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ITERATIVE REGULARIZATION METHOD FOR AN ABSTRACT INVERSE GOURSAT PROBLEM

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Abstract. We deal with a problem of identification of an unknown source in the abstract inverse Goursat problem with two-time variables. We show that the considered problem is ill-posed according to the Hadamard sense. That is, the solution does not depend continuously on the data. In order to overcome the instability of the solution, we propose a regularization method via an iterative procedure, with the help of an extra measurement at an internal point. Some convergence results are established under a priori bound assumptions on the exact solution. Finally, numerical tests are presented to illustrate the accuracy and efficiency of the proposed regularization method.

1. Introduction

In classical evolution equations, the spatial variable was naturally accepted as multidimensional, since the temporal variable was unidimensional. Recently a new idea has been appeared: The mathematical models for certain natural phenomena can be formulated by means of multi-time evolution partial differential equations (PDEs). The term “multi-time” (“multi-temporal”) was introduced in physics by Dirac in 1932 [8], considering multi-temporal wave functions described by evolution PDEs. It was assumed in mathematics by Friedman (see [13]). The classical physical examples of inverse Goursat problems are unidimensional in time. For this category of problems there are several models, we can mention as an example the age structured population dynamics with integral condition with

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respect to one of time variables was studied by many researchers; see [32, 36] and their references.

However nonclassical problems for abstract inverse Goursat problems with two-time variables are not widely investigated. In the literature these problems are encountered in various models of science and technology, particularly, mechanics, physics [3, 4], biomathematics, and cosmology (see, for example, the works of Baez, Hillion, and Uglum [2, 15, 16, 35]), and in mathematical models of diffusion of pollutants in water flows [17], mathematical models of age structured biological population dynamics [19, 20, 38] and mathematical finance [12]. In these models one of time variables is usual time and others might denote various quantities, for example, coordinates, temperature, and age or size of individuals of biological population.

In the present paper, we deal with a problem of identification of an unknown source for an abstract inverse Goursat problem in abstract Hilbert spaces with two-time variables. We will show that the problem is ill-posed in the sense that the solution (if it exists) does not depend continuously on the data. In order to obtain a stable numerical solution, we propose a regularization method via an iterative regularization procedure based on the Kozlov–Maz'ya approach. We show rigorously, with error estimates provided, that the corresponding regularized solutions converge to the exact solution under some priori assumptions on the solution.

The paper is organized as follows. In Section 2, we formulate the abstract inverse Goursat problem with two-time variables and some preliminaries and basic results are given. Section 3 is intended for giving a necessary and sufficient condition for the solvability of our problem. In Section 4, the regularization method will be given. We prove our main theorems of the paper with exact and noisy data, and some error estimates are established under a priori regularity assumption on the problem data. We present numerical experiments in Section 5 that verify the effectiveness of the proposed regularization method, and finally conclusions are summarized in the last section.

2. Formulation of the problem and basic results

Throughout this paper, H denotes a complex separable Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$ and $\mathcal{L}(H)$ stands for the Banach algebra of bounded linear operators on H. Let $A:\mathcal{D}(A)\subset H\longrightarrow H$ be a positive, self-adjoint operator with compact resolvent. Then A has an orthonormal basis of eigenvectors $(\phi_n)\subset H$ with real eigenvalues $(\lambda_n)\subset\mathbb{R}_+$, that is,

$$A\phi_n=\lambda_n\phi_n, n\in\mathbb{N}^*,\quad \langle\phi_i,\phi_j\rangle=\delta_{ij}=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{if } i\neq j,\end{cases}$$

$$0<\nu\leq\lambda_1\leq\lambda_2\leq\lambda_3\leq\dots,\quad \lim_{n\to\infty}\lambda_n=\infty,$$

$$\text{for all } h\in H,\quad h=\sum_{n=1}^{\infty}h_n\phi_n,\quad h_n=\langle h,\phi_n\rangle.$$

2.1. **Formulation of the problem.** In this paper, we consider the following inverse source problem of determining the unknown source term $p \in H$ and the function $u(t)$ for $t = (t_1, t_2) \in]0, T_1] \times]0, T_2]$ from the system of equations

$$\begin{cases} \frac{\partial^2 u(t)}{\partial t_1 \partial t_2} + Au(t) = p, & t = (t_1, t_2) \in]0, T_1] \times]0, T_2], \\ u(t_1, 0) = 0, & t_1 \in [0, T_1], \\ u(0, t_2) = 0, & t_2 \in [0, T_2], \end{cases} \tag{2.1}$$

where $0 < T_1, T_2 < \infty$ and g is a given H-valued function. The source function p is unknown. We use the additional condition

$$u(a, b) = g, \quad (a, b) \in]0, T_1[\times]0, T_2[, \tag{2.2}$$

to identify the unknown source p.

Remark 2.1. In this study, we consider the homogeneous case to simplify calculations. For the nonhomogeneous case,

$$\begin{cases} \frac{\partial^2 u(t)}{\partial t_1 \partial t_2} + Au(t) = p, & t = (t_1, t_2) \in]0, T_1] \times]0, T_2], \\ u(t_1, 0) = \varphi(t_1), & t_1 \in [0, T_1], \\ u(0, t_2) = \psi(t_2), & t_2 \in [0, T_2], \end{cases} \tag{2.3}$$

we assume that φ and ψ satisfy the following assumptions:

$$\varphi(t_1) \in \mathcal{D}(A), \quad t_1 \in [0, T_1], \tag{$H1$}$$

$$\psi(t_2) \in \mathcal{D}(A), \quad t_2 \in [0, T_2], \tag{$H2$}$$

and

$$\varphi(0) = \psi(0) = \chi \in \mathcal{D}(A). \tag{$H3$}$$

Under these conditions, we introduce the following function:

$$v(t_1, t_2) = u(t_1, t_2) - \varphi(t_1) - \psi(t_2) + \chi.$$

we obtain a homogeneous problem with a new source $\hat{p}$,

$$\begin{cases} \frac{\partial^2 v(t)}{\partial t_1 \partial t_2} + Av(t) = \hat{p} = p + \eta(t_1, t_2), & t = (t_1, t_2) \in]0, T_1] \times]0, T_2], \\ v(t_1, 0) = 0, & t_1 \in [0, T_1], \\ v(0, t_2) = 0, & t_2 \in [0, T_2], \end{cases} \tag{2.4}$$

with $\eta(t_1, t_2) = A(\chi - \varphi(t_1) - \psi(t_2))$.

To determine the unknown source p, we apply the same iterative procedure as in the homogeneous case with an additional calculation in the numerical implementation.

In practice, the measurable data g is never known exactly. We assume that the exact data g and the measured data g_δ satisfy $\|g - g^\delta\| \leq \delta$, where δ is a noise level.

In this paper, we continue the investigation started by Aksen [1], Boussetila and Rebbani [5] for an ill-posed evolution problem with two-time variables t_1 and t_2. We note that the case of multi-time variables does not seem to have been widely investigated and that the literature devoted to this class of problems is quite scarce. The study of this case is caused not only by theoretical interest, but also by practical necessity. Consequently, this paper wants to give a contribution for multi-time inverse problems [19, 20, 27, 28, 34, 43].

As we know, there are several works on the subject of inverse source problems by using numerical algorithms. There are also some regularization methods, with strict theoretical analysis, such as the Fourier method, the quasi-reversibility method, the simplified Tikhonov method, and the wavelet dual least squares method. For more details, we refer the reader to [9, 18, 30, 31, 39, 41].

For regularizing problem (2.1), we propose an iterative regularization procedure based on the Kozlov–Maz'ya approach with the help of an extra measurement at an internal point given by (2.2). In [21, 22], Kozlov and Maz'ya proposed an alternating iterative method to solve boundary value problems for general strongly elliptic and formally self-adjoint systems. After that, this method has attracted considerable attention of a lot of mathematicians and the idea has been successfully used for solving various classes of ill-posed (elliptic, parabolic, biparabolic, hyperbolic and fractional evolution) equations; see, for example, [6, 14, 24, 40].

2.2. **Preliminaries and basic results.** In this section, we present the notation and functional setting and prepare some materials, which will be used in our analysis.

For ease of reading, we summarize some well-known facts for nonexpansive operators.

Definition 2.2. A linear operator $M \in \mathcal{L}(H)$ is called nonexpansive if

$$\|M\| \leq 1.$$

For more details concerning the theory of nonexpansive operators, we refer the reader to Krasnosel'skii et al. [23, p. 66]. Let us consider the operator equation

$$S\varphi = (I - M)\varphi = \psi, \tag{2.5}$$

for nonexpansive operators M.

Theorem 2.3. *Let M be a linear self-adjoint, positive, and nonexpansive operator on H. Let $\hat{\psi} \in H$ be such that equation (2.5) has a solution $\hat{\varphi}$. If 1 is not an eigenvalue of M, that is, $(I - M)$ is injective, then the successive approximations*

$$\varphi_{n+1} = M\varphi_n + \hat{\psi}, \quad n = 0, 1, 2, \dots,$$

converge to $\hat{\varphi}$ for any initial data $\varphi_0 \in H$, and we have

$$\hat{\varphi} - \varphi_n = M^n(\varphi_0 - \hat{\varphi}) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.6}$$

Theorem 2.4 (Generalized Picard theorem, Prilepko [26, p. 502])**.** *Let H be a Hilbert space, let S be a positive self-adjoint, unbounded linear operator on H, and let $\Theta : \sigma(A) \longrightarrow \mathbb{R}$ be a continuous function not identically equal to zero, such that*

$$\Theta(S) = \int_0^{+\infty} \Theta(\lambda) dE_\lambda \in \mathcal{L}(H),$$

where $\{E_\lambda,\ \lambda \geq 0\}$ *is the spectral resolution of the identity associated to* S. *Let* $Z(\Theta) = \{\lambda \in \sigma(A) : \Theta(\lambda) = 0\}$ *be the set of zeros of the characteristic function* $\Theta(\lambda)$ *supposed to be either empty or contains isolated points only. Then, the equation*

$$\Theta(S)u = v,$$

is correctly solvable if and only if

(1) $Z(\Theta) \cap \sigma(A) = \emptyset$ (*uniqueness condition*).

(2) $\int\limits_0^{+\infty} \frac{1}{|\Theta(\lambda)|^2} d\|E_\lambda v\|^2 < +\infty$ (*existence condition*).

We denote by

$$J_n(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!(n+k)!}, \quad x \in \mathbb{R}, \quad n = 0, 1, 2, \dots,$$

the Bessel function of the first kind. Some basic properties of this function are listed in the following lemma.

Lemma 2.5 ([25]). *We have the following properties:*

(1) $J_{-n}(x) = (-1)^n J_n(x)$,
(2) $J_0(0) = 1,\ J_n(0) = 0,\ n \in \mathbb{N}^*$,
(3) $|J_0(x)| \leq 1,\ |J_n(x)| \leq \frac{1}{2}, n \in \mathbb{N}^*$,
(4) $\lim\limits_{x \longrightarrow 0} \frac{J_1(x)}{x} = \frac{1}{2},\ \lim\limits_{x \longrightarrow +\infty} J_0(x) = 0$.

For $a \geq 0$, we define the Riemann function by the following formula:

$$R(a; t_1, t_2) = J_0(2\sqrt{a}\sqrt{t_1.t_2}), \quad t_1, t_2 \in \mathbb{R}_+. \tag{2.7}$$

By a simple calculation, we show the following expressions:

$$\int\limits_0^{t_2}\int\limits_0^{t_1} R(a; t_1 - s_1, t_2 - s_2) ds_1 ds_2 = \frac{1 - R(a; t_1, t_2)}{a}, \tag{2.8}$$

$$\frac{\partial}{\partial t_1}\{R(a; t_1 - s_1, t_2 - s_2)\} = -\sqrt{a}\frac{\sqrt{t_2}}{\sqrt{t_1}} J_1(2\sqrt{a}\sqrt{t_1}\sqrt{t_2}), \quad t_1 > 0, \tag{2.9}$$

$$\frac{\partial}{\partial t_2}\{R(a; t_1 - s_1, t_2 - s_2)\} = -\sqrt{a}\frac{\sqrt{t_1}}{\sqrt{t_2}} J_1(2\sqrt{a}\sqrt{t_1}\sqrt{t_2}), \quad t_2 > 0, \tag{2.10}$$

$$\text{for all } t_2 > 0,\ \lim_{t_1 \longrightarrow 0}\left\{-\sqrt{a}\frac{\sqrt{t_2}}{\sqrt{t_1}} J_1(2\sqrt{a}\sqrt{t_1}\sqrt{t_2})\right\} = -at_2, \tag{2.11}$$

$$\text{for all } t_1 > 0,\ \lim_{t_2 \longrightarrow 0}\left\{-\sqrt{a}\frac{\sqrt{t_1}}{\sqrt{t_2}} J_1(2\sqrt{a}\sqrt{t_1}\sqrt{t_2})\right\} = -at_1, \tag{2.12}$$

$$\text{for all } t_1 > 0,\ \text{for all } t_2 > 0, \quad \lim_{(t_1,t_2) \longrightarrow (0,0)} \frac{\partial}{\partial t_1}\{R(a; t_1, t_2)\} = 0, \tag{2.13}$$

$$\text{for all } t_1 > 0,\ \text{for all } t_2 > 0, \quad \lim_{(t_1,t_2) \longrightarrow (0,0)} \frac{\partial}{\partial t_2}\{R(a; t_1, t_2)\} = 0, \tag{2.14}$$

$$\frac{\partial^2}{\partial t_1 \partial t_2}\{R(a;t_1,t_2)\} = \frac{\partial^2}{\partial t_2 \partial t_1}\{R(a;t_2,t_1)\} = -aR(a;t_1,t_2). \tag{2.15}$$

On the basis of (ϕ_n) we introduce the Hilbert scale $\{H^s, s\in\mathbb{R}\}$ induced by A as follows:

$$H^s = \mathcal{D}(A^s) = \{h \in H : \|h\|^2_{H^s} = \sum_{n=1}^{\infty} \lambda_n^{2s} |\langle h,\phi_n\rangle|^2 < +\infty\}.$$

Let $0 < \theta_1 < \theta_2$ and let $0 < \theta_3 < \theta_4$. Then we have the following topological inclusions:

$$H^{\theta_2} \subset H^{\theta_1} \subset H^0 = H \subset H^{-\theta_3} \subset H^{-\theta_4}.$$

Remark 2.6. For $s > 0$, the Hilbert space H^{-s} is the topological dual space of H^s, that is, $H^{-s} = (H^s)'$.

3. Analysis of the problem

3.1. **The direct problem.** Let us consider the following well-posed problem:

$$\begin{cases} \frac{\partial^2 v(t)}{\partial t_1 \partial t_2} + Av(t) = p, & t = (t_1,t_2) \in]0,T_1]\times]0,T_2], \\ v(t_1,0) = 0, & t_1 \in [0,T_1], \\ v(0,t_2) = 0, & t_2 \in [0,T_2], \end{cases} \tag{3.1}$$

where $0 < T_1, T_2 < \infty$ and p is a given H-valued function. By using the Fourier expansion and the given function p,

$$p = \sum_{n=1}^{+\infty} p_n\phi_n, \quad p_n := \langle p,\phi_n\rangle,$$

$$v(t) = \sum_{n=1}^{\infty} v_n(t)\phi_n, \quad v_n(t) := \langle v(t),\phi_n\rangle,$$

we obtain

$$\begin{cases} \frac{\partial^2 v_n(t)}{\partial t_1 \partial t_2} + \lambda_n v_n(t) = p_n, & t = (t_1,t_2) \in]0,T_1]\times]0,T_2], \\ v_n(t_1,0) = 0, & t_1 \in [0,T_1], \\ v_n(0,t_2) = 0, & t_2 \in [0,T_2]. \end{cases} \tag{3.2}$$

We recall here the following results.

Let $D = (0,T_1)\times(0,T_2)$ be a bounded rectangle in the plane $\mathbb{R}^2$ with coordinates $t = (t_1,t_2) \in D$. We consider the equation

$$\begin{cases} \dfrac{\partial^2 U(t)}{\partial t_1 \partial t_2} + AU(t) = f(t), & t \in D, \\ U(t_1,0) = \varphi(t_1), & t_1 \in [0,T_1], \\ U(0,t_2) = \psi(t_2), & t_2 \in [0,T_2], \end{cases} \tag{3.3}$$

where U and f are H-valued functions on D, φ (resp. ψ) is H-valued function on $[0,T_1]$ (resp. $[0,T_2]$), and

$$\varphi(0) = \psi(0) = \eta.$$

It was shown (see [7], [11, Theorem 4.1, Formula (4.10)], and [29, Formula 6]) that under natural conditions on φ, ψ, and f with the help of the Riemann function, the Goursat problem (3.3) is well-posed and its solution is given by

$$\begin{aligned} U(t_1,t_2) =& R(A;t_1,t_2)\eta + \int_0^{t_1} R(A;t_1-s_1,t_2)\varphi'(s_1)\,ds_1 \\ &+ \int_0^{t_2} R(A;t_1,t_2-s_2)\psi'(s_2)\,ds_2 \\ &+ \int_0^{t_2}\int_0^{t_1} R(A;t_1-s_1,t_2-s_2)f(s_1,s_2)\,ds_1 ds_2, \end{aligned} \tag{3.4}$$

where $R(A;t_1,t_2) = J_0\left(2\sqrt{t_1t_2A}\right) \in \mathcal{L}(H)$ is the *Riemann function* defined in terms of its spectral representation:

$$R(A;t_1,t_2)h = \sum_{n=1}^{+\infty} R(\lambda_n;t_1,t_2)\langle h,\phi_n\rangle\phi_n, \quad h = \sum_{n=1}^{+\infty}\langle h,\phi_n\rangle\phi_n \in H.$$

By virtue of properties (2.7), (2.8), (2.15), and (3.4), we deduce that the exact solution of (3.2) is given by

$$v_n(t) = \lambda_n^{-1}(1-R(\lambda_n;t_1,t_2)p_n = \lambda_n^{-1}(1-J_0(2\sqrt{\lambda_n}\sqrt{t_1t_2}))p_n. \tag{3.5}$$

By using properties of the Bessel function $J_0(\cdot)$ (Lemma 2.5(1), (2) and (3)), we have

$$\|R(A;t_1,t_2)\| = \sup_{n\geq 1}|R(\lambda_n;t_1,t_2)| \leq 1, \tag{3.6}$$

$$\|R(A;t_1=0,t_2)\| = \|R(A;t_1,t_2=0)\| = \|R(A;t_1=0,t_2=0)\| = 1, \tag{3.7}$$

$$\|R(A;t_1,t_2)\| < 1, \quad \text{for all } t_1>0, t_2>0, \tag{3.8}$$

$$\|I-R(A;t_1,t_2)\| = \sup_{n\geq 1}|1-R(\lambda_n;t_1,t_2)| \leq 2, \tag{3.9}$$

$$\|F(A;t_1,t_2)\| = \|A^{-1}(I-R(A;t_1,t_2))\| = \sup_{n\geq 1}|\lambda_n^{-1}(1-R(\lambda_n;t_1,t_2))| \leq \frac{2}{\lambda_1}. \tag{3.10}$$

Now we are in position to state our main results.

Theorem 3.1. *For any $p \in H$ (or H^{-1}), Problem (3.2) admits a unique solution given by*

$$v(t_1,t_2) = F(A;t_1,t_2) = A^{-1}(I-R(A;t_1,t_2))p = \sum_{n=1}^{+\infty}\lambda_n^{-1}(1-R(\lambda_n;t_1,t_2))\langle p,\phi_n\rangle\phi_n. \tag{3.11}$$

Moreover, we have the following stability estimates:

$$\sup_{t=(t_1,t_2)\in\overline{Q}}\|v(t)\| \leq \frac{2}{\lambda_1}\|p\|, \tag{3.12}$$

$$\sup_{t=(t_1,t_2)\in\overline{Q}} \|v(t)\| \leq 2\|p\|_{H^{-1}}, \tag{3.13}$$

where $Q =]0, T_1[\times]0, T_2[$ *and* $\overline{Q} = [0, T_1] \times [0, T_2]$.

Remark 3.2. By virtue of (2.9)–(2.15), the function $F(A; t_1, t_2) = A^{-1}(I - R(A; t_1, t_2))$ is strongly continuous, and for all $p \in H$, the function $v(t) = F(A; t_1, t_2)p$ satisfies the following regularity properties:

(1) for all $t \in \overline{Q},\ v(t) \in H^1,\ v(t) \in C(\overline{Q}, H^1)$;
(2) $\frac{\partial v(t)}{\partial t_i} \in C(]0, T1]\times]0, T_2], H^{\frac{1}{2}}) \cap C(\overline{Q}, H),\ i = 1, 2$;
(3) $\frac{\partial^2 v(t)}{\partial t_1 \partial t_2} \in C(\overline{Q}, H)$.

3.2. **The inverse problem.** Using the internal condition $u(a, b) = F(A; a, b)p = g$, we conclude that our inverse problem (2.1)–(2.2) is equivalent to the operator equation

$$F(A; a, b)p = g. \tag{3.14}$$

The theoretical analysis of problem (3.14) is essentially based on the characteristic function $F(\lambda; a, b)$ and the generalized Picard theorem 2.4. We easily check that

$$\text{for all } \lambda \geq \lambda_1, \quad 0 < F(\lambda; a, b) = \frac{1 - J_0(2\sqrt{\lambda}\sqrt{ab})}{\lambda} \leq \frac{2}{\lambda_1} \tag{3.15}$$

and

$$\lim_{\lambda\longrightarrow+\infty} F(\lambda; a, b) = 0, \tag{3.16}$$

which implies that $F(A; a, b)$ is an injective, compact positive self-adjoint operator, and its inverse $F(A; a, b)^{-1}$ is unbounded operator given by

$$F(A; a, b)^{-1}g = \sum_{n=1}^{+\infty} \frac{\lambda_n}{1 - J_0(2\sqrt{\lambda_n}\sqrt{ab})} \langle g, \phi_n\rangle\phi_n. \tag{3.17}$$

This series is convergent if and only if

$$\|F(A; a, b)^{-1}g\|^2 = \sum_{n=1}^{+\infty} \frac{\lambda_n^2}{(1 - J_0(2\sqrt{\lambda_n}\sqrt{ab}))^2} |\langle g, \phi_n\rangle|^2 < +\infty.$$

By a simple calculation, $\mu_n = \frac{\lambda_n}{1-J_0(2\sqrt{\lambda_n}\sqrt{ab})}$ can be estimated as follows:

$$\frac{\lambda_n}{2} \leq \mu_n \leq \frac{\lambda_n}{1 - J_0(2\sqrt{\lambda^*}\sqrt{ab})}, \tag{3.18}$$

where at λ^*, the function $J_0(2\sqrt{\lambda}\sqrt{ab})$ achieves its maximum positive value. From (3.18) and by virtue of the generalized Picard theorem, we deduce that the inverse problem (3.14) is correctly solvable if and only if $g \in H^1$, that is,

$$\sum_{n=1}^{+\infty} \lambda_n^2 |\langle g, \phi_n\rangle|^2 < +\infty. \tag{3.19}$$

Theorem 3.3. *For all $g \in H^1$, problem (3.14) admits a unique solution given by*

$$p = F(A; a, b)^{-1} g = \sum_{n=1}^{+\infty} \frac{\lambda_n}{1 - J_0(2\sqrt{\lambda_n}\sqrt{ab})} \langle g, \phi_n \rangle \phi_n. \tag{3.20}$$

Theorem 3.4. *For all $g \in H^1$, problem (2.1)–(2.2) admits a unique solution given by*

$$u(t) = F(A; t_1, t_2) p = \sum_{n=1}^{+\infty} \frac{1 - J_0(2\sqrt{\lambda_n}\sqrt{t_1 t_2})}{1 - J_0(2\sqrt{\lambda_n}\sqrt{ab})} \langle g, \phi_n \rangle \phi_n. \tag{3.21}$$

Remark 3.5. From (3.20) we see that p is unstable. This follows from the high-frequency

$$\mu_n = \frac{\lambda_n}{1 - J_0(2\sqrt{\lambda_n}\sqrt{ab})} \longrightarrow +\infty, \quad n \longrightarrow +\infty.$$

4. Iterative regularization and error estimates

Description of the method. The iterative algorithm for solving the ill-posed problem (3.14) (resp. (2.1)–(2.2)) starts by letting $p^0 \in H$ be arbitrary. The first approximation $u^0(t)$ is the solution to the direct (well-posed) problem

$$\begin{cases} \frac{\partial^2 u^0(t)}{\partial t_1 \partial t_2} + A u^0(t) = p^0, & t = (t_1, t_2) \in]0, T_1] \times]0, T_2], \\ u^0(t_1, 0) = 0, & t_1 \in [0, T_1], \\ u^0(0, t_2) = 0, & t_2 \in [0, T_2]. \end{cases} \tag{4.1}$$

We solve (4.1) and find

$$u^0(t_1, t_2) = F(A; t_1, t_2) p^0 = \sum_{n=1}^{+\infty} \lambda_n^{-1} (1 - R(\lambda_n; t_1, t_2)) \langle p^0, \phi_n \rangle \phi_n. \tag{4.2}$$

The second approximation $(u^1(t), p^1)$ is obtained by solving

$$\begin{cases} \frac{\partial^2 u^1(t)}{\partial t_1 \partial t_2} + A u^1(t) = p^1, & t = (t_1, t_2) \in]0, T_1] \times]0, T_2], \\ u^1(t_1, 0) = 0, & t_1 \in [0, T_1], \\ u^1(0, t_2) = 0, & t_2 \in [0, T_2], \end{cases} \tag{4.3}$$

where

$$p^1 = p^0 - \omega \Big(u^0(a, b) - g \Big), \tag{4.4}$$

$$0 < \omega < \omega^* = \frac{1}{||F(A; a, b)||},$$

and

$$||F(A; a, b)|| = \sup_{n \geq 1} \frac{1 - J_0(2\sqrt{\lambda_n}\sqrt{ab})}{\lambda_n} \leq \frac{2}{\lambda_1}. \tag{4.5}$$

We solve (4.3) and find

$$u^1(t_1,t_2) = F(A;t_1,t_2)p^1 = \sum_{n=1}^{+\infty} \lambda_n^{-1}(1-R(\lambda_n;t_1,t_2))\langle p^1,\phi_n\rangle\phi_n. \tag{4.6}$$

Finally, we get u^{k+1} by solving the problem

$$\begin{cases} \frac{\partial^2 u^{k+1}(t)}{\partial t_1 \partial t_2} + Au^{k+1}(t) = p^{k+1}, & t=(t_1,t_2)\in]0,T_1]\times]0,T_2], \\ u^{k+1}(t_1,0) = 0, & t_1 \in [0,T_1], \\ u^{k+1}(0,t_2) = 0, & t_2 \in [0,T_2], \end{cases} \tag{4.7}$$

where

$$p^{k+1} = p^k - \omega\Big(u^k(a,b) - g\Big). \tag{4.8}$$

We note here that k plays the role of regularization parameter and ω is an accelerated factor for this iterative procedure.

We set $G = I - \omega F(A;a,b)$. If we iterate backwards in (4.8), then we obtain

$$p^k = G^k p^0 + (I - G^k)p,$$

$$p^k - p = G^k(p^0 - p),$$

$$u^k(t) - u(t) = F(A;t_1,t_2)G^k(p^0-p).$$

Proposition 4.1. *The operator $G = I - \omega F(A;a,b)$ is self-adjoint and nonexpansive on H (1 is not an eigenvalue of G). Moreover, let $k \in \mathbb{N}^*$. Then*

$$\left|\left|\sum_{i=0}^{k-1} G^i\right|\right| \leq \sum_{i=0}^{k-1} ||G^i|| \leq k.$$

Proof.
- The self-adjointness follows from the definition of G, since we have the inequality

  $$0 < G(\lambda) < 1, \quad \text{for all } \lambda \geq \lambda_1,$$

  which implies that $\sigma_p(G) \subset]0,1[$. Consequently 1 it is not an eigenvalue of G.
- Since $\|G\| \leq 1$, it follows immediately that $\| \sum_{i=0}^{k-1} G^i \| \leq k$.

□

We provide the following lemma, which will be used in the proof of the convergence estimates.

Lemma 4.2 ([37, 42])**.** *For $0 < \mu < 1$, define*

$$p_k(\mu) = \sum_{i=0}^{k-1}(1-\mu)^i,$$

and

$$r_k(\mu) = 1 - p_k(\mu) = (1-\mu)^k,$$

there hold

$$\begin{array}{lll} p_k(\mu)\mu^\theta & \leq & k^{1-\theta}, \qquad 0 \leq \theta \leq 1, \\ r_k(\mu)\mu^\theta & \leq & \kappa_\theta (k+1)^{-\theta}, \end{array} \tag{4.9}$$

where

$$\kappa_\theta = \begin{cases} 1, & 0 \le \theta \le 1, \\ \theta^\theta, & \theta > 1. \end{cases}$$

Now we are in a position to state the main result of this method.

Theorem 4.3. *Let p^0 be an arbitrary element for the iterative procedure suggested above, and let u^k be the kth approximate solution. Then we have*

$$\sup_{t\in\overline{Q}} ||u(t) - u^k(t)|| \longrightarrow 0, k \longrightarrow +\infty. \tag{4.10}$$

Moreover, if $(p - p^0) \in H^\theta$, $\theta > 0$, then the rate of convergence of the method is given by

$$\sup_{t\in\overline{Q}} ||u(t) - u^k(t)|| \le \frac{\tau}{(1+k)^\theta}, \tag{4.11}$$

where $\tau = \kappa_\theta \left(\frac{1}{\omega}\right)^\theta \left(\frac{1}{M}\right)^\theta$.

Proof. (i) It follows immediately from Theorem 2.3 and (3.15) that

$$\begin{aligned} ||u(t) - u^k(t)|| &= ||F(A;t_1,t_2)G^k(p^0 - p)|| \\ &\le ||F(A;t_1,t_2)||\,||G^k(p^0 - p)|| \\ &\le \tfrac{2}{\lambda_1}||G^k(p^0 - p)|| \longrightarrow 0, \quad k \longrightarrow +\infty. \end{aligned} \tag{4.12}$$

(ii) For notational convenience and simplicity, we denote

$$0 < \mu_n = 1 - \omega F(\lambda_n; a, b) < 1 \tag{4.13}$$

and

$$M = \sup_{\lambda \ge \lambda_1} \frac{1}{1 - J_0(2\sqrt{\lambda}\sqrt{ab})}. \tag{4.14}$$

We have

$$\begin{aligned} ||u(t) - u^k(t)||^2 &= ||F(A;t_1,t_2)G^k(p^0 - p)||^2 \\ &\le \left(\tfrac{2}{\lambda_1}\right)^2 \sum_{n=1}^{+\infty} (1 - \omega F(\lambda_n, a, b))^{2k} \left|(p - p^0)_n\right|^2 \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{2}{\lambda_1}\right)^2 \sum_{n=1}^{+\infty} (1 - \omega F(\lambda_n, a, b))^{2k} \left|(p - p^0)_n\right|^2 \\ &= \left(\frac{2}{\lambda_1}\right)^2 \sum_{n=1}^{+\infty} (1 - \omega F(\lambda_n, a, b))^{2k} (\omega F(\lambda_n, a, b))^{2\theta} (\omega F(\lambda_n, a, b))^{-2\theta} \left|(p - p^0)_n\right|^2. \end{aligned}$$

By using (4.9) and (4.14), we derive the following estimates

$$(1 - \omega F(\lambda_n, a, b))^{2k} (\omega F(\lambda_n, a, b))^{2\theta} \le \left(\kappa_\theta \frac{1}{(k+1)^\theta}\right)^2, \tag{4.15}$$

$$\sum_{n=1}^{+\infty}(\omega F(\lambda_n,a,b))^{-2\theta}\left|(p-p^0)_n\right|^2 \leq \left(\frac{1}{\omega}\right)^{2\theta}\left(\frac{1}{M}\right)^{2\theta}\sum_{n=1}^{+\infty}\lambda_n^{2\theta}\left|(p-p^0)_n\right|^2. \quad (4.16)$$

Combining (4.15) and (4.15), we obtain the desired estimate. □

Remark 4.4. Let $u_\delta^k(t)$ be the kth approximate solution associated to inexact data g^δ such that $\|g-g^\delta\| \leq \delta$. Then we have

$$\|u(t)-u_\delta^k(t)\| \leq \triangle_1+\triangle_2,$$

where

$$\triangle_1 = \|u(t)-u^k(t)\| \leq \frac{\tau}{(1+k)^\theta}$$

and

$$\triangle_2 = \|u^k(t)-u_\delta^k(t)\| = \|\omega F(A;t_1,t_2)\sum_{i=0}^{k-1}(g-g^\delta)\| \leq \omega\frac{2}{\lambda_1}k\delta.$$

If we choose $k=k(\delta)$ such that $\omega\frac{2}{\lambda_1}k\delta \longrightarrow 0$ as $\delta \longrightarrow 0$, then the total error estimate is given by

$$\sup_{t\in\overline{Q}}\|u(t)-u_\delta^k(t)\| \leq \frac{\tau}{(1+k)^\theta}+\omega\frac{2}{\lambda_1}k\delta. \quad (4.17)$$

Remark 4.5. To speed up the proposed iterative method, we use a preconditioning variant of this method [10, 33], which is described as

$$p_{k+1} = p_k - \omega S\Big(u^k(a,b)-g\Big), \quad (4.18)$$

where $S=A^{-r}$ is the preconditioner, and $r \geq 0$. The relaxation parameter ω is chosen such that

$$0<\omega<\omega^* = \frac{1}{\|SF(A;a,b)\|}.$$

5. Numerical tests

In this section, we give a three-dimensional numerical test to show the feasibility and efficiency of the proposed methods. Numerical experiments were carried out by using MATLAB. We consider the following inverse problem:

$$\begin{cases} u_{t_1t_2}(x,t)-u_{xx}(x,t)=p(x), & x\in(0,\pi),\ t=(t_1,t_2)\in]0,2[\times]0,2[,\\ u(0,t_1,t_2)=u(\pi,t_1,t_2)=0, & (t_1,t_2)\in\overline{Q}=[0,2]\times[0,2],\\ u(x,0,t_2)=0, & t_2\in[0,2],\\ u(x,t_1,0)=0, & t_1\in[0,2],\\ u(x,1,1)=g(x), & x\in[0,\pi], \end{cases} \quad (5.1)$$

where $p(x)$ is the unknown source and $u(x,1,1)=g(x)$ is the supplementary condition. We know that

$$A=-\frac{\partial^2}{\partial x^2}, \quad \mathcal{D}(A)=H_0^1(0,\pi)\cap H^2(0,\pi)\subset H=L^2(0,\pi),$$

is positive, self-adjoint with compact resolvent (A is diagonalizable). The eigenpairs (λ_n, ϕ_n) of A are

$$\lambda_n = n^2, \quad \phi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), \quad n \in \mathbb{N}^*.$$

In this case, formula (3.20) takes the form

$$p(x) = \frac{2}{\pi} \sum_{k=1}^{+\infty} \frac{k^2}{1 - J_0(2k)} \left(\int_0^{\pi} g(x) \sin(kx) dx \right) \sin(kx). \tag{5.2}$$

In the following, we consider an example that has an exact expression of solutions $(u(x, t_1, t_2), p(x))$:

$$g(x) = u(x, 1, 1) = \sqrt{\frac{2}{\pi}} (1 - J_0(2)) \sin(x) + \sqrt{\frac{2}{\pi}} \frac{1}{4} (1 - J_0(4)) \sin(2x),$$

$$p(x) = \sqrt{\frac{2}{\pi}} \sin(x) + \sqrt{\frac{2}{\pi}} \sin(2x).$$

Adding a random distributed perturbation (obtained by the MATLAB command `randn`) to each data function, we obtain the vector g^δ:

$$g^\delta = g + \varepsilon \mathrm{randn}(\mathrm{size}(g)),$$

where ε indicates the noise level of the measurement data and the function "randn(.)" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$. "randn(size(g))" returns an array of random entries that is the same size as g. The bound on the measurement error δ can be measured in the sense of root mean square error according to

$$\delta = \|g^\delta - g\|_* = \left(\frac{1}{M+1} \sum_{i=1}^{M+1} \left(g(x_i) - g^\delta(x_i) \right)^2 \right)^{1/2}.$$

The relative error $RE(p)$ is given by

$$RE(p) = \frac{\|p_k^\delta - p\|_*}{\|p\|_*}. \tag{5.3}$$

Iteration method. By using the central difference with step length $h = \frac{\pi}{N+1}$ to approximate the first derivative u_x and the second derivative u_{xx}, we can get the following semi-discrete problem:

$$\begin{cases} u_{t_1 t_2}(x_i, t) - \mathbb{A}_h(x_i, t) = 0, & x_i = ih, i = 1, \ldots, N, \\ & t = (t_1, t_2) \in]0, 2[\times]0, 2[, \\ u(x_0 = 0, t_1, t_2) = u(x_{N+1} = \pi, t_1, t_2) = 0, & (t_1, t_2) \in [0, 2] \times [0, 2], \\ u(x_i, 0, t_2) = 0 = u(x_i, t_1, 0), & i = 1, \ldots, N, (t_1, t_2) \in [0, 2] \times [0, 2], \\ u(x_i, 1, 1) = g^\delta(x_i), & i = 1, \ldots, N, \end{cases} \tag{5.4}$$

where $\mathbb{A}_h$ is the discretization matrix stemming from the operator $A = -\frac{d^2}{dx^2}$:

$$\mathbb{A}_h = \frac{1}{h^2}\text{Tridiag}(-1, 2, -1) \in \mathcal{M}_N(\mathbb{R})$$

is a symmetric, positive definite matrix. We assume that it is fine enough so that the discretization errors are small compared to the uncertainty δ of the data; this means that $\mathbb{A}_h$ is a good approximation of the differential operator $A = -\frac{d^2}{dx^2}$, whose unboundedness is reflected in a large norm of $\mathbb{A}_h$. The eigenpairs (μ_k, e_k) of $\mathbb{A}_h$ are given by

$$\mu_k = 4\left(\frac{N+1}{\pi}\right)^2 \sin^2\left(\frac{k\pi}{2(N+1)}\right), \quad e_k = \left(\sin\left(\frac{jk\pi}{N+1}\right)\right)_{j=1}^{N}, \quad k = 1, \dots, N.$$

The discrete iterative approximation of (5.4) takes the form

$$p_k^\delta(x_j) = \left(I - \omega A_h^{-r} F(A_h;1,1)\right)^k p_0(x_j) + \omega \sum_{i=0}^{k-1} \left(I - \omega A_h^{-r} F(A_h;1,1)\right)^i A_h^{-r} g^\delta(x_j), \tag{5.5}$$

where $j = 1, \dots, N = 1000$, $r \geq 0$, and $F(A_h; 1, 1) = (I - J_0(2\sqrt{A_h}))A_h^{-1} \in \mathcal{M}_N(\mathbb{R})$.

5.1. **Tables.**

TABLE 1. Basic KM iteration method: The relative errors RE for fixed ω, r and for various values of δ

N	k	ε	ω	r	RE
1000	10	0.001	1.1590	0	0.0178
1000	10	0.01	1.1590	0	0.1266
1000	10	0.1	1.1590	0	1.2728
1000	4	0.1	1.1590	0	0.4558

TABLE 2. Preconditioning KM iteration method: The relative errors RE for fixed ω, r, δ and for various values of k

N	k	ε	ω	r	RE
1000	40	0.1	1.1567	1	0.0142
1000	35	0.1	1.1567	1	0.0128

5.2. **Conclusion and discussion.** The numerical results (Figures 5–6) are quite satisfactory. Even with the aggressive noise level $\delta = 0.1$, the numerical solutions are still in good agreement with the exact solution. In the case of KM iteration without preconditioning (Figures 1–4), the method is sensitive, and the numerical results obtained are far from the exact solutions, but for a low noise ($\delta = 0.001$), they can be improved for certain optimal choices of the parameters involved in the method.

In this study, a convergent and stable reconstruction of an unknown right-hand side has been obtained using an iterative regularization method. Both theoretical and numerical studies have been provided.

5.3. **Generalization.** By using Mittag–Leffler and Wright-type functions, this study can be extended to the fractional case, that is, we can generalize the obtained results to the following fractional Goursat problem:

The inverse problem here is to determine the unknown source term $p \in H$ and the function $u(t)$ for $t = (t_1, t_2) \in]0, T_1] \times]0, T_2]$ from the additional data

$$u(a, b) = g, \quad (a, b) \in]0, T_1[\times]0, T_2[, \tag{5.6}$$

and the system of equations

$$\begin{cases} D^{\alpha}_{t_1} D^{\beta}_{t_2} u + Au(t) = p, & t = (t_1, t_2) \in]0, T_1] \times]0, T_2], \\ u(t_1, 0) = 0, & t_1 \in [0, T_1], \\ u(0, t_2) = 0, & t_2 \in [0, T_2], \end{cases} \tag{5.7}$$

where $0 < T_1, T_2 < \infty$, g is a given H-valued function, $0 < \alpha < 1$, $0 < \beta < 1$, and the notation D^{θ}_{y} means the Djrbashian–Caputo derivative operator of order $\theta \in (0, 1)$ for differentiable function, defined by

$$D^{\alpha}_{y} u(y) = \frac{1}{\Gamma(1-\theta)} \int_0^y (y-s)^{-\theta} u'_y(s) ds, \qquad 0 < \theta < 1.$$

5.4. **Figures.**

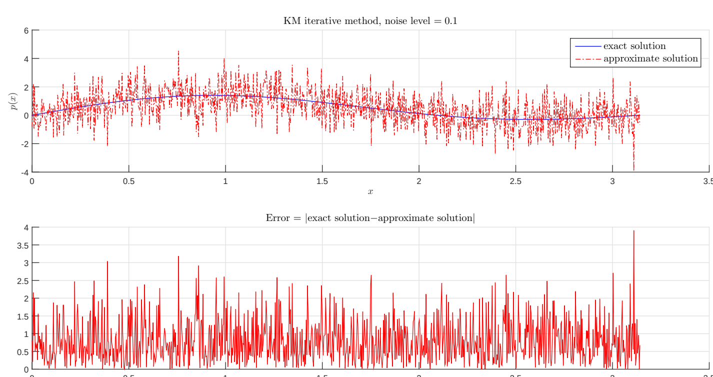


FIGURE 1. Basic KM iteration method: $\delta = 0.1$ (noise level), $k = 10$ (iteration number), $r = 0$ (preconditioner parameter).

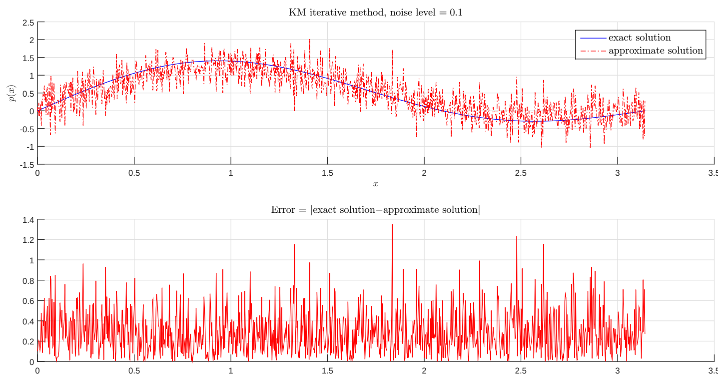


FIGURE 2. Basic KM iteration method: $\delta = 0.1$ (noise level), $k = 4$ (iteration number), $r = 0$ (preconditioner parameter).

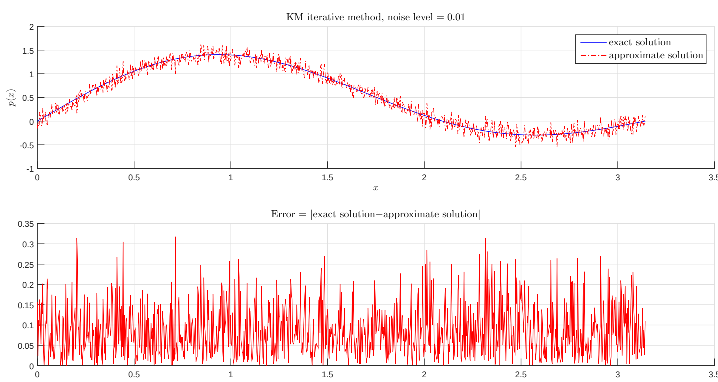


FIGURE 3. Basic KM iteration method: $\delta = 0.01$ (noise level), $k = 10$ (iteration number), $r = 0$ (preconditioner parameter).

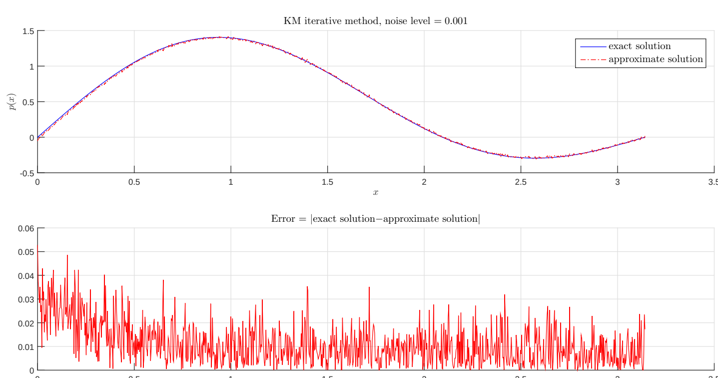


FIGURE 4. Basic KM iteration method: $\delta = 0.001$ (noise level), $k = 4$ (iteration number), $r = 0$ (preconditioner parameter).

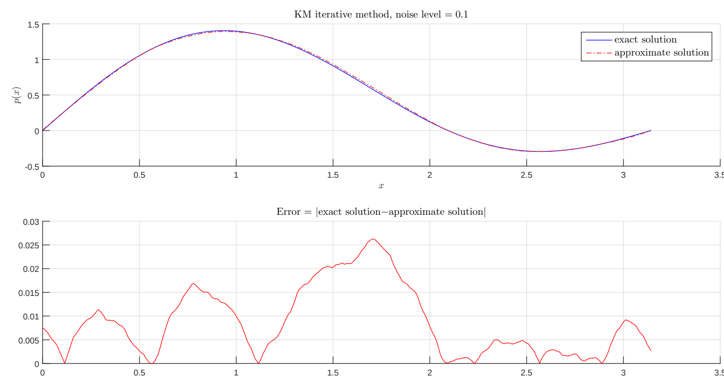
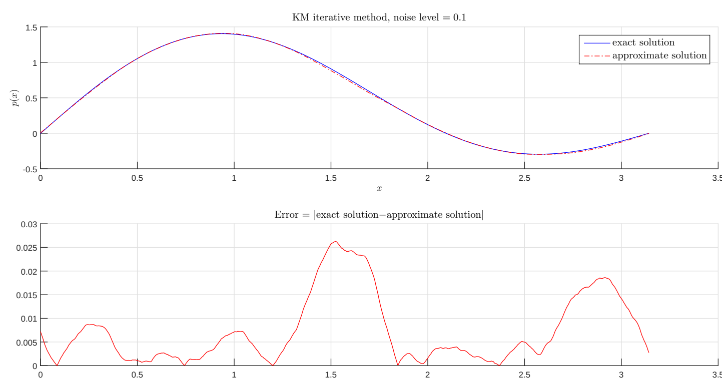


FIGURE 5. Preconditioning KM iteration method: $\delta = 0.1$ (noise level), $k = 35$ (iteration number), $r = 1$ (preconditioner parameter).



h!

FIGURE 6. Preconditioning KM iteration method: $\delta = 0.1$ (noise level), $k = 40$ (iteration number), $r = 1$ (preconditioner parameter).

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