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ON THE DISCRETE GROUP ANALYSIS FOR THE EXACT SOLUTIONS OF SOME CLASSES OF THE NONLINEAR ABEL AND BURGERS EQUATIONS

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Abstract. This article presents an account of the fundamentals of the discrete group approach for analysis and integration of practical differential equations. In this article, by means of appropriate transformations, the nonlinear Burgers equation is transformed into the other class of the second-order differential equation of the Emden–Fowler type, and this Emden–Fowler equation reduces to nonlinear Abel equations. This approach shows that, under these transformations of discrete group, the solution of reference equation can be transformed into the solution of the transformed equation. Under such conditions, we approach to determine some solutions for the Abel, Burgers, Emden–Fowler, and heat equations.

1. Introduction and preliminaries

It is well known that the theory of differential equations takes a central place among possible instruments for the modeling of different processes and phenomena. The classical concepts of groups introduced by Lie and Bäcklund, which constitute the foundation of modern group analysis, are responsible for out standing achievements in the theory of partial differential equations. However, a similar approach based on a search for continuous transformation groups, which map the equation under investigation into itself (i.e., exactly into the same equation), was proved to be ineffective for solving ordinary differential equations (ODEs); see [1, 5, 9–13].

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In [10,11,13], the authors analyzed the discrete group analysis for some classes of ODEs. Also, we study some classes of fundamentals of the discrete group approach for analysis and integration of practical differential equations to the exact solution of the nonlinear partial differential equations [7], nonlinear ODEs [6], and nonlinear Volterra integral equations [3].

This article presents new methods for the analysis of ODEs based on the search for discrete transformation groups closed on the class of equations under consideration (i.e., the original equation here may turn in to another equation of the same class). This approach enables us to find a great number of new integrable equations, which thus far, could not be integrated by using the classical methods.

Definition 1.1. The class of generalized Emden–Fowler equations is written as

$$y''_{xx} = Ax^n y^m y'^l, \qquad a = (n, m, l), \tag{1.1}$$

where it is determined by a three-dimensional parameter vector $a = (n, m, l) \in \mathbb{R}^3$.

Definition 1.2. The general Abel equation is written as

$$[\phi_1(x)y + \phi_0(x)]y'_x = \psi_2(x)y^2 + \psi_1(x)y + \psi_0(x).$$

Definition 1.3. A set D of ODEs is referred to as a class of equations $D(x, y, a) \in D$ uniquely defined by a vector a of parameters.

Let D be a class of ODE and let

$$D(x, y, a) = 0 \tag{1.2}$$

be an equation in this class, where a is a vector parameters.

We shall seek the transformations F_i that are closed in the class (1.2), that is, they change only the vector a as follows:

$$F_i : D(x, y, a) \to D(t, u, b_i). \tag{1.3}$$

If each F_i has inverses, then the collection $\{F_i\}$ defines a D.T.G (Discrete Transformation Group) on the class (1.2).

All the existing methods of exact solution of ODEs can be conditionally divided in two groups (see [6]):

(A) A search for transformation of the original class D of ODE to some other class D_1 in which a method of solutions is available.

(B) A search for transformations leaving the original class D invariant, that is, a transformation into itself, that gives independent information about solution.

Practically, all the classic methods of exact solution of ODE use approach **(A)** based on a rather restricted number of standard soluble ODE; therefore transformations often are artificial. To develop approach **(A)**, one can apply different transformations to standard soluble equations infinitely extending the set of such equations. However the probability that the ODE chosen for the investigation belonging to the extended set is very small; see [1,5,9,12].

The DGM does not operate with a single equation as in applications of Lie method (see [14]) but operates with a class of equations D, depending on a vector

a of parameters, containing the investigated equation, but contrary to approach **(A)**, one considers the transformations of the given class D, which are closed in itself on a chosen class of ODE.

There are two methods for searching discrete group transformations: point transformations and Bäcklund transformation. In the class of point transformations

$$y = f(t, u), \qquad x = g(t, u), \qquad J = f_u g_t - g_u f_t \neq 0, \tag{1.4}$$

the following two methods are effectively applicable: The direct method and the method based on Lie Algorithm (LA) [6, 11, 13]. The direct method is based on the substitution of the transformation (1.3) into (1.1). Imposing condition (1.3) leads to a partial differential equation with unknown functions f and g, which can be split with respect to the independent variables into the lower order differential equations with respect to u and t. As a result, we obtain an over determined system of nonlinear partial differential equations with respect to f and g; for more details see [14].

There are many methods of search for Bäcklund transformations such as the direct method, RF-pair method, and support equation method. Each of which is fully discussed in [5, 12].

Definition 1.4. An RF-Pair is an operation of consecutive raising and lowering the order of equation.

Now, we define the following R-operations and F-operations:

i) Termwise m-fold differentiation of the original equation, type RD^m.
ii) Termwise one- or two-fold differentiation of original equation with respect to the independent variable, type accordingly RX or RX^2.
iii) The equation is an exact derivative of the mth-order: termwise integration m times, type FI^m.
iv) The equation is autonomous, that is, it does not conclude an independent variable in an explicit form, type FX:

$$FX : y'_x = u(y), \qquad y''_{xx} = uu'_y.$$

v) The equation is homogeneous in the extended sense, type FU: the transformation $x = e^t$ and $y = ue^{kt}$, with an appropriate choice of k, leads to an autonomous form followed by a transformation FX.

If an $R(F)$-operating $RZ^m(FZ^m)$ is inverted, then it is denoted by $RZ^{-m}(FZ^{-m})$. The RF-pair will be written in a contracted form by means of an ordered pair, the second letters used in the designation of the operation symbol and the left letter corresponding to the transformation performed first

$$RF(D, X) \equiv (FX) \otimes (RD) \equiv FX(RD).$$

This article presents some transformations for the analysis of differential equations based on the search for discrete transformation groups closed on the class of equations under consideration (i.e., the original equation here may turn in to another equation of the same class). This approach enables us to find a great number of new integrable equations, which thus far, could not be integrated by using the classical methods.

Also, in this work, we present a new analytical method based on Lie group method and Bäcklund transformations [5, 10–13] to simulate Abel and Burgers equations, by using Cole–Hopf transformation [2, 8]. In fact, using a nonlinear Cole–Hopf transformation, the nonlinear Burgers equation is reduced to diffusion equation. Under the Lie group method and Bäcklund transformations, the solution of the transformed equation can be converted into the solution of the reference equation.

2. Analysis of the method

Consider the nonlinear Burgers equations (see [4]):

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} = \nu\frac{\partial^2 U}{\partial X^2}, \tag{2.1}$$

subject to the initial conditions

$$U(X,0) = U_0(X), \tag{2.2}$$

and boundary conditions

$$U(0,t) = F_1(t), \tag{2.3}$$

$$U(L,t) = F_2(t), \tag{2.4}$$

where $0 \leq X \leq L,\ t \geq 0$ and $F_1,\ F_2$ are known functions, ν is a given viscosity coefficient defined by $\nu = \dfrac{1}{Re}$, and Re is the Reynolds number.

The nonlinear coupled Burgers equation is a special form of incompressible Navier–Stokes equation without having pressure term and continuity equation. Burgers equation is an important partial differential equation from fluid dynamics and widely used for various physical applications, such as shock flows, wave propagation in combustion chambers, vehicular traffic movement, acoustic transmission, and so on; see [2, 4, 8].

Here a procedure is developed for generating analytical exact solutions of the Burgers equations. It has been pointed out by Cole [2] and Hopf [8] that the Cole–Hopf transformation can be interpreted as a multi-dimensional transformation. In one dimensions, the Cole–Hopf transformation $\mathcal{T}_{CF}$ relates a function T to U in the following way:

$$\mathcal{T}_{CF}:\ U(X,t) = -2\nu\frac{\frac{\partial}{\partial X}T(X,t)}{T(X)}. \tag{2.5}$$

Then equations (2.1) become

$$\frac{\partial T(X,t)}{\partial t} = \nu\frac{\partial^2 T(X,t)}{\partial X^2}, \tag{2.6}$$

subject to the initial-boundary conditions

$$\begin{array}{lll} T(X,0) = & \exp\left(-\frac{1}{2\nu}\int_0^X U_0(s)\mathrm{d}s\right), & 0 \le X \le L, \\ \frac{\partial T(X,t)}{\partial X}\ |_{X=0}= & \frac{\partial T(X,t)}{\partial X}\ |_{X=L}= 0, & t \ge 0. \end{array} \tag{2.7}$$

Now, the nonlinear problem under investigation is described by the following equation with initial and boundary conditions (2.7):

$$\frac{\partial T}{\partial t} = \nu \frac{\partial^2 T}{\partial X^2},\ 0 \le X \le L,\ t \ge 0. \tag{2.8}$$

We introduce dimensionless variables

$$\mathcal{T}_1:\ x = \frac{X}{L}, \qquad \tau = \frac{\nu}{L^2}t, \qquad y = T, \tag{2.9}$$

where L is a constant chosen as the length scale. Problem (2.8) by using $\mathcal{T}_1$ of (2.9) is written in the following form:

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}. \tag{2.10}$$

Equation (2.10), by employment of a new self-similarity variable

$$\mathcal{T}_2:\ w = \frac{x}{\sqrt{\tau}}, \tag{2.11}$$

reduces to a two-point boundary value problem for an ODE of second-order of the following Emden–Fowler equations:

$$y''_{ww} = -\frac{1}{2}wy'_w,\ a = (1,1,0). \tag{2.12}$$

This boundary value problem via substitution of hodograph transformation

$$\mathcal{T}_3:\ y = t,\ w = u, \tag{2.13}$$

is transformed to

$$u''_{tt} = \frac{1}{2}u(u'_t)^2,\ a = (0,1,2). \tag{2.14}$$

Therefore, it follows that the boundary value problem (2.14) is integrable by quadratures, as a consequence of which, we have been able to construct some solutions for the Burgers equation (for further details see [8]).

Now, we shall describe some transformations [14], in which equation (2.14) is reduced to the Abel equations. Applying the operation

$$\mathcal{T}_4:\ z = \frac{t}{u}u'_t,\ v = \frac{1}{2}u^2, \tag{2.15}$$

into (2.14) leads to the equation

$$(z^2 v - z^2 + z)v'_z = 2zv, \tag{2.16}$$

and by using the substitution

$$\mathcal{T}_5:\ \eta = v + \frac{1}{z}, \tag{2.17}$$

equation (2.16) reduces to the following equation:

$$\eta\eta' = (2z-1)z^{-2}\eta + (2z^2 - 2z)z^{-3}. \tag{2.18}$$

Also, substituting

$$\mathcal{T}_6 : \begin{cases} \theta = \sqrt{2} t u' \exp(-u^2), \\ \varpi = \int \exp(\frac{s^2}{2}) \mathrm{d}s, \\ s = \sqrt{2} u, \end{cases}$$

transforms equation (2.14) to the Abel equation

$$\theta\theta'_{\varpi} - \theta = \quad 0. \tag{2.19}$$

Note that, all the transformations applied for the conversion of Burgers equation are invertible, and this invertibility clearly allows us to avoid some lengthy computations for the conversion of the initial and boundary condition. Also, under these transformations, the solution of the transformed equation can be converted into the solution of the reference equation.

3. Exact solutions

In this section, we give some applications of the proposed scheme obtaining the solution of the Burgers equation.

Now, by using the operators $\mathcal{T}_{cf}, \mathcal{T}_1, \ldots, \mathcal{T}_6$ with the consideration of equations (2.1)–(2.19), we obtain Table 1. It is necessary to pay attention to the point that using the properties of discrete groups and inevitability of the applied transformations, we can reach the analytic solution of any equation appearing in Table 1, if we can obtain the solution of a single equation by using classical methods.

The exact solution of equation (2.14) is given by [6]

$$t = \quad C_1 f(\zeta) + C_2, \quad u = 2\zeta, \tag{3.1}$$

where $C_i, i = 1, 2$, are arbitrary constants and

$$f(\zeta) = \int \exp(-\zeta^2) \mathrm{d}\zeta, \tag{3.2}$$

where $f(\zeta)$ is the Gaussian integral.

Now, by the application of inverse transformations on (3.1) and (3.2), we obtain the exact solution of Burgers equation. We summarize the selected results form this procedure for exact solution of Burgers equation and Abel equations in Tables 2 and 3, respectively.

Now, to integrate equation (2.1), we start from (2.14). By using the operators and inverse operators on (3.1) and (3.2), according to Table 1, we obtain the analytical solution of Burgers and Abel equations. Some results of this method are summarized in Table 2.

4. Conclusion

This approach can reveal that, under the group action, the solution of the reference equation can be transformed into the solution of the transformed equation. The action of the group upon an element of the class generates the orbit of this element, that is, the set of all equations of the given class obtained through the action of transformations of the group. By using an application of the discrete transformation group analysis, an ODE belonging to a class can be transformed

Table 1.

Equation		Exact Solution (E.S.)
Eq. 7		E.S. of Eq. 7
$\downarrow \mathcal{T}_{cf}$		$\uparrow \mathcal{T}_{cf}^{-1}$
Eq. 14		E.S. of Eq. 14
$\downarrow \mathcal{T}_1$		$\uparrow \mathcal{T}_1^{-1}$
Eq. 16		E.S. of Eq. 16
$\downarrow \mathcal{T}_2$		$\uparrow \mathcal{T}_2^{-1}$
Eq. 18		E.S. of Eq. 18
$\downarrow \mathcal{T}_3$		$\uparrow \mathcal{T}_3^{-1}$
Eq. 20	$\xrightarrow[\text{Solvable}]{}$	E.S. Eq. 20
$\downarrow \mathcal{T}_4$		$\downarrow \mathcal{T}_4$
Eq. 22		E.S. of Eq. 22
$\downarrow \mathcal{T}_5$		$\downarrow \mathcal{T}_5$
Eq. 24		E.S. of Eq. 24
$\downarrow \mathcal{T}_6$		$\downarrow \mathcal{T}_6$
Eq. 25		E.S. of Eq. 25

Table 2.

Equation	Exact solution
$u''_{tt} = \frac{1}{2}u(u'_t)^2$	$t = C_1 f(\zeta) + C_2$ $u = 2\zeta$ $f(\zeta) = \int \exp(-\zeta^2)\mathrm{d}\zeta$ $C_i, i = 1, 2,$ are arbitrary constants
$y''_{ww} = -\frac{1}{2}wy'_w$	$w = 2\zeta$ $y(w) = C_1 f(\zeta) + C_2$
$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$	$\zeta = \frac{1}{2}\frac{x}{\sqrt{\tau}}$ $y(x,\tau) = C_1 f(\frac{1}{2}\frac{x}{\sqrt{\tau}}) + C_2$
$\frac{\partial T}{\partial t} = \nu\frac{\partial^2 T}{\partial X^2}$	$T(X,t) = C_1 f(\frac{1}{2\sqrt{\nu}}\frac{X}{\sqrt{t}}) + C_2$
$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} = \nu\frac{\partial^2 U}{\partial X^2}$	$U(X,t) = -2\nu C_1 \left[C_1 f(\frac{1}{2\sqrt{\nu}}\frac{X}{\sqrt{t}}) + C_2\right]^{-1} \frac{\partial}{\partial X}\left[f(\frac{1}{2\sqrt{\nu}}\frac{X}{\sqrt{t}})\right]$

TABLE 3.

Equation	Exact solution
$u''_{tt} = \frac{1}{2}u(u'_t)^2$	$t = C_1 f(\zeta) + C_2$ $u = 2\zeta$ $f(\zeta) = \int \exp(-\zeta^2)\mathrm{d}\zeta$ $C_i, i = 1, 2,$ are arbitrary constants
$(z^2 v - z^2 + z)v'_z = 2zv$	$v = 2\zeta^2$ $z = \frac{C_1 f(\zeta)+C_2}{C_1\zeta\frac{\mathrm{d}f(\zeta)}{\mathrm{d}\zeta}}$
$\eta\eta' = (2z-1)z^{-2}\eta + (2z^2-2z)z^{-3}$	$\eta = 2\zeta^2 + \frac{C_1\zeta\frac{\mathrm{d}f(\zeta)}{\mathrm{d}\zeta}}{C_1 f(\zeta)+C_2}$
$\theta\theta'_{\varpi} - \theta = 0$	$\theta = \frac{2\sqrt{2}(C_1 f(\zeta)+C_2)\exp(-4\zeta^2)}{C_1\zeta\frac{\mathrm{d}f(\zeta)}{\mathrm{d}\zeta}}$ $s = 2\sqrt{2}\zeta$ $\varpi = \int \exp(4s^2)\mathrm{d}s$

to a reduced form in some other classes, which may be integrated by using classical methods. By introducing some useful transformations, the nonlinear Burgers equation in applied physics and Abel equations, can be transformed into the classical Emden–Fowler equation, which may be integrated by using classical methods. This approach shows that, under these transformations, the solution of transformed equation can be converted into the solution of the reference equation.

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