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METALLIC STRUCTURES ON THE TANGENT BUNDLE OF P-SASAKIAN MANIFOLDS

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ABSTRACT. In this article, we introduce some metallic structures on the tangent bundle of a P-Sasakian manifold by the complete lift, horizontal lift, and vertical lift of a P-Sasakian structure (ϕ, η, ξ) on a tangent bundle. Then we investigate the integrability and parallelity of these metallic structures.

1. INTRODUCTION

The lift of geometrical objects, vector fields, and forms, has an important role in differential geometry. By the method of lift, we can generalize the differentiable structure on any manifold to its tangent bundle and any other bundles on manifold; see [7, 8, 10]. In this article, we study the metallic structures on the tangent bundle of a P-Sasakian Riemannian manifold. The metallic structure is a generalization of the almost product structure. A *metallic structure* is a polynomial structure as defined by Goldenberg et al. [2, 3]. In [5, 6], the authors introduced the notation of metallic structure on a Riemannian manifold. Suppose that p and q are two positive integers. The positive solution of the equation $x^2 - px - q = 0$, among other common characteristics, is a member of the metallic means family. This number is shown by $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$, where it is a generalization of golden proportions.

Definition 1.1. Let M be a manifold. A metallic structure on M is a $(1, 1)$ tensor field J that satisfies the equation $J^2 = pJ + qI$, where p and q are positive integers and I is the identity operator on the Lie algebra $\mathcal{X}(M)$ of vector fields

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on M. If g is a Riemannian metric on M, then we say that g is J-compatible whenever

$$g(JX, Y) = g(X, JY), \quad \text{for all } X, Y \in \mathcal{X}(M),$$

or equivalently

$$g(JX, JY) = pg(X, JY) + qg(X, Y), \quad \text{for all } X, Y \in \mathcal{X}(M).$$

In this case, (M, J, g) is named a metallic Riemannian manifold (see [6]).

Let J be a metallic structure on M. Then the Nijenhuise tensor N_J of J is a tensor field of type $(1, 2)$ and given by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y],$$

for $X, Y \in \mathcal{X}(M)$.

On the other hand, at first time, Sato in [9] introduced the P-Sasakian structure on manifolds and studied several properties of these manifolds. An n-dimensional smooth manifold M is called an *almost paracontact manifold* if it admits an almost paracontact structure (ϕ, η, ξ), consisting of a $(1, 1)$ tensor field ϕ, a 1-form η, and a vector field ξ that satisfy the conditions

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0.$$

Let g be a Riemannian metric compatible with (ϕ, η, ξ), that is,

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \quad \text{for all } X, Y \in \mathcal{X}(M),$$

or equivalently

$$g(X, \phi Y) = g(\phi X, Y), \quad g(X, \xi) = \eta(X), \quad \text{for all } X, Y \in \mathcal{X}(M),$$

where $\mathcal{X}(M)$ is the collection of all smooth vector fields on M. Then, M is said to be an *almost paracontact* Riemannian manifold.

An almost paracontact Riemannian manifold (M, g) is called a *P-Sasakian manifold* if it satisfies

$$(\nabla_X \phi)(Y) = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{1.1}$$

where ∇ is the Levi-Civita connection of the Riemannain manifold. We have

$$\nabla_X \xi = \phi X, \quad (\nabla_X \eta)(Y) = g(\phi X, Y) = (\nabla_Y \eta)(X), \quad \text{for all } X, Y \in \mathcal{X}(M).$$

1.1. **Lifts of geometric structure on tangent bundle.** Let (M, g) be a smooth n-dimensional Riemannian manifold, and let TM denote its tangent bundle. We denote the natural projection by $\pi : TM \to M$, where it defines the natural bundle structure of TM over M and denote the set of all tensor fields of the type (k, l) in M by $T^k_l(M)$. For any point $(x, y) \in TM$, let $V_y = \ker\{\pi_*(y) : T_y(TM) \to T_x M\}$ and $VTM = \cup_{y \in TM} V_y$. Also, suppose that HTM is a complement of VTM in TM, that is,

$$TTM = VTM \oplus HTM.$$

Also, VTM and HTM are called *vertical distribution* and *horizontal distribution*, respectively. Suppose that the space M is covered by a system of coordinate neighborhood $(U, \varphi) = (U, x^1, x^2, \ldots, x^n)$. Then the corresponding induced local chart on TM is $(\pi^{-1}(U), x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n)$. If in any point of $x \in M$,

$\Gamma^h_{ki}(x)$ is the Christoffel symbols of g, then the sets of vector fields $\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\}$ and $\{\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}\}$ on $\pi^{-1}(U)$ define local frame fields for VTM and HTM, respectively, where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^k \Gamma^l_{ki} \frac{\partial}{\partial y^l}$. Note that the set $\{\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\}$ defines a local frame on TM. In the following, we recall from [7, 10] some lifts of geometrical objects of a manifold to its tangent bundle.

1.1.1. *Vertical lifts.* Let f be a function on M. Then the vertical lift of f to TM is the function f^v on TM given by $f^v = f \circ \pi$. For any vector field $X \in \mathcal{X}(M)$, we define a vector field X^v in TM by $X^v(\iota\omega) = (\omega(X))^v$, where ω is an arbitrary 1-form in M, so we call X^v the vertical lift of X. Note, $X^v \in VTM$, and for all function on M, we define $X^v(df) = X.f$. Let F be a tensor field of type $(1, r)$ or $(0, r)$, $r \geq 1$, on M. Then, the vertical lift of F on TM is defined by

$$F^v_y(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_r) = \Big(F_y\big(\pi_\star(\tilde{X}_1), \pi_\star(\tilde{X}_2), \dots, \pi_\star(\tilde{X}_r)\big)\Big)^v,$$

where $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_r \in T_y(TM)$, $y \in T_xM$, $x \in M$. Hence, for any $X_1, \dots, X_r \in \mathcal{X}(M)$, we have

$$F^v(X^v_1, X^v_2, \dots, X^v_r) = 0, \ F^v(X^c_1, X^c_2, \dots, X^c_r) = \Big(F(X_1, X_2, \dots, X_r)\Big)^v.$$

1.1.2. *Complete lifts.* If f is a function on M, then the complete lift of f is the function f^c on TM and defined by

$$f^c(x, y) = df(x)(y), \ \ y \in T_xM, \ x \in M.$$

Also, the complete lift of a vector field $X = X^i \frac{\partial}{\partial x^i}$ on M is defined by

$$X^c = X^i \frac{\partial}{\partial x^i} - y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial y^i}.$$

Therefore, we obtain $(\frac{\partial}{\partial x^i})^c = \frac{\partial}{\partial x^i}$ and $X^c f^c = (Xf)^c$ for any function f on M. Suppose that ω is a 1-form on M. The complete lift of ω on TM is defined by $\omega^c(X^c) = (\omega(X))^c$, $\omega^c(X^v) = (\omega(X))^v$, for each $X \in \mathcal{X}(M)$. In general case, the complete lift of a tensor field F of type $(1, r)$ or $(0, r)$, $r \geq 1$, on M is defined by $F^c(X^c_1, \dots, X^c_r) = (F(X_1, \dots, X_r))^c$ for any $X_1, \dots, X_r \in \mathcal{X}(M)$. Then the complete lift of a Riemannian metric g is defined by

$$g^c = \begin{pmatrix} y^k \frac{\partial g_{ij}}{\partial x^k} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}.$$

From [7] and [10], we have the following result.

Proposition 1.2. *Let M be a manifold with a Riemannain metric g. For any $X, Y \in \mathcal{X}(M)$, $f \in C^\infty(M)$, and $(1,1)$ tensor field F, we have*

- $X^v f^v = 0, \ X^v f^c = X^c f^v = (Xf)^v, \ X^c f^c = (Xf)^c,$
- $F^c(X^v) = (F(X))^v,$
- *g^c is a semi-Riemannian metric and*

$$g^c(X^v, Y^c) = g^c(X^c, Y^v) = (g(X, Y))^v, \ g^c(X^v, Y^v) = 0,$$
$$g^c(X^c, Y^c) = (g(X, Y))^c,$$

- *if $P(x)$ is a polynomial in one variable x, then $P(F^c) = (P(F))^c$.*

We define the complete lift of a linear connection ∇ to TM as the unique linear connection ∇^c on TM as $\nabla^c_{X^c}Y^c = (\nabla_X Y)^c$ for $X, Y \in \mathcal{X}(M)$. Therefore

$$\begin{aligned} \nabla^c_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} &= \Gamma^k_{ij}\frac{\partial}{\partial x^k} + y^l \frac{\partial \Gamma^k_{ij}}{\partial x^l}\frac{\partial}{\partial y^k},\ \nabla^c_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 0, \\ \nabla^c_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} &= \Gamma^k_{ij}\frac{\partial}{\partial y^k},\ \nabla^c_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}\frac{\partial}{\partial y^k}. \end{aligned}$$

Proposition 1.3 (see [7, 10])**.** *Let T and R be the torsion and curvature tensors of ∇, respectively. Then T^c and R^c are the torsion and curvature tensors of ∇^c, respectively, and*

- *∇ is symmetric if and only if ∇^c is symmetric,*
- *∇ is flat if and only if ∇^c is flat.*

Proposition 1.4 (see [7, 10])**.** *Let F be a tensor field of type $(1, r)$ or $(0, r)$, $r \geq 1$, on M and let $X, Y \in \mathcal{X}(M)$. Then*

$$\begin{aligned} \nabla^c_{X^v}Y^v &= (\nabla_X Y)^v,\ \nabla^c_{X^c}Y^c = (\nabla_X Y)^c,\ \nabla^c_{X^v}Y^c = \nabla^c_{X^c}Y^v = (\nabla_X Y)^v, \\ \nabla^c F^v &= (\nabla F)^v,\ \nabla^c F^c = (\nabla F)^c. \end{aligned}$$

1.1.3. *Horizontal lifts.* The horizontal lift f^h of a function f on M is given by $f^h = f^c - \nabla_\gamma f$, where $\nabla_\gamma f = \gamma(\nabla f)$, and for any tensor field F of type $(1, r)$ or $(0, r)$, $r \geq 1$, on M, $\gamma_X F = (F_X)^v$ and $F_X(X_1, \ldots, F_{r-1}) = F(X_1, \ldots, X_{r-1}, X)$. For any vector field $X = X^i \frac{\partial}{\partial x^i}$ on M, there exists a unique vector $X^h \in HVM$ such that $\pi_* X^h = X$, that is, if $X = X^i \frac{\partial}{\partial x^i}$, then $X^h = X^i \frac{\partial}{\partial x^i} - y^j X^i \Gamma^l_{ij} \frac{\partial}{\partial y^l}$. We call X^h the horizontal lift of X in the point $(x, y) \in TM$. Let ω be a 1-form on M. Then the horizontal lift ω^h of ω is defined by $\omega^h = \omega^c - \nabla_\gamma \omega$. Then for any $X \in \mathcal{X}(M)$, we have $\omega^h(X^h) = 0$ and $\omega^h(X^v) = (\omega(X))^v$. The horizontal lift of a $(1, 1)$ tensor field F on M is defined by $F^h(X^h) = (FX)^h$ and $F^h(X^v) = (FX)^v$. The horizontal lifts of a Riemannian metric g is defined by $g^h = g_{ij}\theta^i \otimes \eta^j + g_{ij}\eta^i \otimes \theta^j$, where $\theta^i = dx^i$, $\eta^i = y^j \Gamma^i_{jk} dx^k + dy^i$.
From [7, 10], we have the following result.

Proposition 1.5. *Let M be a manifold with a Riemannain metric g. For any $X, Y \in \mathcal{X}(M)$, $f \in C^\infty(M)$, and $(1, 1)$ tensor field F, we have*

- *g^h is a semi-Riemannian metric and $g^h(X^v, Y^h) = (g(X, Y))^v$, $g^h(X^v, Y^v) = g^h(X^h, Y^h) = 0$,*
- *if $P(x)$ is a polynomial in one variable x, then $P(J^h) = (P(J))^h$,*
- *$g^h = g^c$ if and only if $\nabla g = 0$,*
- *$[X^v, Y^v] = 0$, $[X^v, Y^c] = [X, Y]^v$, $[X^c, Y^c] = [X, Y]^c$, $[X^v, Y^h] = -(\nabla_Y X)^v$, $[X^h, Y^h] = [X, Y]^h - \gamma R(X, Y)$, where R is a curvature tensor of g and the vertical vector lift γF is defined by $(\gamma F)(y) = (F(y))^v$.*

Let ∇ be a linear connection on M. Then we define the horizontal lift of ∇ to TM as the unique linear connection ∇^h on TM given by

$$\nabla^h_{X^v}Y^v = \nabla^h_{X^v}Y^h = 0,\ \nabla^h_{X^h}Y^v = (\nabla_X Y)^v,\ \nabla^h_{X^h}Y^h = (\nabla_X Y)^h,$$

for $X,Y \in \mathcal{X}(M)$. Hence $\nabla^h_{X^c}Y^c = (\nabla_X Y)^c - \gamma R(\cdot\,,X,Y)$, where $R(\cdot\,,X,Y)Z = R(Z,X,Y)$.

2. Metallic structures on the tangent bundle of a P-Sasakian manifold

In [1,4], the authors have studied some geometric structures on tangent bundle. Let M be an n-dimensional P-Sasakian manifold with structure tensor (ϕ,η,ξ,g). In this part of this section, we introduce a metallic structure induced on TM by the complete lift of a P-Sasakian structure, and then we show that this metallic structure is integrable.

Proposition 2.1. *On the tangent bundle of a P-Sasakian manifold with structure tensor (ϕ,η,ξ,g), there exists a metallic structure given by*

$$J = \frac{p}{2}I - (\frac{2\sigma_{p,q}-p}{2})(\phi^c + \eta^v \otimes \xi^v + \eta^c \otimes \xi^c). \tag{2.1}$$

Proof. From the definition of the almost paracontact structure of a P-Sasakian manifold, we obtain the following relations:

$$\begin{aligned}&(\phi^c)^2 = (\phi^2)^c = I - \eta^c \otimes \xi^v - \eta^v \otimes \xi^c,\\ &\eta^v(\xi^c) = \eta^c(\xi^v) = 1, \quad \eta^v(\xi^v) = \eta^c(\xi^c) = 0,\\ &\phi^c(\xi^v) = \phi^c(\xi^c) = 0, \quad \eta^v \circ \phi^c = \eta^c \circ \phi^c = 0.\end{aligned}$$

Therefore, using these relations for any $\tilde{X} \in \mathcal{X}(TM)$, we have

$$\begin{aligned}J(\xi^v) &= \frac{p}{2}I(\xi^v) - (\frac{2\sigma_{p,q}-p}{2})(\phi^c + \eta^v \otimes \xi^v + \eta^c \otimes \xi^c)(\xi^v)\\ &= \frac{p}{2}\xi^v - (\frac{2\sigma_{p,q}-p}{2})(\phi^c(\xi^v) + \eta^v(\xi^v)\xi^v + \eta^c(\xi^v)\xi^c)\\ &= \frac{p}{2}\xi^v - \frac{2\sigma_{p,q}-p}{2}\xi^c.\end{aligned}$$

Similarly, we get

$$\begin{aligned}J(\xi^c) &= \frac{p}{2}\xi^c - \frac{2\sigma_{p,q}-p}{2}\xi^v,\\ J(\phi^c\tilde{X}) &= \frac{p}{2}\phi^c\tilde{X} - \frac{2\sigma_{p,q}-p}{2}(\tilde{X} - \eta^c(\tilde{X})\xi^v - \eta^c(\tilde{X})\xi^c).\end{aligned}$$

Now, we obtain

$$J(\tilde{X}) = \frac{p}{2}\tilde{X} - \frac{2\sigma_{p,q}-p}{2}(\phi^c\tilde{X} + \eta^v(\tilde{X})\xi^v + \eta^c(\tilde{X})\xi^c)$$

and

$$J^2(\tilde{X}) = \frac{p}{2}J(\tilde{X}) - \frac{2\sigma_{p,q}-p}{2}(J(\phi^c\tilde{X}) + \eta^v(\tilde{X})J(\xi^v) + \eta^c(\tilde{X})J(\xi^c)) = pJ(\tilde{X}) + q\tilde{X},$$

which completes the proof. □

Proposition 2.2. *If M is a P-Sasakian manifold with structure tensor (ϕ,η,ξ,g) and J is defined by (2.1), then*

$$g^c(J\tilde{X}, J\tilde{Y}) = pg^c(\tilde{X}, J\tilde{Y}) + qg^c(\tilde{X},\tilde{Y}), \quad \textit{for all } \tilde{X},\tilde{Y} \in \mathcal{X}(TM).$$

Proof. For any $X, Y \in \mathcal{X}(M)$, we have

$$g^c(X^v, Y^v) = 0, \;\; g^c(X^v, \xi^c) = (g(X,\xi))^v = (\eta(X))^v,$$
$$g^c((\phi X)^v, \xi^c) = (g(\phi X, \xi))^v = (g(X, \phi\xi))^v = 0,$$
$$g^c(\xi^c, \xi^c) = (g(\xi,\xi))^c = 0.$$

Therefore,

$$g^c(JX^v, JY^v) = -\frac{(2\sigma_{p,q} - p)p}{2}(\eta X)^v(\eta Y)^v$$

and

$$g^c(X^v, JY^v) = -\frac{2\sigma_{p,q} - p}{2}(\eta X)^v(\eta Y)^v.$$

Thus,

$$g^c(JX^v, JY^v) = pg^c(X^v, JY^v) + qg^c(X^v, Y^v).$$

Also, using $g^c(X^v, Y^c) = (g(X,Y))^v$ and $g^c(X^c, Y^c) = (g(X,Y))^c$, we have

$$g^c(JX^v, JY^c) = (\frac{p^2}{2} + q)(g(X,Y))^v - \frac{(2\sigma_{p,q} - p)p}{2}[(g(X, \phi Y))^v - (\eta X)^v(\eta Y)^v]$$

and

$$g^c(X^v, JY^c) = \frac{p}{2}(g(X,Y))^v - \frac{(2\sigma_{p,q} - p)p}{2}[(g(X, \phi Y))^v - (\eta X)^v(\eta Y)^v].$$

Hence,

$$g^c(JX^v, JY^c) = pg^c(X^v, JY^c) + qg^c(X^v, Y^c).$$

The other cases are similar. □

Theorem 2.3. *Let M be a P-Sasakian manifold with structure tensor (ϕ, η, ξ, g) and let J be defined by (2.1). Then the metallic structure J is integrable.*

Proof. The 1-form η defines an $(n-1)$-dimensional distribution $\mathcal{D}$ by

$$\text{for all } p \in M, \quad \mathcal{D}_p = \{v \in T_pM : \eta(v) = 0\}, \tag{2.2}$$

and the complement of $\mathcal{D}$ is the 1-dimensional distribution spanned by ξ. Suppose that

$$N^1 = N_\phi - 2d\eta \otimes \xi, \;\; N^2(X,Y) = (L_{\phi X}\eta)Y - (L_{\phi Y}\eta)X, \;\; N^3 = L_\xi \phi, \;\; N^4 = L_\xi \eta.$$

By Proposition 1.5, for any X, Y of $C^\infty(M)$-module of all sections of distribution $\mathcal{D}$, we have

$$
\begin{aligned}
N_J(X^v, Y^v) &= 0,\\
N_J(X^v, Y^c) &= A\Big([N^1(X,Y)]^v + N^2(X,Y)\xi^c\Big),\\
N_J(X^c, Y^c) &= A\Big([N^1(X,Y)]^c + N^2(X,Y)\xi^v\Big),\\
N_J(X^v, \xi^v) &= A\Big(-(N^3(X))^v + N^4(X)\xi^c\Big),\\
N_J(X^v, \xi^c) &= A\Big([\phi(N^3(X)) - N^4(X)\xi]^v + N^2(X,\xi)\xi^c\Big),\\
N_J(X^c, \xi^v) &= A\Big(-(N^3(X))^c + (\phi(N^3(X)))^v - [N^4(\phi X) - N^4(X)]^c\xi^c\Big),\\
N_J(X^c, \xi^c) &= A\Big(-(N^3(X))^v + [N^4(X) + N^2(X,\xi)]\xi^c\\
&\quad + [\phi(N^3(X)) - N^4(X)\xi]^c\Big),\\
N_J(\xi^v, \xi^v) &= N_J(\xi^c, \xi^c) = N_J(\xi^v, \xi^c) = 0,
\end{aligned}
$$

where $A = (\frac{2\sigma_{p,q}-p}{2})^2$. Indeed the tensor N^1 of a P-Sasakian manifold vanishes. On the other hand, if $N^1 = 0$, then also N^2, N^3, and N^4 vanish. Hence $N_J(\tilde{X}, \tilde{Y}) = 0$ for all $\tilde{X}, \tilde{Y} \in \mathcal{X}(TM)$, that is J is integrable. $\square$

Theorem 2.4. *Let M be a P-Sasakian manifold with structure tensor (ϕ, η, ξ, g) and let J be defined by (2.1). Then the metallic structure J is never parallel with respect to ∇^c.*

Proof. We have

$$
\begin{aligned}
(\nabla^c_{X^c} J)\xi^c &= \nabla^c_{X^c}(J\xi^c) - J(\nabla^c_{X^c}\xi^c)\\
&= -\frac{2\sigma_{p,q}-p}{2}\left[\nabla^c_{X^c}\big((\phi\xi)^c + \eta(\xi)^v\xi^v + \eta(\xi)^c\xi^c\big) - (\phi\nabla_X\xi)^c\right.\\
&\quad \left. -(\eta(\nabla_X\xi))^v\xi^v - (\eta(\nabla_X\xi))^c\xi^c\right].
\end{aligned}
$$

Using $\phi X = \nabla_X \xi$, we get

$$
(\nabla^c_{X^c} J)\xi^c = -\frac{2\sigma_{p,q}-p}{2}[(\phi X)^v - X^c] \neq 0, \quad \text{for all } X \in \mathcal{D} \setminus \{0\},
$$

where $\mathcal{D}$ is a distribution and defined by (2.2). $\square$

Proposition 2.5. *Let M be a P-Sasakian manifold with structure tensor (ϕ, η, ξ, g), let $\nabla\phi = 0$, and let J be defined by (2.1). Then, the fundamental 2-form Φ, given by*

$$
\Phi(\tilde{X}, \tilde{Y}) = g^c(\tilde{X}, J\tilde{Y}) - \frac{p}{2} g^c(\tilde{X}, \tilde{Y}), \quad \tilde{X}, \tilde{Y} \in \mathcal{X}(TM),
$$

is closed if and only if

$$
g(\nabla_Y X, \phi Z) + g(\nabla_Z Y, \phi X) + g(\nabla_X Z, \phi Y) = 0, \quad \textit{for all } X, Y, Z \in \mathcal{X}(M). \tag{2.3}
$$

Proof. The coboundary formula for d on a 2-form Φ is

$$\begin{aligned} 3d\Phi(\tilde{X},\tilde{Y},\tilde{Z}) &= \tilde{X}\Phi(\tilde{Y},\tilde{Z})+\tilde{Y}\Phi(\tilde{Z},\tilde{X})+\tilde{Z}\Phi(\tilde{X},\tilde{Y}) \\ &\quad -\Phi([\tilde{X},\tilde{Y}],\tilde{Z})-\Phi([\tilde{Z},\tilde{X}],\tilde{Y})-\Phi([\tilde{Y},\tilde{Z}],\tilde{X}), \end{aligned}$$

for any $\tilde{X},\tilde{Y},\tilde{Z}\in\mathcal{X}(TM)$. Hence, for any $X,Y,Z\in\mathcal{X}(M)$, we have

$$\begin{aligned} 3d\Phi(X^c,Y^c,Z^v) &= X^c g^c(Y^c,JZ^v)-\frac{p}{2}X^c g^c(Y^c,Z^v)+Y^c g^c(Z^v,JX^c) \\ &\quad -\frac{p}{2}Y^c g^c(Z^v,X^c)+Z^v g^c(X^c,JY^c)-\frac{p}{2}Z^V g^c(X^c,Y^c) \\ &\quad -g^c([X,Y]^c,JZ^v)+\frac{p}{2}g^c([X,Y]^c,Z^v)-g^c([Z,X]^v,JY^c) \\ &\quad +\frac{p}{2}g^c([Z,X]^v,Y^c)-g^c([Y,Z]^v,JX^c)+\frac{p}{2}g^c([Y,Z]^v,X^c). \end{aligned}$$

On the other hand,

$$JZ^v=\frac{p}{2}Z^v-\frac{2\sigma_{p,q}-p}{2}\big((\phi(Z))^v+(\eta(Z))^v\xi^c\big)$$

and

$$JX^c=\frac{p}{2}X^c-\frac{2\sigma_{p,q}-p}{2}\big((\phi(X))^v+(\eta(X))^v\xi^v+(\eta(X))^c\xi^c\big).$$

Therefore,

$$\begin{aligned} -\frac{6}{2\sigma_{p,q}-p}d\Phi(X^c,Y^c,Z^v) &= \{Xg(Y,\phi Z)+Yg(Z,\phi X)+Zg(X,\phi Y) \\ &\quad -g([X,Y],\phi Z)-g([Z,X],\phi Y) \\ &\quad -g([Y,Z],\phi X)\}^v \\ &\quad +X^c[(\eta(Z))^v(\eta(Y))^c]+Y^c[(\eta(X))^c(\eta(Z))^v] \\ &\quad +Z^v[(\eta(Y))^v(\eta(X))^v]+Z^v[(\eta(Y))^c(\eta(X))^c] \\ &\quad -(\eta(Z))^v g([X,Y],\xi)^c-(\eta(Y))^c g([Z,X],\xi)^v \\ &\quad -(\eta(X))^c g([Y,Z],\xi)^v. \end{aligned}$$

Since $\nabla\phi=0$ and $g(X,\phi Y)=g(Y,\phi X)$, we get

$$Xg(Y,\phi Z)-g([X,Y],\phi Z)=g(\nabla_Y X,\phi Z)+g(\nabla_X Z,\phi Y).$$

Also, $\nabla_X\xi=\phi X$ results that

$$\begin{aligned} g([X,Y],\xi) &= g(\nabla_X Y,\xi)-g(\nabla_Y X,\xi) \\ &= Xg(Y,\xi)-g(Y,\nabla_X\xi)-Yg(X,\xi)+g(X,\nabla_Y\xi) \\ &= X(\eta(Y))-g(Y,\phi X)-Y(\eta(X))+g(X,\phi Y) \\ &= X(\eta(Y))-Y(\eta(X)). \end{aligned}$$

Thus,

$$\begin{aligned} -\frac{6}{2\sigma_{p,q}-p}d\Phi(X^c,Y^c,Z^v) &= 2\left\{g(\nabla_Y X,\phi Z)+g(\nabla_Z Y,\phi X)+g(\nabla_X Z,\phi Y)\right\} \\ &\quad +2\left\{(\eta(Y))^c(X\eta(Z))^v+(\eta(Z))^v(Y\eta(X))^c\right. \\ &\quad \left.+(\eta(X))^c(Z\eta(Y))^v\right\}. \end{aligned}$$

Now, if $X, Y, Z \in \mathcal{D}$, then $d\Phi(X^c, Y^c, Z^v) = 0$ is equivalent with (2.3), where $\mathcal{D}$ is a distribution and defined by (2.2). Also, if $X = \xi$ or $Z = \xi$, then we get the same result. The other cases are reducible to (2.3). □

Note. We recall that $\frac{p}{2}I - (\frac{2\sigma_{p,q}-p}{2})(\phi^c \pm \eta^v \otimes \xi^v \pm \eta^c \otimes \xi^c)$ are also metallic structures. For these structures, we can obtain the similar results as for the metallic structure (2.1).
In the following, we study a metallic structure on TM induced by the horizontal lift.

Proposition 2.6. *Let M be a P-Sasakian manifold with structure tensor (ϕ, η, ξ, g). Then there exists a metallic structure on its tangent bundle, given by*

$$F = \frac{p}{2}I - (\frac{2\sigma_{p,q} - p}{2})(\phi^h + \eta^h \otimes \xi^h + \eta^v \otimes \xi^v). \tag{2.4}$$

Proof. By the definition of vertical lift and horizontal lift of the almost paracontact structure of a P-Sasakian manifold M, we have

$$(\phi^h)^2 = (\phi^2)^h = I - \eta^h \otimes \xi^v - \eta^v \otimes \xi^h,$$
$$\eta^v(\xi^h) = \eta^h(\xi^v) = 1, \quad \eta^v(\xi^v) = \eta^h(\xi^h) = 0,$$
$$\phi^h(\xi^v) = \phi^h(\xi^h) = 0, \quad \eta^v \circ \phi^h = \eta^h \circ \phi^h = 0.$$

Therefore for any $\tilde{X} \in \mathcal{X}(TM)$, we get

$$F(\xi^v) = \frac{p}{2}\xi^v - \frac{2\sigma_{p,q} - p}{2}\xi^h, \quad J(\xi^h) = \frac{p}{2}\xi^h - \frac{2\sigma_{p,q} - p}{2}\xi^v,$$
$$F(\phi^h\tilde{X}) = \frac{p}{2}\phi^h\tilde{X} - \frac{2\sigma_{p,q} - p}{2}(\tilde{X} - \eta^h(\tilde{X})\xi^v - \eta^v(\tilde{X})\xi^h).$$

Now, we obtain

$$F(\tilde{X}) = \frac{p}{2}\tilde{X} - \frac{2\sigma_{p,q} - p}{2}(\phi^h\tilde{X} + \eta^v(\tilde{X})\xi^v + \eta^h(\tilde{X})\xi^h)$$

and

$$\begin{aligned} F^2(\tilde{X}) &= \frac{p}{2}F(\tilde{X}) - \frac{2\sigma_{p,q} - p}{2}(F(\phi^h\tilde{X}) + \eta^v(\tilde{X})F(\xi^v) + \eta^h(\tilde{X})F(\xi^h)) \\ &= pF(\tilde{X}) + q\tilde{X}, \end{aligned}$$

which finishes the proof. □

Definition 2.7 (Sasakian metric)**.** Let (M, g) be a Riemannian manifold. The Sasakian metric on TM is defined as follows:

$$G(X^v, Y^h) = 0, \; G(X^v, Y^v) = [g(X, Y)]^v, \; G(X^h, Y^h) = [g(X, Y)]^v,$$

for any $X, Y \in \mathcal{X}(M)$.

Proposition 2.8. *Let M be a P-Sasakian manifold with structure tensor (ϕ, η, ξ, g). If F is defined by (2.4) on TM and G is the Sasakian metric, then*

$$G(F\tilde{X}, F\tilde{Y}) = pG(\tilde{X}, F\tilde{Y}) + qG(\tilde{X}, \tilde{Y}),$$

for any $\tilde{X}, \tilde{Y} \in \mathcal{X}(TM)$.

Proof. For any $X, Y \in \mathcal{X}(M)$, we have

$$\eta^h(X^v) = \eta^v(X^v) = 0,\ \eta^h X^h = \eta^v X^h = (\eta(X))^h,$$
$$\phi^h X^v = (\phi X)^v,\ \phi^h X^h = (\phi X)^h.$$

Hence,

$$FX^v = \frac{p}{2}X^v - \frac{2\sigma_{p,q} - p}{2}(\phi X)^v.$$

Now by the definition of the Sasakian metric G, we get

$$G(FX^v, FY^v) = \left\{(q + \frac{p^2}{2})g(X, Y) - \frac{2\sigma_{p,q} - p}{2}pg(X, \phi Y)\right\}^v$$

and

$$G(X^v, FY^v) = \left\{\frac{p}{2}g(X, Y) - \frac{2\sigma_{p,q} - p}{2}g(X, \phi Y)\right\}^v.$$

Therefore,

$$G(FX^v, FY^v) = pG(X^v, FY^v) + qG(X^v, Y^v).$$

The other cases are similar. □

Definition 2.9. Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection with respect to the Riemannian metric g. Let $\mathcal{D}$ be a distribution defined by (2.2). The connection ∇ is called $\mathcal{D}$-flat if $\nabla_X Y \in \mathcal{D}$ for all $X, Y \in \mathcal{D}$.

Theorem 2.10. *Let M be a P-Sasakian manifold with structure tensor (ϕ, η, ξ, g). Then the metallic structure F defined by (2.4) on TM is integrable if and only if ∇ is $\mathcal{D}$-flat and*

$$R(\phi X, \phi Y) + R(X, Y) - \phi\{R(\phi X, Y) + R(X, \phi Y)\} = 0, \tag{2.5}$$

where R is a curvature tensor of M.

Proof. Let $X, Y \in \mathcal{D}$, let $a = -\frac{2}{2\sigma_{p,q}-p}$, and let $U \in TM$. Then

$$\begin{aligned} N_F(X^h, Y^h)U &= [N^1(X, Y)]^h U + \{\eta R(\phi X, Y)U + \eta R(X, \phi Y)U\}\xi^h \\ &\quad -\{R(\phi X, \phi Y)U + R(X, Y)U - \phi R(\phi X, Y)U \\ &\quad +R(X, \phi Y)U\}^v + N^2(X, Y)\xi^v. \end{aligned}$$

Also,

$$\begin{aligned} aN_F(X^h, Y^v) &= \big(\nabla_{\phi X}\phi Y - \phi\nabla_{\phi X}Y - \phi\nabla_X\phi Y + \nabla_X Y\big)^v \\ &\quad -\{\eta(\nabla_{\phi X}Y) + \eta(\nabla_X\phi Y)\}\xi^h, \\ N_F(X^v, Y^v) &= 0, \\ aN_F(X^h, \xi^h)U &= \big(\nabla_{\phi X}\xi - \phi(\nabla_X\xi) + \phi(N^3(X))\big)^h U + [\eta R(\phi X, \xi)U]\xi^h \\ &\quad +\{N^2(X, \xi) - \eta(\nabla_X\xi)\}\xi^v + [\phi R(\phi X, \xi)U]^v, \\ aN_F(X^h, \xi^v)U &= -[N^3(X)]^h U + \{N^2(X, \xi) + \eta R(X, \xi)U\}\xi^h \\ &\quad -\{R(\phi X, \xi)U - \phi R(X, \xi)U + \phi(\nabla_{\phi X}\xi) - \nabla_X\xi\}^v \\ &\quad -N^4(X)\xi^v, \end{aligned}$$

and

$$aN_F(X^v,\xi^v)=-(\nabla_\xi\phi X-\phi\nabla_\xi X)^v+\eta(\nabla_\xi X)\xi^h.$$

Hence, $N_F=0$ if and only if (2.5) is true and

$$\nabla_{\phi X}\phi Y-\phi\nabla_{\phi X}Y-\phi\nabla_X\phi Y+\nabla_X Y=0. \tag{2.6}$$

From (1.1), Eq. (2.6) is equivalent with $\eta(\nabla_X Y)=0$, for any $X,Y\in\mathcal{D}$, that is, ∇ is $\mathcal{D}$-flat. □

Theorem 2.11. *Let M be a P-Sasakian manifold with structure tensor (ϕ,η,ξ,g). Then the metallic structure F defined by (2.4) on TM is never parallel with respect to ∇^h.*

Proof. We have

$$(\nabla^h_{X^h}F)\xi^h=\nabla^h_{X^h}(F\xi^h)-F(\nabla^h_{X^h}\xi^h)=-\frac{2\sigma_{p,q}-p}{2}[(\phi X)^v-(\phi^2X)^h].$$

If $X\in\mathcal{D}\setminus\{0\}$, then $(\nabla^h_{X^h}F)\xi^h\neq 0$, where $\mathcal{D}$ is defined by (2.2). □

We define the fundamental 2-form Φ' by

$$\Phi'(\tilde{X},\tilde{Y})=G(\tilde{X},F\tilde{Y})-\frac{p}{2}G(\tilde{X},\tilde{Y}),\quad \tilde{X},\tilde{Y}\in\mathcal{X}(TM).$$

Proposition 2.12. *Let M be a P-Sasakian manifold with structure tensor (ϕ,η,ξ,g) and let F be defined by (2.4). Then fundamental 2-form Φ' is never closed.*

Proof. Let $X\in\mathcal{D}$ be a unit vector field, that is, $g(X,X)=1$, where $\mathcal{D}$ is defined by (2.2). Then by a similar proof as that of Proposition 2.5, we have

$$\begin{aligned}-\frac{6}{2\sigma_{p,q}-p}d\Phi'(X^h,X^v,\xi^v) &= -G([\xi^v,X^h],(\phi X)^v)=-g(\nabla_X\xi,\phi X)^v\\ &= -g(X,X)^v=-1.\end{aligned}$$

□

References

1. S. Azami, *General natural metallic structure on tangent bundle*,Iran. J. Sci. Technol. Trans. A Sci. **42** (2018), no. 1, 81–88.
2. S.I. Goldberg and N.C. Petridis, *Differentiable solutions of algebraic equations on manifolds*, Kodai Math. Sem. Rep. **25** (1973) 111–128.
3. S.I. Goldberg and K. Yano, *Polynomial structures on manifolds*, Kodai Math. Sem. Rep. **22** (1970) 199–218.
4. S. Gudmundson and E. Kappos, *On the geometry of the tangent bundle with the Cheeger Gromoll metric*, Tokyo J. Math. **25** (2002), no. 1, 75–83.
5. C.E. Hretcanu and A.M. Blaga, *Hemi-slant submanifolds in metallic Riemannian manifolds*, Carpathian J. Math. **35** (2019), no. 1, 59–68.
6. C.E. Hretcanu and M. Crasmareanu, *Metallic structures on Riemannian manifolds*, Rev. Un. Mat. Argentina **54** (2013), no. 2, 15–27.
7. M. de León and P.R. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics*, North-Holland Mathematics Studies 158, North-Holland Publishing Co. Amsterdam, 1989.
8. V. Oproiu, *A generalization of natural almost Hermitian structures on the tangent bundles*, Math. J. Toyama Univ. **22** (1999) 1–14.

9. I. Sato, *On a structure similar to the almost contact structure II*, Tensor (N.S.) **31** (1977) 199–205.
10. K. Yano and S. Ishihara, *Tangent and Cotangent Bundle: Differential Geometry*, Marcel Dekker, Inc. New York, 1973.

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