Khayyam J. Math. 2 (2016), no. 1, 1–5
DOI: 10.22034/kjm.2016.13993

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ON THE CHEBYSHEV POLYNOMIAL BOUNDS FOR CLASSES OF UNIVALENT FUNCTIONS

ŞAHSENE ALTINKAYA*,[1] AND SIBEL YALÇIN[2]

Communicated by J. Brzdęk

Abstract. In this work, by considering a general subclass of univalent functions and using the Chebyshev polynomials, we obtain coefficient expansions for functions in this class.

1. Introduction and definitions

Let D be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, and let A be the class of all analytic functions on D, satisfying the conditions

$$f(0) = 0 \text{ and } \quad f'(0) = 1.$$

Then each function f in A has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, by S we shall denote the class of all functions in A which are univalent on D.

If the functions f and g are analytic on D, then f is said to be subordinate to g, written as

$$f(z) \prec g(z), \qquad (z \in D)$$

if there exists a Schwartz function $w(z)$, analytic on D, with

$$w(0) = 0 \text{ and } \quad |w(z)| < 1 \qquad (z \in D)$$

Date: Received: 04 November 2015; Accepted: 16 February 2016.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C50.

Key words and phrases. Chebyshev polynomials; Analytic and univalent functions; coefficient bounds; subordination.

such that

$$f(z) = g(w(z)) \qquad (z \in D).$$

Denote by K the subclass of S consisting of convex functions, so that $f \in K$ if and only if for $z \in D$

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0.$$

The Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for normalized univalent functions

$$f(z) = z + a_2 z^2 + \cdots$$

is well known for its rich history in the theory of geometric functions. Its origin was in the disproof by Fekete and Szegö of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see [3]).

Chebyshev polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. The majority of books and research papers dealing with specific orthogonal polynomials of Chebyshev family, contain mainly results of Chebyshev polynomials of first and second kinds $T_n(x)$ and $U_n(x)$ and their numerous uses in different applications, see for example, Doha [2] and Mason [4].

The Chebyshev polynomials of the first and second kinds are well known. In the case of a real variable x in $(-1, 1)$, they are defined by

$$T_n(x) = \cos n\theta,$$

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},$$

where the subscript n denotes the polynomial degree and where $x = \cos\theta$.

Definition 1.1. A function $f \in A$ is said to be in the class $K(\lambda, t)$, $\lambda \geq 0$ and $t \in \left(\frac{1}{2}, 1\right]$, if the following subordination hold

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec H(z,t) := \frac{1}{1 - 2tz + z^2} \qquad (z \in D). \tag{1.2}$$

We note that if $t = \cos\alpha$, $\alpha \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, then

$$\begin{aligned} H(z,t) &= \frac{1}{1 - 2tz + z^2} \\ &= 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} z^n \quad (z \in D). \end{aligned}$$

Thus

$$H(z,t) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + \cdots \quad (z \in D).$$

Following [5], we write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad \big(z \in D, \;\; t \in (-1,1)\big),$$

where $U_{n-1} = \dfrac{\sin(n \arccos t)}{\sqrt{1-t^2}}$ $(n \in \mathbb{N})$ are the Chebyshev polynomials of the second kind. Also it is known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$\begin{aligned} &U_1(t) = 2t, \\ &U_2(t) = 4t^2 - 1, \\ &U_3(t) = 8t^3 - 4t, \\ &\vdots \end{aligned} \tag{1.3}$$

The Chebyshev polynomials $T_n(t),\ t \in [-1, 1]$, of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1-tz}{1-2tz+z^2} \quad (z \in D).$$

However, the Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$ are well connected by the following relationships

$$\begin{aligned} &\frac{dT_n(t)}{dt} = nU_{n-1}(t), \\ &T_n(t) = U_n(t) - tU_{n-1}(t), \\ &2T_n(t) = U_n(t) - U_{n-2}(t). \end{aligned}$$

In this paper, motivated by the earlier work of Dziok et al. [1], we use the Chebyshev polynomial expansions to provide estimates for the initial coefficients of univalent functions in $K(\lambda, t)$.

2. Coefficient bounds for the function class $K(\lambda, t)$

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class $K(\lambda, t)$. Then*

$$|a_2| \le \frac{2t}{1+\lambda}$$

and

$$|a_3| \le \frac{(2\lambda^2 + 10\lambda + 4)\, t^2}{(1+2\lambda)(1+\lambda)^2} + \frac{t}{1+2\lambda} - \frac{1}{2(1+2\lambda)}.$$

Proof. Let $f \in K(\lambda, t)$. From (1.2), we have

$$(1-\lambda) \frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + U_1(t) w(z) + U_2(t) w^2(z) + \cdots, \tag{2.1}$$

for some analytic function w such that $w(0) = 0$ and $|w(z)| < 1$ for all $z \in D$. From the equalities (2.2) and (2.3), we obtain that

$$(1-\lambda) \frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + U_1(t) c_1 z + \left[U_1(t) c_2 + U_2(t) c_1^2\right] z^2 + \cdots. \tag{2.2}$$

It is fairly well-known that if $|w(z)| = \left|c_1 z + c_2 z^2 + c_3 z^3 + \cdots\right| < 1,\ z \in D$, then

$$|c_j| \le 1, \text{ for all } j \in \mathbb{N}; \tag{2.3}$$

and

$$\left|c_2 - \mu c_1^2\right| \le \max\{1, |\mu|\}, \quad \text{for all} \quad \mu \in \mathbb{R}. \tag{2.4}$$

It follows from (2.2) that

$$(1+\lambda) a_2 = U_1(t) c_1, \tag{2.5}$$

$$2(1+2\lambda) a_3 - (1+3\lambda) a_2^2 = U_1(t) c_2 + U_2(t) c_1^2. \tag{2.6}$$

From (1.3) and (2.5) we obtain

$$|a_2| \leq \frac{2t}{1+\lambda}. \tag{2.7}$$

Next, in order to find the bound on $|a_3|$, by using (2.5) in (2.6), we obtain

$$2\left(1+2\lambda\right)a_3 = U_1(t)c_2 + \left\{U_2(t) + \frac{(1+3\lambda)}{(1+\lambda)^2}U_1^2(t)\right\}c_1^2 \tag{2.8}$$

Then, in view of (1.3) and (2.3) , we have from (2.8)

$$|a_3| \leq \frac{\left(2\lambda^2+10\lambda+4\right)t^2}{(1+2\lambda)\left(1+\lambda\right)^2} + \frac{t}{1+2\lambda} - \frac{1}{2\left(1+2\lambda\right)}.$$

□

3. Fekete-Szegö inequalities for the function class $K(\lambda, t)$

Theorem 3.1. *Let f given by (1.1) be in the class $K(\lambda, t)$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{t}{1+2\lambda}; & \mu \in [\mu_1, \mu_2] \\[2ex] \dfrac{t}{1+2\lambda}\left|\frac{4t^2-1}{2t} + \frac{2(1+3\lambda)}{(1+\lambda)^2}t - 4\mu\frac{(1+2\lambda)}{(1+\lambda)^2}t\right|; & \mu \notin [\mu_1, \mu_2] \end{cases}$$

where

$$\mu_1 = \frac{4t^2\left(\lambda^2+5\lambda+2\right)-(1+\lambda)^2(1+2t)}{8(1+2\lambda)t^2}, \quad \mu_2 = \frac{4t^2\left(\lambda^2+5\lambda+2\right)-(1+\lambda)^2(1-2t)}{8(1+2\lambda)t^2}.$$

Proof. From (2.5) and (2.8)

$$\left|a_3 - \mu a_2^2\right| = \frac{U_1(t)}{2\left(1+2\lambda\right)}\left|c_2 + \left\{\frac{U_2(t)}{U_1(t)} + \frac{1+3\lambda}{(1+\lambda)^2}U_1(t) - 2\mu\frac{\left(1+2\lambda\right)U_1(t)}{(1+\lambda)^2}\right\}c_1^2\right|.$$

Then, in view of (2.4), we conclude that

$$\left|a_3 - \mu a_2^2\right| \leq \frac{U_1(t)}{2\left(1+2\lambda\right)}\max\left\{1, \left|\frac{U_2(t)}{U_1(t)} + \frac{1+3\lambda}{(1+\lambda)^2}U_1(t) - 2\mu\frac{\left(1+2\lambda\right)U_1(t)}{(1+\lambda)^2}\right|\right\}. \tag{3.1}$$

Finally, by using (1.3) in (3.1)

$$\left|a_3 - \mu a_2^2\right| \leq \frac{t}{1+2\lambda}\max\left\{1, \left|\frac{4t^2-1}{2t} + \frac{2\left(1+3\lambda\right)}{(1+\lambda)^2}t - 4\mu\frac{\left(1+2\lambda\right)}{(1+\lambda)^2}t\right|\right\}.$$

Because $t > 0$, we have

$$\begin{aligned} &\left|\tfrac{4t^2-1}{2t} + \tfrac{2(1+3\lambda)}{(1+\lambda)^2}t - 4\mu\tfrac{(1+2\lambda)}{(1+\lambda)^2}t\right| \leq 1 \\ &\Leftrightarrow \left\{\tfrac{4t^2\left(\lambda^2+5\lambda+2\right)-(1+\lambda)^2(1+2t)}{8(1+2\lambda)t^2} \leq \mu \leq \tfrac{4t^2\left(\lambda^2+5\lambda+2\right)-(1+\lambda)^2(1-2t)}{8(1+2\lambda)t^2}\right\} \\ &\Leftrightarrow \mu_1 \leq \mu \leq \mu_2 \end{aligned}$$

□

Taking $\lambda = 1$ in the above theorem we get the following corollary.

Corollary 3.2. *If* $f \in K(t)$, *then*

$$|a_2| \le t;$$

$$|a_3| \le \frac{4t^2}{3} + \frac{t}{3} - \frac{1}{6};$$

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{t}{3}; & \mu \in [\mu_1, \mu_2] \\ \left|\frac{8t^2 - 1 - 6\mu t^2}{6}\right|; & \mu \notin [\mu_1, \mu_2] \end{cases}$$

References

1. J. Dziok, R.K. Raina and J. Sokol, *Application of Chebyshev polynomials to classes of analytic functions*, C. R. Acad. Sci. Paris, Ser. I, **353** (2015), 433-438.
2. E.H. Doha, *The first and second kind Chebyshev coefficients of the moments of the general-order derivative of an infinitely differentiable function*, Int. J. Comput. Math. **51** (1994), 21-35.
3. M. Fekete and G. Szegö, *Eine bemerkung über ungerade schlichte funktionen*, J. Lond. Math. Soc. **2** (1933), 85-89.
4. J.C. Mason, *Chebyshev polynomials approximations for the L-membrane eigenvalue problem*, SIAM J. Appl. Math. **15** (1967), 172-186.
5. E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes of Analytic Functions; with an Account of the Principal Transcendental Functions*, 4th ed., Cambridge University Press, 1963.

[1] Department of Mathematics, Faculty of Arts and Science, Uludag University, Bursa, Turkey.
E-mail address: `sahsene@uludag.edu.tr`

[2] Department of Mathematics, Faculty of Arts and Science, Uludag University, Bursa, Turkey.
E-mail address: `syalcin@uludag.edu.tr`