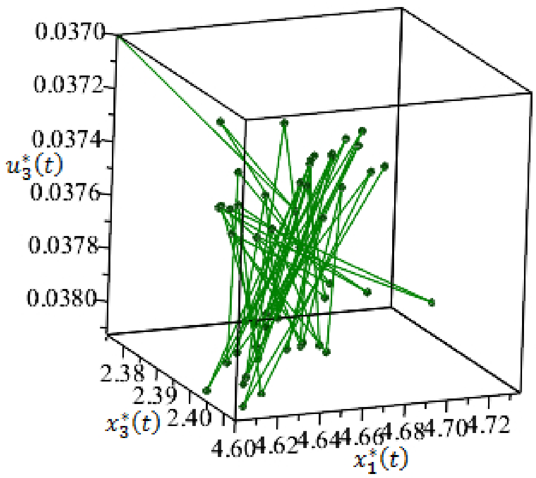

FIGURE 11. Positive almost periodic solution of system (5.1). Time series of $(x_1^*(t), x_3^*(t), u_3^*(t))$ with initial value $(x_1^*(0), x_3^*(0), u_3^*(0)) = (4.6, 2.375, 0.037)$.

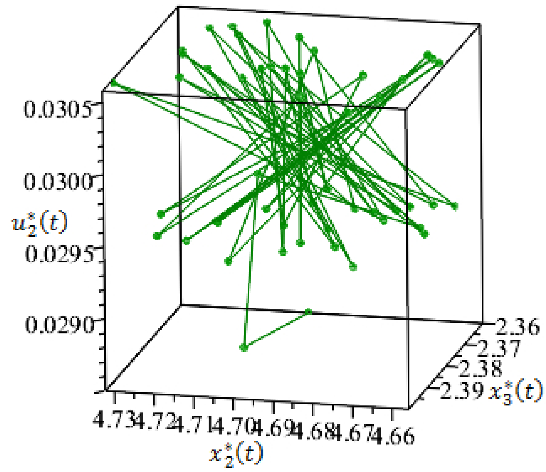

FIGURE 12. Positive almost periodic solution of system (5.1). Time series of $(x_2^*(t), x_3^*(t), u_2^*(t))$ with initial value $(x_2^*(0), x_3^*(0), u_2^*(0)) = (4.7, 2.36, 0.0285)$.

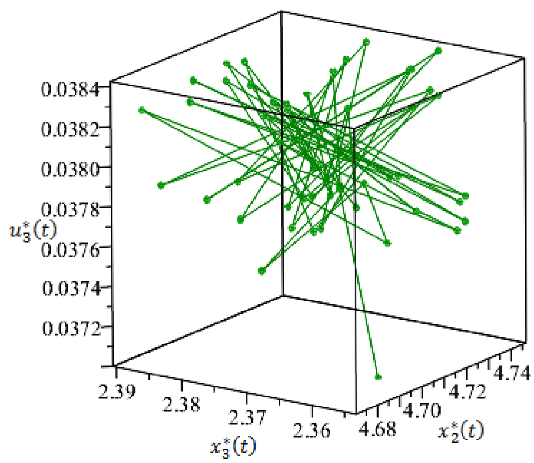

FIGURE 13. Positive almost periodic solution of system (5.1). Time series of $(x_2^*(t), x_3^*(t), u_3^*(t))$ with initial value $(x_2^*(0), x_3^*(0), u_3^*(0)) = (4.7, 2.36, 0.037)$.

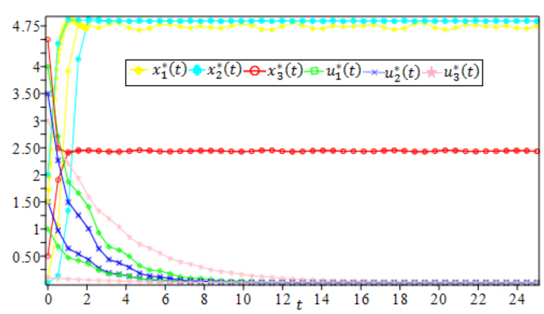

FIGURE 14. Dynamic behaviors of the system (5.1) with initial values $(x_1^*(0), x_2^*(0), x_3^*(0), u_1^*(0), u_2^*(0), u_3^*(0)) = (0.1, 0.01, 0.5, 4, 1.5, 3), (1.5, 2, 4.5, 1, 3.5, 0.1)$.

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