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TOEPLITZ AND HANKEL OPERATORS ON A VECTOR-VALUED BERGMAN SPACE

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Abstract. In this paper we derive certain algebraic properties of Toeplitz and Hankel operators defined on the vector-valued Bergman spaces $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ where $\mathbb{D}$ is the open unit disk in $\mathbb{C}$ and $n \geq 1$. We show that the set of all Toeplitz operators $T_\Phi, \Phi \in L^\infty_{M_n}(\mathbb{D})$ is strongly dense in the set of all bounded linear operators $\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))$ and characterize all finite rank little Hankel operators.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$ and let $dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$ be the area measure on $\mathbb{D}$ normalised so that the area of $\mathbb{D}$ is 1. For $1 \leq p < \infty$, the Bergman space $L_a^p(\mathbb{D})$ is the space of all holomorphic functions f in $\mathbb{D}$ for which

$$\|f\|_{L_a^p(\mathbb{D})} = \left(\int_{\mathbb{D}} |f(z)|^p dA(z)\right)^{\frac{1}{p}} < \infty.$$

The quantity $\|.\|_{L_a^p(\mathbb{D})}$ is a norm if $p \geq 1$. Thus $L_a^p(\mathbb{D})$ is the subspace of holomorphic functions that are in the space $L^p(\mathbb{D}, dA)$. The Bergman spaces are Banach spaces, which is a consequence of the estimate:

$$\sup_{z\in K} |f(z)| \leq C_K \|f\|_{L_a^p(\mathbb{D})}$$

valid on compact subsets K of $\mathbb{D}$. If $p = 2$, then $L_a^p(\mathbb{D})$ is a Hilbert space. Since point evaluation at $z \in \mathbb{D}$ is a bounded linear functional [12] on the Hilbert

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space $L_a^2(\mathbb{D})$, the Riesz representation theorem implies that there exists a unique function K_z in $L_a^2(\mathbb{D})$ such that

$$f(z) = \int_{\mathbb{D}} f(w)\overline{K_z(w)}dA(w)$$

for all f in $L_a^2(\mathbb{D})$. Let $K(z,w)$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by

$$K(z,w) = \overline{K_z(w)}.$$

The function $K(z,w)$ is thus the reproducing kernel for the Bergman space $L_a^2(\mathbb{D})$ and is called the Bergman kernel. The sequence $\{e_n(z)\}_{n\geq 0} = \{\sqrt{n+1}z^n\}_{n\geq 0}$ of functions [12] form the standard orthonormal basis for $L_a^2(\mathbb{D})$ and

$$K(z,w) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n(w)}.$$

The Bergman kernel is independent of the choice of orthonormal basis and $K(z,w) = \frac{1}{(1-z\bar{w})^2}$. Let $k_a(z) = \frac{K(z,a)}{\sqrt{K(a,a)}} = \frac{1-|a|^2}{(1-\bar{a}z)^2}$. These functions k_a are called the normalized reproducing kernels of $L_a^2(\mathbb{D})$; it is clear that they are unit vectors in $L_a^2(\mathbb{D})$. Let $L^\infty(\mathbb{D}, dA)$ denote the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\|f\|_\infty = \operatorname{esssup}\{|f(z)| : z \in \mathbb{D}\} < \infty$$

and $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$.

Let $L_a^{2,\mathbb{C}^n}(\mathbb{D}) = L_a^2(\mathbb{D}) \otimes \mathbb{C}^n$ and $L_{M_n}^\infty(\mathbb{D}) = L^\infty(\mathbb{D}) \otimes M_n$ where $M_n(\mathbb{C}) = M_n, n \geq 1$ is the set of all $n\times n$ matrices with entries in $\mathbb{C}$. The space $L_a^{2,\mathbb{C}^n}(\mathbb{D}), n \geq 1$ is called the vector-valued Bergman space. The inner product on $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ is defined as

$$\langle f, g\rangle_{L_a^{2,\mathbb{C}^n}(\mathbb{D})} = \int_{\mathbb{D}} \langle f(z), g(z)\rangle_{\mathbb{C}^n} dA(z).$$

With this inner product $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ is a Hilbert space. The norm defined on $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ is given by

$$\|f\|^2_{L_a^{2,\mathbb{C}^n}(\mathbb{D},dA)} = \int_{\mathbb{D}} \|f(z)\|^2_{\mathbb{C}^n} dA(z).$$

It is a closed subspace of $L^{2,\mathbb{C}^n}(\mathbb{D}, dA) = L^2(\mathbb{D}, dA) \otimes \mathbb{C}^n$. Let P denote the orthogonal projection from $L^{2,\mathbb{C}^n}(\mathbb{D}, dA)$ onto $L_a^{2,\mathbb{C}^n}(\mathbb{D})$. For $\Phi \in L_{M_n}^\infty(\mathbb{D})$, we define the Toeplitz operator T_Φ from $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ into itself as $T_\Phi f = P(\Phi f)$ and the Hankel operator H_Φ from $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ into $(L_a^{2,\mathbb{C}^n}(\mathbb{D}))^\perp = L^{2,\mathbb{C}^n}(\mathbb{D}, dA) \ominus L_a^{2,\mathbb{C}^n}(\mathbb{D})$ as $H_\Phi f = (I-P)(\Phi f)$. For $\Phi \in L_{M_n}^\infty(\mathbb{D})$, define $\|\Phi\|_\infty = \operatorname{esssup}_{z\in\mathbb{D}}\|\Phi(z)\|$. If $\Phi \in L_{M_n}^\infty(\mathbb{D})$, then it is not difficult to see that $\|T_\Phi\| \leq \|\Phi\|_\infty$ and $\|H_\Phi\| \leq \|\Phi\|_\infty$. This is so as $\|P\| \leq 1$ and $\|I-P\| \leq 1$.

For $\Phi \in L_{M_n}^\infty(\mathbb{D})$, we define the little Hankel operator S_Φ from $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ into itself as $S_\Phi f = PJ(\Phi f)$ where $J : L^{2,\mathbb{C}^n}(\mathbb{D}, dA) \to L^{2,\mathbb{C}^n}(\mathbb{D}, dA)$ is defined as $Jf(z) = f(\overline{z})$. The map J is unitary. There are also many equivalent ways of defining little Hankel operators. Let $\overline{L_a^{2,\mathbb{C}^n}(\mathbb{D})} = \overline{L_a^2(\mathbb{D})} \otimes \mathbb{C}^n$. For $\Phi \in L_{M_n}^\infty(\mathbb{D})$, define h_Φ from $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ into $\overline{L_a^{2,\mathbb{C}^n}(\mathbb{D})}$ as $h_\Phi f = \overline{P}(\Phi f)$ where $\overline{P}$ is the orthogonal projection from $L^{2,\mathbb{C}^n}(\mathbb{D}, dA)$ onto $\overline{L_a^{2,\mathbb{C}^n}}(\mathbb{D})$. It is not difficult to verify that $h_\Phi = JS_\Phi$.

Let $\mathcal{L}(H)$ be the set of all bounded linear operators from the Hilbert space H into itself and $\mathcal{L}C(H)$ be the set of all compact operators in $\mathcal{L}(H)$.

Consider the direct sum $\displaystyle\sum_{k=1}^{n} \oplus L_k$, with each L_k the same Hilbert space $L_a^2(\mathbb{D})$. Define the bounded linear operators

$$U_i : L_a^2(\mathbb{D}) \longrightarrow \sum_{k=1}^{n} \oplus L_k,\ V_i : \sum_{k=1}^{n} \oplus L_k \longrightarrow L_a^2(\mathbb{D}),$$

for each $i \in \{1, 2, \cdots, n\}$ as follows. When $f \in L_a^2(\mathbb{D})$ and $g = \{g_k\} \in \sum_{k=1}^{n} \oplus L_k, V_i g = g_i$ and $U_i f$ is the family $\{h_k\}$ in which $h_i = f$ and all other h_k are 0. Let L_i' be the range of U_i. It consists of all elements $\{h_k\}$ of $\sum_{k=1}^{n} \oplus L_k$ in which $h_k = 0$ when $k \neq i$. The space L_i' is a closed subspace of $\sum_{k=1}^{n} \oplus L_k$ and observe that $V_i U_i$ is the identity operator on $L_a^2(\mathbb{D})$ and $U_i V_i$ is the projection E_i from $\sum_{k=1}^{n} \oplus L_k$ onto L_i'. Since the subspace $L_i', i \in \{1, 2, \cdots, n\}$ are pairwise orthogonal, and $\bigvee_{i=1}^{n} L_i' = \sum_{k=1}^{n} \oplus L_k$, it follows that the sum $\sum_{i=1}^{n} E_i = I$. Note that $U_i = V_i^*$, since

$$\langle U_i f, \{f_k\}\rangle = \langle f, f_i\rangle = \langle f, V_i\{f_k\}\rangle$$

whenever $f \in L_a^2(\mathbb{D})$ and $\{f_k\} \in \sum_{k=1}^{n} \oplus L_k$. With each bounded linear operator T acting on $\sum_{k=1}^{n} \oplus L_k$, we associate a matrix $(T_{ij})_{1\leq i,j\leq n}$, with entries T_{ij} in $\mathcal{L}(L_a^2(\mathbb{D}))$ defined by

$$T_{ij} = V_i T U_j. \tag{1.1}$$

If $g = \{g_k\} \in \sum_{k=1}^{n} \oplus L_k$, then Tg is an element $\{p_k\}$ of $\sum_{k=1}^{n} \oplus L_k$ and

$$p_i = V_i T g = V_i T \left(\sum_{k=1}^{n} E_k g\right) = \sum_{k=1}^{n} V_i T U_j V_j g = \sum_{j=1}^{n} T_{ij} g_j.$$

Thus

$$T\left(\sum_{k=1}^{n} \oplus g_k\right) = \sum_{k=1}^{n} \oplus p_k \text{ where } p_i = \sum_{j=1}^{n} T_{ij} g_j, i \in \{1, 2, \cdots, n\}. \tag{1.2}$$

The usual rules of matrix algebra have natural analogues in this situation. From (1.1), the matrix elements T_{ij} depend linearly on T. Since

$$V_i T^* U_j = U_i^* T^* V_j^* = (V_j T U_i)^* = (T_{ji})^*,$$

the matrix of T^* has $(T_{ji})^*$ in the (i, j) position. If S and T are bounded linear operators acting on $\sum_{k=1}^{n} \oplus L_k$, and $R = ST$, then

$$\begin{aligned} R_{ij} = V_i R U_j = V_i S T U_j &= \sum_{k=1}^{n} V_i S E_k T U_j \\ &= \sum_{k=1}^{n} V_i S U_k V_k T U_j = \sum_{k=1}^{n} S_{ik} T_{kj}. \end{aligned}$$

Thus we establish a one-to-one correspondence between elements of $\mathcal{L}\left(\sum_{k=1}^{n} \oplus L_k\right)$ and certain matrices $(T_{ij})_{i,j=1}^{n}$ with entries T_{ij} in $\mathcal{L}(L_a^2(\mathbb{D}))$. Each such matrix corresponds to some bounded operator T acting on $\sum_{k=1}^{n} \oplus L_k$; indeed, T is defined by (1.2), and its boundedness follows at once from the relations

$$||\{p_k\}||^2 = \sum_{i=1}^{n} ||p_i||^2 = \sum_{i=1}^{n} ||\sum_{j=1}^{n} T_{ij} g_j||^2 \leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} ||T_{ij}||\ ||g_j||\right)^2$$

$$\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} ||T_{ij}||^2\right) \left(\sum_{j=1}^{n} ||g_j||^2\right) = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} ||T_{ij}||^2\right) ||\{g_k\}||^2.$$

In this paper we derive certain algebraic properties of Toeplitz and Hankel operators defined on the vector-valued Bergman spaces $L_a^{2,\mathbb{C}^n}(\mathbb{D}), n \geq 1$. We have shown that if there exists $A, B \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))$ such that $AT_\Phi B = T_\Phi$ for all $\Phi \in L_{M_n}^{\infty}(\mathbb{D})$, then $A = \alpha I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))}, B = \beta I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))}, \alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 1$ and that the set of all Toeplitz operators $T_\Phi, \Phi \in L_{M_n}^{\infty}(\mathbb{D})$ is strongly dense in the set of all bounded linear operators $\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))$ and characterize all finite rank little Hankel operators defined on the vector-valued Bergman space. The layout of this paper is as follows. In section 2, we establish that if $AT_\Phi B = T_\Phi$ for all $\Phi \in L_{M_n}^{\infty}(\mathbb{D})$, then $A = \alpha I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))}, B = \beta I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))}, \alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 1$. Furthermore, it is shown that the set of all Toeplitz operators $T_\Phi, \Phi \in L_{M_n}^{\infty}(\mathbb{D})$ from $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ into itself is strongly dense in the Banach space $\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))$. In section 3, we prove that there exists no finite rank Hankel operator H_Φ with nonconstant matrix-valued symbol Φ that is diagonal. We further establish certain elementary properties of little Hankel operators and characterize all finite rank little Hankel operators with diagonal matrix-valued symbols.

2. Toeplitz operators with symbols in $L_{M_n}^{\infty}(\mathbb{D})$

In this section we have shown that if there exists $A, B \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))$ such that $AT_\Phi B = T_\Phi$ for all $\Phi \in L_{M_n}^{\infty}(\mathbb{D})$, then $A = \alpha I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))}, B = \beta I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))}, \alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 1$. Here $I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))}$ is the identity operator from the space $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ into itself. Further, we show that the set of all Toeplitz operators $T_\Phi, \Phi \in L_{M_n}^{\infty}(\mathbb{D})$ from $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ into itself is strongly dense in the Banach space $\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))$.

Theorem 2.1. *If $A, B \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D})), n \geq 1$ and $AT_\Phi B = T_\Phi$ for all $\Phi \in L_{M_n}^{\infty}(\mathbb{D})$, then $A = \alpha I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))}, B = \beta I_{\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))}, \alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 1$.*

Proof. Suppose $A, B \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D})), n \geq 1$ and $AT_\Phi B = T_\Phi$ for all $\Phi \in L_{M_n}^{\infty}(\mathbb{D})$. Since $L_a^{2,\mathbb{C}^n}(\mathbb{D}) = L_a^2(\mathbb{D}) \otimes \mathbb{C}^n$, we obtain

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \text{ and } B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix}, \text{ where}$$

$A_{ij}, B_{ij} \in \mathcal{L}(L_a^2(\mathbb{D}))$ for all $i, j \in \{1, 2, \cdots, n\}$. Here $A_{ij} = V_i A U_j$ and $B_{ij} =$

V_iBU_j for all $i,j \in \{1,2,\cdots,n\}$. Further, as $\Phi \in L^\infty_{M_n}(\mathbb{D}) = L^\infty(\mathbb{D})\otimes M_n$, we have $\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix}$, where $\phi_{ij} \in L^\infty(\mathbb{D})$ for all $i,j \in \{1,2,\cdots,n\}$. Hence

$$T_\Phi = \begin{pmatrix} T_{\phi_{11}} & T_{\phi_{12}} & \cdots & T_{\phi_{1n}} \\ T_{\phi_{21}} & T_{\phi_{22}} & \cdots & T_{\phi_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ T_{\phi_{n1}} & T_{\phi_{n2}} & \cdots & T_{\phi_{nn}} \end{pmatrix}.$$

By considering elementary matrices of the type

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & T_{\phi_{ij}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{pmatrix},$$

with just one nonzero (i,j)th entry $T_{\phi_{ij}}$, $\phi_{ij} \in L^\infty(\mathbb{D})$, $i,j \in \{1,2,\cdots,n\}$ and using the operator equations

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & T_{\phi_{ij}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & T_{\phi_{ij}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{pmatrix},$$

it follows from [5] that $V_iAU_j = V_iBU_j = 0$ if $i \neq j$, $i,j = 1,2,\cdots,n$ and $V_iAU_i = \alpha I_{\mathcal{L}(L^2_a(\mathbb{D}))}$, $V_iBU_i = \beta I_{\mathcal{L}(L^2_a(\mathbb{D}))}$ for all $i = 1,2,\cdots,n$ and for some $\alpha,\beta \in \mathbb{C}$ such that $\alpha\beta = 1$. This implies $A = \alpha I_{\mathcal{L}(L^{2,\mathbb{C}^n}_a(\mathbb{D}))}$ and $B = \beta I_{\mathcal{L}(L^{2,\mathbb{C}^n}_a(\mathbb{D}))}$. The theorem follows. $\square$

Theorem 2.2. *Let* $T \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D})), n \geq 1, F_i = \begin{pmatrix} F_{i1} \\ \vdots \\ F_{in} \end{pmatrix} \in L_a^{2,\mathbb{C}^n}(\mathbb{D}), G_i = \begin{pmatrix} G_{i1} \\ \vdots \\ G_{in} \end{pmatrix} \in L_a^{2,\mathbb{C}^n}(\mathbb{D}), i = 1, \cdots, N$. *Then there exists* $\Phi \in L^\infty_{M_n}(\mathbb{D})$ *such that* $\langle T_\Phi F_i, G_i \rangle = \langle T F_i, G_i \rangle, i = 1, \cdots, N$.

Proof. Let $f_1, f_2, \cdots, f_k$ and $g_1, g_2, \cdots, g_m$ respectively be bases of the finite-dimensional subspaces of $L_a^{2,\mathbb{C}^n}(\mathbb{D})$ generated by $F_1, \cdots, F_N$ and $G_1, \cdots, G_N$. We shall find $\Phi \in L^\infty_{M_n}(\mathbb{D})$ such that $\langle T_\Phi f_i, g_j \rangle = \langle T f_i, g_j \rangle$ for all $i = 1, \cdots, k$ and $j = 1, \cdots, m$.

Consider the operator $R : L^\infty_{M_n}(\mathbb{D}) \to \mathbb{C}^{k\times m}$, defined by $(R\Phi)_{ij} = \langle T_\Phi f_i, g_j \rangle$, $i = 1, \cdots, k$ and $j = 1, \cdots, m$. Suppose $u \in \mathbb{C}^{k\times m}$ is orthogonal to the range of R. That is, let

$$\sum_{i=1}^{k} \sum_{j=1}^{m} (R\Phi)_{ij} \overline{u_{ij}} = 0$$

for all $\Phi \in L^\infty_{M_n}(\mathbb{D})$. This implies (taking $\Phi = I_{n\times n}$, the identity matrix)

$$\sum_{i=1}^{k} \sum_{j=1}^{m} \langle f_i, g_j \rangle_{L_a^{2,\mathbb{C}^n}(\mathbb{D})} \overline{u_{ij}} = 0.$$

Hence

$$\sum_{i=1}^{k} \sum_{j=1}^{m} \langle f_i(z), g_j(z) \rangle_{\mathbb{C}^n} \overline{u_{ij}} = 0$$

almost everywhere on $\mathbb{D}$. Since the left hand side is obviously continuous on $\mathbb{D}$, this equality holds, in fact, on the whole of $\mathbb{D}$. Thus the function

$$\Omega(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{m} \langle f_i(x), g_j(\overline{y}) \rangle_{\mathbb{C}^n} \overline{u_{ij}}$$

which is analytic in $\mathbb{D} \times \mathbb{D}$, equals zero when $x = \overline{y}$. By the uniqueness theorem [11], this implies that $\Omega \equiv 0$ on $\mathbb{D} \times \mathbb{D}$. Because, functions $f_i, i = 1, 2, \cdots, k$, are linearly independent, we obtain

$$\sum_{j=1}^{m} u_{ij} g_j(\overline{y}) = 0$$

for all $y \in \mathbb{D}, i = 1, 2, \cdots, k$; but $g_j, j = 1, 2, \cdots, m$, are also linearly independent, and so $u_{ij} = 0$ for all i, j; i.e., $u = 0$. This means that the range of R is all of $\mathbb{C}^{k\times m}$ and the result follows. □

Theorem 2.3. *The set of all Toeplitz operators* $T_\Phi, \Phi \in L^\infty_{M_n}(\mathbb{D})$ *is dense in* $\mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))$ *in the strong operator topology.*

Proof. From Theorem 2.2, it follows that the collection $\mathcal{N} = \{T_\Phi : \Phi \in L^\infty_{M_n}(\mathbb{D})\}$ is dense in $\mathcal{L}(L^{2,\mathbb{C}^n}_a(\mathbb{D}))$ in the weak operator topology. As $\mathcal{N}$ is a subspace, i.e., a convex set, its weak operator topology and strong operator topology closures coincide. Hence $\mathcal{N}$ is dense in $\mathcal{L}(L^{2,\mathbb{C}^n}_a(\mathbb{D}))$ in the strong operator topology. Let $T \in \mathcal{L}(L^{2,\mathbb{C}^n}_a(\mathbb{D}))$. Then there exists $\Phi_N \in L^\infty_{M_n}(\mathbb{D})$ such that $T_{\Phi_N} \to T$ in the strong operator topology. This can also be verified as follows: Let $T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}$ where $T_{ij} = V_i T U_j \in \mathcal{L}(L^2_a(\mathbb{D}))$. From [6] and [7], it follows that $\{T_\phi : \phi \in L^\infty(\mathbb{D})\}$ is dense in $\mathcal{L}(L^2_a(\mathbb{D}))$ in the strong operator topology. Thus there exists a sequence $T_{\phi^{ij}_m}$ that converges to T_{ij} strongly for all $i, j \in \{1, 2, \cdots, n\}$. Let $\Phi_m = (\phi^{ij}_m)^n_{i,j=1}$. Then for $F = (f_1, f_2, \cdots, f_n)^T \in L^{2,\mathbb{C}^n}_a(\mathbb{D})$, we obtain

$$\|T_{\Phi_m}F - TF\|^2 = \left\| \begin{pmatrix} T_{\phi^{11}_m} - T_{11} & T_{\phi^{12}_m} - T_{12} & \cdots & T_{\phi^{1n}_m} - T_{1n} \\ T_{\phi^{21}_m} - T_{21} & T_{\phi^{22}_m} - T_{22} & \cdots & T_{\phi^{2n}_m} - T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ T_{\phi^{n1}_m} - T_{n1} & T_{\phi^{n2}_m} - T_{n2} & \cdots & T_{\phi^{nn}_m} - T_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \right\|^2$$

$$= \left\| \begin{pmatrix} (T_{\phi^{11}_m} - T_{11})f_1 + (T_{\phi^{12}_m} - T_{12})f_2 + \cdots + (T_{\phi^{1n}_m} - T_{1n})f_n \\ (T_{\phi^{21}_m} - T_{21})f_1 + (T_{\phi^{22}_m} - T_{22})f_2 + \cdots + (T_{\phi^{2n}_m} - T_{2n})f_n \\ \vdots \\ (T_{\phi^{n1}_m} - T_{n1})f_1 + (T_{\phi^{n2}_m} - T_{n2})f_2 + \cdots + (T_{\phi^{nn}_m} - T_{nn})f_n \end{pmatrix} \right\|^2$$

$$\le \sum_{i,j=1}^{n} \|T_{\phi^{ij}_m} f_j - T_{ij} f_j\|^2 \to 0$$

as $m \to \infty$. Hence the set of all Toeplitz operators $\{T_\Phi, \Phi \in L^\infty_{M_n}(\mathbb{D})\}$ is dense in $\mathcal{L}(L^{2,\mathbb{C}^n}_a(\mathbb{D}))$ in the strong operator topology. $\square$

3. Hankel operators with matrix-valued symbols

Suppose $\Phi \in L^\infty_{M_n}(\mathbb{D})$. In this section we show that $H_\Phi \equiv 0$ if and only if $\Phi \in H^\infty_{M_n}(\mathbb{D})$ and that there exists no finite rank Hankel operator H_Φ with nonconstant matrix-valued symbol Φ that is diagonal. We further establish certain elementary properties of little Hankel operators and characterize all finite rank little Hankel operators with diagonal matrix-valued symbols.

Theorem 3.1. *Let* $\Phi \in L^\infty_{M_n}(\mathbb{D})$ *and* $\Phi = \begin{pmatrix} \phi_{11} & 0 & \cdots & 0 \\ 0 & \phi_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \phi_{nn} \end{pmatrix}$, *where* $\phi_{ii} \in L^\infty(\mathbb{D}), 1 \le i \le n$. *The following hold:*

(i) *The operator* $H_\Phi \equiv 0$ *if and only if* $\Phi \in H^\infty_{M_n}(\mathbb{D})$.

(ii) *The operator* $H_{\phi_{jj}} \neq 0$ *for all* $j \in \{1, 2, \cdots, n\}$ *if and only if* $\ker H_\Phi = \{0\}$. *Further* $H_\Phi \equiv 0$ *if and only if* $\ker H_\Phi = L_a^{2,\mathbb{C}^n}(\mathbb{D})$.

(iii) *If in addition,* $\Phi \in H^\infty_{M_n}(\mathbb{D})$, *then the operator* H_{Φ^*} *is a finite rank Hankel operator if and only if* Φ *is a diagonal matrix with entries in* $\mathbb{C}$.

Proof. It is not difficult to see that $H_\Phi = \begin{pmatrix} H_{\phi_{11}} & 0 & \cdots & 0 \\ 0 & H_{\phi_{22}} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & H_{\phi_{nn}} \end{pmatrix}$ where $H_{\phi_{ii}} \in \mathcal{L}(L_a^2(\mathbb{D}))$ is a Hankel operator with symbol $\phi_{ii} \in L^\infty(\mathbb{D})$.

Suppose $\phi \in L^\infty(\mathbb{D})$. Before we begin the proof of the theorem, the points to note are the following:

(a)If $\phi f \in L_a^2(\mathbb{D})$ for all $f \in L_a^2(\mathbb{D})$ then $\phi \in H^\infty(\mathbb{D})$.

(b)$H_\phi \equiv 0$ if and only if $\phi \in H^\infty(\mathbb{D})$.

The statement (a) can be verified as follows: Suppose $\phi L_a^2(\mathbb{D}) \subset L_a^2(\mathbb{D})$. Then $T_\phi f = \phi f$ and therefore $\phi(z) = \frac{T_\phi f(z)}{f(z)}$. Hence ϕ is analytic on $\mathbb{D} - \{\text{ zeros of } f\}$. Each isolated singularity of ϕ in $\mathbb{D}$ is removable, since ϕ is assumed to be bounded. Thus ϕ is analytic on $\mathbb{D}$. Since $\phi \in L^\infty(\mathbb{D})$, we have $\phi \in H^\infty(\mathbb{D})$.

To establish (b), suppose $H_\phi \equiv 0$. Then $H_\phi f = 0$ for all $f \in L_a^2(\mathbb{D})$. That is, $T_\phi f = \phi f$. From (a) it follows that $\phi \in H^\infty(\mathbb{D})$. Conversely, if $\phi \in H^\infty(\mathbb{D})$, then $\phi f \in L_a^2(\mathbb{D})$ for all $f \in L_a^2(\mathbb{D})$. Hence $H_\phi f = 0$ for all $f \in L_a^2(\mathbb{D})$. Therefore $H_\phi \equiv 0$.

Now (i) follows from (a) and (b) since $H_\Phi \equiv 0$ if and only if $H_{\phi_{jj}} \equiv 0$ for all $j \in \{1, 2, \cdots, n\}$. That is, if and only if $\phi_{jj} \in H^\infty(\mathbb{D})$ for all $j \in \{1, 2, \cdots, n\}$. Thus $H_\Phi \equiv 0$ if and only if $\Phi \in H^\infty_{M_n}(\mathbb{D})$.

To prove (ii), suppose $\phi \in L^\infty(\mathbb{D})$. Then

$$\begin{aligned} \ker H_\phi &= \{f \in L_a^2(\mathbb{D}) : (I - P)(\phi f) = 0\} \\ &= \{f \in L_a^2(\mathbb{D}) : \phi f \in L_a^2(\mathbb{D})\}. \end{aligned}$$

Now if $\ker H_\phi \neq \{0\}$, then $\phi \in H^\infty(\mathbb{D})$ (proceed as in (a)). This implies H_ϕ is equivalent to zero and $\ker H_\phi = L_a^2(\mathbb{D})$. Thus if $H_\phi \neq 0$, then $\ker H_\phi = \{0\}$. Further, if $\ker H_\phi = \{0\}$ then it follows that $\phi \notin H^\infty(\mathbb{D})$ and $H_\phi \not\equiv 0$. To prove (ii), let $\Phi \in L^\infty_{M_n}(\mathbb{D})$. Then $\ker H_\Phi$ is equal to

$$\left\{ (f_1, f_2, \cdots, f_n) \in L_a^{2,\mathbb{C}^n}(\mathbb{D}) : \begin{pmatrix} H_{\phi_{11}} & 0 & \cdots & 0 \\ 0 & H_{\phi_{22}} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & H_{\phi_{nn}} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$
$$= \{(f_1, f_2, \cdots, f_n) \in L_a^{2,\mathbb{C}^n}(\mathbb{D}) : H_{\phi_{jj}} f_j = 0 \text{ for all } j \in \{1, 2, \cdots, n\}\}.$$

Thus it follows that $\ker H_\Phi = \{0\}$ if and only if $\ker H_{\phi_{jj}} = \{0\}$ for all $j \in \{1, 2, \cdots, n\}$. But $\ker H_{\phi_{jj}} = \{0\}$ for all $j \in \{1, 2, \cdots, n\}$ if and only if $H_{\phi_{jj}} \neq 0$ for all $j \in \{1, 2, \cdots, n\}$.

To prove (iii), we shall first show that if $\phi \in H^\infty(\mathbb{D})$, then $H_{\overline{\phi}}$ is a finite rank Hankel operator if and only if ϕ is a constant. This can be verified as follows:

Sufficiency is obvious. For the necessity, suppose that $H_{\overline{\phi}}$ is a finite rank operator, where ϕ is analytic on $\mathbb{D}$. Then

$$\ker H_{\overline{\phi}} = \{f \in L^2_a(\mathbb{D}) : (I-P)(\overline{\phi} f) = 0\} = \{f \in L^2_a(\mathbb{D}) : \overline{\phi} f \in L^2_a(\mathbb{D})\}$$

has finite codimension and is invariant under multiplication by z. By the result of Axler and Bourdon [1], there exists a polynomial q whose roots lie in $\mathbb{D}$ such that $\ker H_\phi = qL^2_a(\mathbb{D})$. Let $\phi(z) = \sum c_k z^k$; then $\overline{\phi}(z)q(z) \in L^2_a(\mathbb{D})$ implies that either ϕ is a constant or $q = 0$. If $q = 0$ then $\ker H_{\overline{\phi}} = \{0\}$. This implies $(\text{Range}H^*_{\overline{\phi}})^\perp = \{0\}$. Hence $\text{Range}H^*_{\overline{\phi}} = L^2_a(\mathbb{D})$. This implies $H_{\overline{\phi}}$ is not of finite rank. Hence $q \neq 0$ since $H_{\overline{\phi}}$ has finite rank, so the claim is verified.

Now if $\Phi \in H^\infty_{M_n}(\mathbb{D})$ then H_{Φ^*} is a finite rank Hankel operator if and only if $H_{\overline{\phi_{jj}}}$ is of finite rank for all $j \in \{1, 2, \cdots, n\}$. That is, if and only if $\overline{\phi_{jj}}$ is a constant for all $j \in \{1, 2, \cdots, n\}$. That is, if and only if Φ is a diagonal matrix with entries in $\mathbb{C}$. $\square$

Definition-3.1 A function $G \in L^2_a(\mathbb{D})$ is called an inner function in $L^2_a(\mathbb{D})$ if $|G|^2 - 1$ is orthogonal to H^∞.

This definition of inner function in a Bergman space was given by Korenblum and Stessin [10]. If N is a subspace of $L^2_a(\mathbb{D})$, let $Z(N) = \{z \in \mathbb{D} : f(z) = 0 \text{ for all } f \in N\}$, which is called the common zero set of functions in N. Hence if z_1 is a zero of multiplicity at most n of all functions in N, then z_1 appears n times in the set $Z(N)$, and each z_1 is treated as a distinct element of $Z(N)$.

Theorem 3.2. *Let $\Phi = (\phi_{ij})$ where $\phi_{ij} \in L^\infty(\mathbb{D}), 1 \leq i, j \leq n$. Suppose $\phi_{ij} = 0$ if $i \neq j$ and let $S_\Phi \in \mathcal{L}(L^{2,\mathbb{C}^n}_a(\mathbb{D}))$ be the little Hankel operator with symbol Φ. The following hold:*

(i) *The operator $S_\Phi \equiv 0$ if and only if $\Phi \in \overline{(L^{2,\mathbb{C}^n}_a(\mathbb{D}))}^\perp$.*

(ii) *The operator $S \in \mathcal{L}(L^{2,\mathbb{C}^n}_a(\mathbb{D}))$ is a little Hankel operator if and only if $T^*_{zI_{n\times n}}S = ST_{zI_{n\times n}}$ where $I_{n\times n}$ is the identity matrix of order n.*

(iii) *If $\Psi \in L^\infty_{M_n}(\mathbb{D})$, then the subspace $\ker S_\Psi$ is an invariant subspace of $T_{zI_{n\times n}}$.*

(iv) *Let $\Psi = (\psi_{ij})$, $\psi_{ij} \in L^\infty(\mathbb{D})$ and $\psi^+_{ij}(z) = \overline{\psi_{ij}(\overline{z})}, 1 \leq i, j \leq n$. Then $S^*_\Psi = S_{\Psi^+}$ where $\Psi^+ = (\psi^+_{ij})_{1\leq i,j\leq n}$.*

(v) *If for $j \in \{1, 2, \cdots, n\}$, $\ker S_{\phi_{jj}} = \{f \in L^2_a(\mathbb{D}) : f = 0$ on $\boldsymbol{b_{jj}}\}$ where $\boldsymbol{b_{jj}} = \{b^k_{jj}\}^\infty_{k=1}$ is an infinite sequence of points in $\mathbb{D}$, then there exists an inner function $G \in L^2_a(\mathbb{D})$ such that $\ker S_\Phi = GL^{2,\mathbb{C}^n}_a(\mathbb{D}) \cap L^{2,\mathbb{C}^n}_a(\mathbb{D})$.*

(vi) *If S_Φ is a finite rank little Hankel operator on $L^{2,\mathbb{C}^n}_a(\mathbb{D})$ then $\ker S_\Phi = GL^{2,\mathbb{C}^n}_a(\mathbb{D})$ for some inner function $G \in L^2_a(\mathbb{D})$ and the following hold: (1) G vanishes on $\mathbf{a}= \{a_j\}^N_{j=1}$, a finite sequence of points in $\mathbb{D}$. (2) $\|G\|_{L^2} = 1$. (3) G is equal to a constant plus a linear combination of the Bergman kernel functions $K(z, a_1), K(z, a_2), \ldots, K(z, a_n)$ and certain of their derivatives.(4) $|G|^2 - 1$ is orthogonal to L^1_h, the class of harmonic functions in L^1 of the disc.*

Proof. To prove (i), assume $\phi \in L^\infty(\mathbb{D})$. We shall first verify that $S_\phi \equiv 0$ if and only if $\phi \in (\overline{L^2_a(\mathbb{D})})^\perp$. Suppose $S_\phi \equiv 0$. Then $S_\phi f = 0$ for all $f \in L^2_a(\mathbb{D})$. Thus $PJ(\phi f) = 0$ and hence $\phi f \in (\overline{L^2_a(\mathbb{D})})^\perp$, for all $f \in L^2_a(\mathbb{D})$. Since $1 \in L^2_a(\mathbb{D})$,

$\phi \in (\overline{L_a^2(\mathbb{D})})^\perp$. Now suppose $\phi \in (\overline{L_a^2(\mathbb{D})})^\perp$. This implies $\langle \phi, \overline{g}\rangle = 0$ for all $g \in L_a^2(\mathbb{D})$. Hence $\langle \phi f, \overline{g}\rangle = \langle \phi, \overline{fg}\rangle = 0$ for all $g \in L_a^2(\mathbb{D})$ and $f \in H^\infty(\mathbb{D})$. Thus $\langle h_\phi f, \overline{g}\rangle = \langle \overline{P}(\phi f), \overline{g}\rangle = 0$ for all $g \in L_a^2(\mathbb{D})$ and $f \in H^\infty(\mathbb{D})$. Thus $h_\phi f = 0$ for all $f \in H^\infty(\mathbb{D})$. Since $H^\infty(\mathbb{D})$ is dense in $L_a^2(\mathbb{D})$, we obtain $h_\phi \equiv 0$. That is, $S_\phi = J h_\phi \equiv 0$.

Now to prove (i), notice that $S_\Phi \equiv 0$ if and only if $S_{\phi_{jj}} \equiv 0$ for all $j \in \{1, 2, \cdots, n\}$. This is true if and only if $\phi_{jj} \in (\overline{L_a^2(\mathbb{D})})^\perp$. That is, if $\Phi \in \overline{(L_a^{2,\mathbb{C}^n}(\mathbb{D}))}^\perp$.

Now we prove (ii). Let $S \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))$. Since $L_a^{2,\mathbb{C}^n}(\mathbb{D}) = L_a^2(\mathbb{D}) \oplus L_a^2(\mathbb{D}) \oplus \cdots \oplus L_a^2(\mathbb{D})$, the operator $S = \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{pmatrix}$ for some $S_{ij} \in \mathcal{L}(L_a^2(\mathbb{D})), 1 \leq i, j \leq n$. Suppose $T^*_{zI_{n\times n}} S = S T_{zI_{n\times n}}$. This implies $T_z^* S_{ij} = S_{ij} T_z$. From [8], it follows that $S_{ij} = S_{\psi_{ij}}$ for $\psi_{ij} \in L^\infty(\mathbb{D}), 1 \leq i, j \leq n$. Thus

$$S = \begin{pmatrix} S_{\psi_{11}} & S_{\psi_{12}} & \cdots & S_{\psi_{1n}} \\ S_{\psi_{21}} & S_{\psi_{22}} & \cdots & S_{\psi_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ S_{\psi_{n1}} & S_{\psi_{n2}} & \cdots & S_{\psi_{nn}} \end{pmatrix}.$$

That is, $S = S_\Psi$ where $\Psi = \begin{pmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1n} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \psi_{n1} & \psi_{n2} & \cdots & \psi_{nn} \end{pmatrix}$. Conversely, suppose $S \in \mathcal{L}(L_a^{2,\mathbb{C}^n}(\mathbb{D}))$ is a little Hankel operator. That is, $S = S_\Psi$ where $\Psi \in L^\infty_{M_n}(\mathbb{D})$. Let $\Psi = (\psi_{ij})_{1\leq i,j\leq n}$. Then $S_\Psi = (S_{\psi_{ij}})_{1\leq i,j\leq n}$. From [8], it follows that $T_z^* S_{\psi_{ij}} = S_{\psi_{ij}} T_z$. This implies $T^*_{zI_{n\times n}} S_\Psi = S_\Psi T_{zI_{n\times n}}$.

To prove (iii), let $f \in \ker S_\Psi$. Then $S_\Psi T_{zI_{n\times n}} f = T^*_{zI_{n\times n}} S_\Psi f = 0$. That is, $T_{zI_{n\times n}} f \in \ker S_\Psi$.

To prove (iv), we shall first verify that if $\psi \in L^\infty(\mathbb{D})$ then $S_\psi^* = S_{\psi^+}$ where $\psi^+(z) = \overline{\psi(\overline{z})}$. Let $f, g \in L_a^2(\mathbb{D})$. Then

$$\begin{aligned} \langle S_\psi^* f, g\rangle &= \langle f, S_\psi g\rangle \\ &= \langle f, PJ(\psi g)\rangle \\ &= \langle f, (J\psi) Jg\rangle \\ &= \langle \overline{J\psi} f, Jg\rangle \\ &= \langle \psi^+ f, Jg\rangle \\ &= \langle J(\psi^+ f), g\rangle \\ &= \langle PJ(\psi^+ f), g\rangle \\ &= \langle S_{\psi^+} f, g\rangle. \end{aligned}$$

Thus $S_\psi^* = S_{\psi^+}$. Now if $\Psi = (\psi_{ij})_{1\leq i,j\leq n}$ then $S_\Psi = (S_{\psi_{ij}})_{1\leq i,j\leq n}$. Then $S_\Psi^* = (S^*_{\psi_{ij}})_{1\leq i,j\leq n} = (S_{\psi^+_{ij}})_{1\leq i,j\leq n} = S_{\Psi^+}$.

Now we prove (v). Notice that for $1 \leq j \leq n$, $\ker S_{\phi_{jj}}$ is an invariant subspace of T_z. If $\ker S_{\phi_{jj}}$ can be expressed in terms of its common zero set, i.e., if $\ker S_{\phi_{jj}} =$

$\{f \in L^2_a(\mathbb{D}) : f = 0 \text{ on } \boldsymbol{b_{jj}}\}$, then by [3],[4] and [9], $\ker S_{\phi_{jj}} = G_{jj}L^2_a(\mathbb{D}) \cap L^2_a(\mathbb{D})$ for some inner functions $G_{jj} \in L^2_a(\mathbb{D})$ formed by the corresponding zeros $\{b^k_{jj}\}_{k=1}^{\infty}, j = 1, 2, \cdots, n$. Let G be the inner function formed by the union of zeros of the functions $G_{jj}, j = 1, 2, \cdots, n$ counting multiplicities. It is not difficult to see that $\ker S_\Phi = GL^{2,\mathbb{C}^n}_a(\mathbb{D}) \cap L^{2,\mathbb{C}^n}_a(\mathbb{D})$ as $\ker S_\Phi$ is an invariant subspace of $T_{zI_{n\times n}}$.

To prove (vi), first we shall verify that if $\phi \in L^\infty(\mathbb{D})$ and S_ϕ is a finite rank little Hankel operator on $L^2_a(\mathbb{D})$, then $\ker S_\phi = GL^2_a(\mathbb{D})$ for some inner function $G \in L^2_a(\mathbb{D})$.

Since S_ϕ is a little Hankel operator on $L^2_a(\mathbb{D})$, hence $T^*_z S_\phi = S_\phi T_z$. So $\ker S_\phi$ is invariant under multiplication by z and $\ker S_\phi$ has finite codimension since S_ϕ is of finite rank. Let $\mathbf{a} = \{a_j\}_{j=1}^N$ be the common zeroes (counting multiplicities) of functions in $\ker S_\phi$ i.e., $\mathcal{Z}(\ker S_\phi) = \{a_j\}_{j=1}^N$. Let G be the extremal function for the problem

$$\sup\{Ref^{(k)}(0) : f \in L^2_a, \|f\|_{L^2} \leq 1, f = 0 \text{ on } \mathbf{a}\},$$

where k is the multiplicity of the number of times zero appears in $\mathbf{a} = \{a_j\}_{j=1}^N (k = 0 \text{ if } 0 \notin \{a_j\}_{j=1}^N)$. It is clear from [2],[3], [4] and [9] that G satisfies the conditions (1)-(4) and G vanishes precisely on $\mathbf{a}$ in $\overline{\mathbb{D}}$ counting multiplicities. Moreover, for every function $f \in L^2_a(\mathbb{D})$ that vanishes on $\mathbf{a}= \{a_j\}_{j=1}^N$ there exists $g \in L^2_a(\mathbb{D})$ such that $f = Gg$. Hence $\ker S_\phi = GL^2_a(\mathbb{D})$.

Now suppose $\Phi \in L^\infty_{M_n}(\mathbb{D})$ and $\Phi = \begin{pmatrix} \phi_{11} & 0 & \cdots & 0 \\ 0 & \phi_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \phi_{nn} \end{pmatrix}$, $\phi_{jj} \in L^\infty(\mathbb{D})$ and S_Φ is a finite rank little Hankel operator on $L^{2,\mathbb{C}^n}_a(\mathbb{D})$. Then $S_\Phi = \begin{pmatrix} S_{\phi_{11}} & 0 & \cdots & 0 \\ 0 & S_{\phi_{22}} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & S_{\phi_{nn}} \end{pmatrix}$ and each $S_{\phi_{jj}}, 1 \leq j \leq n$ is a finite rank little Hankel operator on $L^2_a(\mathbb{D})$. From the argument above, it follows that $\ker S_{\phi_{jj}} = G_{jj}L^2_a(\mathbb{D}), 1 \leq j \leq n$ where $G_{jj} \in L^2_a(\mathbb{D})$ is an inner function and each G_{jj} vanishes on a finite set of points in $\mathbb{D}$, $\|G_{jj}\|_{L^2} = 1$ and each G_{jj} is a linear combination of the Bergman kernels and some of their derivatives and $|G_{jj}|^2 - 1$ is orthogonal to L^1_h. Let $\{\gamma_1, \gamma_2, \cdots, \gamma_l\}$ be the union of the zeros of the functions $G_{jj}, 1 \leq j \leq n$ counting multiplicities. Let $G \in L^2_a(\mathbb{D})$ be the inner function formed by the zeros $\gamma_1, \gamma_2, \cdots, \gamma_l$ taking multiplicities into account. It is not difficult to verify that $\ker S_\phi = GL^{2,\mathbb{C}^n}_a(\mathbb{D})$ and G is formed by a linear combination of (see [2],[3], [4] and [9]) the Bergman kernels and some of their derivatives and G satisfies the conditions (1)-(4). $\square$

Theorem 3.3. *If $\Psi = (\psi_{ij}) \in L^\infty_{M_n}(\mathbb{D})$ where $\psi_{ij} = 0, i \neq j$ and S_Ψ is a finite rank little Hankel operator on $L^{2,\mathbb{C}^n}_a(\mathbb{D})$ then $\Psi = \Phi + \chi$ where $\Phi = (\phi_{ij}), \phi_{ij} \in L^\infty(\mathbb{D}), 1 \leq i, j \leq n$, $\phi_{ij} = 0, i \neq j$ and each $\overline{\phi_{jj}}$ is a linear combination of* *the Bergman kernels and some of their derivatives and* $\chi = (\theta_{ij})$ *where* $\theta_{ij} \in (\overline{L_a^2})^\perp \cap L^\infty(\mathbb{D})$ *and* $\theta_{ij} = 0, i \neq j$.

Proof. Since $\Psi = (\psi_{ij})_{1\le i,j\le n} \in L^\infty_{M_n}(\mathbb{D})$ and $\psi_{ij} = 0, i \neq j$, we have $S_\Psi = \begin{pmatrix} S_{\psi_{11}} & 0 & \cdots & 0 \\ 0 & S_{\psi_{22}} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & S_{\psi_{nn}} \end{pmatrix}$. The operator S_Ψ is a finite rank little Hankel operator if and only if each $S_{\psi_{jj}}$ is a finite rank little Hankel operator on $L_a^2(\mathbb{D})$ for all $j \in \{1, 2, \cdots, n\}$. Now let $1 \le j \le n$. Since for each j, $S_{\psi_{jj}}$ is a finite rank little Hankel operator on $L_a^2(\mathbb{D})$, there exist inner functions $G_{jj} \in L_a^2(\mathbb{D})$ such that $\ker S_{\psi_{jj}} = G_{jj}L_a^2(\mathbb{D})$. Thus $\psi_{jj}G_{jj} \in (\overline{L_a^2})^\perp$. So $\langle \psi_{jj}G_{jj}, \bar{h}\rangle = 0$ for all $h \in L_a^2(\mathbb{D})$, that is, $\langle G_{jj}h, \overline{\psi_{jj}}\rangle = 0$ for all $h \in L_a^2(\mathbb{D})$ and so $\overline{\psi_{jj}} = \overline{\phi_{jj}} + \overline{\theta_{jj}}$ where $\overline{\theta_{jj}} \in (L_a^2)^\perp$, the orthogonal complement of $L_a^2(\mathbb{D})$ with respect to $L^2(\mathbb{D}, dA)$ and $\overline{\phi_{jj}} \in (G_{jj}L_a^2)^\perp$, the orthogonal complement of $G_{jj}L_a^2(\mathbb{D})$ with respect to $L_a^2(\mathbb{D})$. Suppose the function G_{jj} vanishes precisely at $\boldsymbol{d}^j = \{d_1^j, d_2^j, \cdots, d_{m_j}^j\}$, a finite number of points in $\mathbb{D}$ counting multiplicities. Since $K_{d_1^j}, K_{d_2^j}, \ldots, K_{d_{m_j}^j}$ and their derivatives (where if the point $\alpha \in \mathbb{D}$ occurs k times in $\boldsymbol{d}^j$ then we include the functions $(1-\bar{\alpha}z)^{-2}, z(1-\bar{\alpha}z)^{-3}, \ldots, z^{k-1}(1-\bar{\alpha}z)^{-k-1}$) form a basis for $(G_{jj}L_a^2(\mathbb{D}))^\perp, j \in \{1, 2, \cdots, n\}$, hence $\overline{\phi_{jj}}$ is a linear combination of the Bergman kernels and some of their derivatives and $\overline{\theta_{jj}} \in (L_a^2(\mathbb{D}))^\perp \cap L^\infty(\mathbb{D})$ since $\overline{\psi_{jj}}, \overline{\phi_{jj}} \in L^\infty(\mathbb{D})$. Thus $\Psi = \Phi + \chi$ where $\Phi = (\phi_{jj}), \chi = (\theta_{jj})$ and $\overline{\phi_{jj}}$ is a linear combination of the Bergman kernels and some of their derivatives and $\theta_{jj} \in (\overline{L_a^2(\mathbb{D})})^\perp \cap L^\infty(\mathbb{D})$. $\square$

Now let $\mathbf{b} = \{b_j\}_{j=1}^\infty$ be an infinite sequence of points in $\mathbb{D}$. Let $\mathcal{I} = I(\mathbf{b}) = \{f \in L_a^2(\mathbb{D}) : f = 0 \text{ on } \mathbf{b}\}$. Let $G_\mathbf{b}$ be the solution of the extremal problem

$$\sup\{Ref^{(n)}(0) : f \in \mathcal{I},\ \|f\|_{L^2} \le 1\}, \tag{3.1}$$

where n is the number of times zero appears in the sequence $\mathbf{b}$ (i.e., the functions in $\mathcal{I}$ have a common zero of order n at the origin). The natural question that arises at this point is to see if it is possible to construct a little Hankel operator $S_\Phi, \Phi \in L^\infty_{M_n}(\mathbb{D})$ whose kernel is $G_\mathbf{b}L_a^{2,\mathbb{C}^n}(\mathbb{D}) \cap L_a^{2,\mathbb{C}^n}(\mathbb{D})$. In the case that $\mathbf{b}= \{b_j\}_{j=1}^N$ is a finite set of points in $\mathbb{D}$, it is possible to construct a little Hankel operator $S_\Phi, \Phi \in L_{M_n^\infty(\mathbb{D})}$ such that $\ker S_\Phi = G_\mathbf{b}L_a^{2,\mathbb{C}^n}(\mathbb{D})$ as follows:

Theorem 3.4. *Let* $\mathbf{b} = (b_j)_{j=1}^N$ *be a finite set of points in* $\mathbb{D}$ *and* $\mathcal{I} = I(\mathbf{b}) = \{f \in L_a^2(\mathbb{D}) : f = 0 \text{ on } \mathbf{b}\}$ *and let* $G_\mathbf{b}$ *be the solution of the extremal problem (3.1). Let*

$$\bar{\phi} = \sum_{j=1}^{N}\sum_{\nu=0}^{m_j-1} c_{j\nu}\frac{\partial^\nu}{\partial\overline{b_j}^\nu}K_{b_j}(z),$$

where $c_{j\nu} \neq 0$ *for all* j, ν *and* m_j *is the number of times* b_j *appears in* $\mathbf{b}$. *Then* $\ker S_\Phi = G_\mathbf{b}L_a^{2,\mathbb{C}^n}(\mathbb{D})$ *where* $\Phi = (\phi_{rs})_{r,s=1}^n$ *and* $\phi_{rs} = \phi$ *if* $r = s$ *and* 0, *if* $r \neq s$.

Proof. The set of vectors $\{K_{b_1}, \ldots, \frac{\partial^{m_1-1}}{\partial \bar{b}_1^{m_1-1}} K_{b_1}, \ldots, K_{b_N}, \ldots, \frac{\partial^{m_n-1}}{\partial \bar{b}_N^{m_n-1}} K_{b_N}\}$ forms a basis [9] for $(G_{\mathbf{b}} L_a^2(\mathbb{D}))^{\perp}$. By the Gram-Schmidt orthogonalization process we can get an orthonormal basis $\{\psi_j\}_{j=1}^{l}$ for $(G_{\mathbf{b}} L_a^2(\mathbb{D}))^{\perp}$. If $\bar{\phi} \in (G_{\mathbf{b}} L_a^2)^{\perp}$ then $\langle \bar{\phi}, G_{\mathbf{b}} t\rangle = 0$ for all $t \in L_a^2(\mathbb{D})$, i.e., $\langle \bar{t}, \phi G_{\mathbf{b}}\rangle = 0$ for all $t \in L_a^2(\mathbb{D})$ and so $G_{\mathbf{b}} \in \ker S_{\phi}$. Since $\ker S_{\phi}$ is invariant under the operator of multiplication by z we have that

$$G_{\mathbf{b}} L_a^2(\mathbb{D}) \subset \ker S_{\phi}. \tag{3.2}$$

Suppose $f \in \ker S_{\phi}$; then $\langle \phi f, \bar{h}\rangle = 0$ for all $h \in L_a^2(\mathbb{D})$, so in particular $\langle \phi f, \overline{K_{b_j}}\rangle = 0$ for all $j = 1, 2, \ldots N$. Therefore, $\langle \bar{\phi} \bar{f}, K_{b_j}\rangle = 0$ for all $j = 1, 2, \ldots N$. Thus $\overline{\phi(b_j) f(b_j)} = 0$ for all $j = 1, 2, \ldots N$. Since $\overline{\phi(b_j)} \neq 0$ for all $j = 1, 2, \ldots N$, hence $\overline{f(b_j)} = 0$ for all $j = 1, 2, \ldots N$. Thus $f \in \mathcal{I}$. Since $G_{\mathbf{b}}$ is the solution of the extremal problem (3.1) therefore, $f \in G_{\mathbf{b}} L_a^2$. Hence

$$\ker S_{\phi} \subset G_{\mathbf{b}} L_a^2. \tag{3.3}$$

From (3.2) and (3.3), $\ker S_{\phi} = G_{\mathbf{b}} L_a^2(\mathbb{D}) = \mathcal{I}$. Now let $\Phi = (\phi_{rs})_{r,s=1}^{n}$ where $\phi_{rs} = \phi$ if $r = s$ and 0, if $r \neq s$. It is not difficult now to verify that $\ker S_{\Phi} = G_{\mathbf{b}} L_a^{2,\mathbb{C}^n}(\mathbb{D})$.

□

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