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ZYGMUND-TYPE INEQUALITIES FOR AN OPERATOR PRESERVING INEQUALITIES BETWEEN POLYNOMIALS

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Abstract. In this paper, we present certain new L_p inequalities for $\mathcal{B}_n$-operators which include some known polynomial inequalities as special cases.

1. Introduction and statement of results

Let $\mathscr{P}_n$ denote the space of all complex polynomials $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n. For $P \in \mathscr{P}_n$, define

$$\|P(z)\|_0 := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\left|P(e^{i\theta})\right| d\theta\right\},$$

$$\|P(z)\|_p := \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}, \ 0<p<\infty,$$

$$\|P(z)\|_\infty := \max_{|z|=1} |P(z)|, \quad m := \min_{|z|=1} |P(z)|,$$

and denote for any complex function $\psi : \mathbb{C} \to \mathbb{C}$ the composite function of P and ψ, defined by $(P \circ \psi)(z) := P(\psi(z)) \quad (z \in \mathbb{C})$, as $P \circ \psi$.

If $P \in \mathscr{P}_n$, then

$$\|P'(z)\|_p \le n\,\|P(z)\|_p, \quad p \ge 1 \tag{1.1}$$

and

$$\|P(Rz)\|_p \le R^n\,\|P(z)\|_p, \ \ R>1, \ \ p>0. \tag{1.2}$$

Inequality (1.1) was found out by Zygmund [20] whereas inequality (1.2) is a simple consequence of a result of Hardy [8]. Arestov [2] proved that (1.1) remains true for $0<p<1$ as well. For $p=\infty$, the inequality (1.1) is due to Bernstein

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(for reference, see [11, 15, 18]) whereas the case $p=\infty$ of inequality (1.2) is a simple consequence of the maximum modulus principle (see [11, 12, 15]). Both the inequalities (1.1) and (1.2) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z|<1$. In fact, if $P\in\mathscr{P}_n$ and $P(z)\neq 0$ in $|z|<1$, then inequalities (1.1) and (1.2) can be respectively replaced by

$$\|P'(z)\|_p \leq n\frac{\|P(z)\|_p}{\|1+z\|_p}, \quad p\geq 0 \tag{1.3}$$

and

$$\|P(Rz)\|_p \leq \frac{\|R^n z+1\|_p}{\|1+z\|_p}\|P(z)\|_p, \quad R>1, \quad p>0. \tag{1.4}$$

Inequality (1.3) is due to De-Bruijn [7](see also [3]) for $p\geq 1$. Rahman and Schmeisser [1] extended it for $0<p<1$, whereas the inequality (1.4) was proved by Boas and Rahman [6] for $p\geq 1$ and later it was extended for $0<p<1$ by Rahman and Schmeisser [14]. For $p=\infty$, the inequality (1.3) was conjectured by Erdös and later verified by Lax [9] whereas inequality (1.4) was proved by Ankeny and Rivlin [1].

As a compact generalization of inequalities (1.3) and (1.4), Aziz and Rather [5] proved that if $P\in\mathscr{P}_n$ and $P(z)$ does not vanish in $|z|<1$, then for $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\leq 1$, $|\beta|\leq 1$, $R>r\geq 1$ and $p>0$,

$$\|P(Rz)+\phi_n(R,r,\alpha,\beta)P(rz)\|_p \leq \frac{C_p}{\|1+z\|_p}\|P(z)\|_p \tag{1.5}$$

where

$$C_p=\left\|\left(R^n+\phi_n(R,r,\alpha,\beta)r^n\right)z+\left(1+\phi_n(R,r,\alpha,\beta)\right)\right\|_p \tag{1.6}$$

and

$$\phi_n(R,r,\alpha,\beta)=\beta\left\{\left(\frac{R+1}{r+1}\right)^n-|\alpha|\right\}-\alpha. \tag{1.7}$$

If we take $\beta=0$, $\alpha=1$ and $r=1$ in (1.5) and divide two sides of (1.5) by $R-1$ then make $R\to 1$, we obtain inequality (1.3). Whereas inequality (1.4) is obtained from (1.5) by taking $\alpha=\beta=0$.

Rahman [13] (see also Rahman and Schmeisser [15, p. 538]) introduced a class $\mathcal{B}_n$ of operators B that maps $P\in\mathscr{P}_n$ into itself. That is, the operator B carries $P\in\mathscr{P}_n$ into a polynomial

$$B[P](z) := \lambda_0 P(z)+\lambda_1\left(\frac{nz}{2}\right)\frac{P'(z)}{1!}+\lambda_2\left(\frac{nz}{2}\right)^2\frac{P''(z)}{2!} \tag{1.8}$$

where λ_0,λ_1 and λ_2 are such that all the zeros of

$$u(z):=\lambda_0+C(n,1)\lambda_1 z+C(n,2)\lambda_2 z^2, \quad C(n,r)=n!/r!(n-r)!,$$

lie in the half plane

$$|z|\leq|z-n/2|. \tag{1.9}$$

While extending Bernstein type inequalities to $\mathcal{B}_n$ operators, they [13] proved that if $P\in\mathscr{P}_n$ and $P(z)$ does not vanish in $|z|<1$, then

$$|B[P\circ\sigma](z)|\leq\frac{1}{2}\left\{R^n|\Lambda_n|+|\lambda_0|\right\}\|P(z)\|_\infty \quad for \quad |z|=1, \tag{1.10}$$

(see [13, Inequalities (5.2) and (5.3)]) where $\sigma(z) = Rz$, $R \geq 1$ and

$$\Lambda_n := \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}. \tag{1.11}$$

As an extension of inequality (1.10) to L_p-norm, recently W.M. Shah and A. Liman [19] while seeking the desired extension, have made an incomplete attempt [19, Theorem 2] by claiming to have proved that if $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for each $R \geq 1$ and $p \geq 1$,

$$\|B[P \circ \sigma](z)\|_p \leq \frac{R^n|\Lambda_n| + |\lambda_0|}{\|1+z\|_p} \|P(z)\|_p\,, \tag{1.12}$$

where $B \in B_n$ and $\sigma(z) = Rz$ and Λ_n is defined by (1.11).

Rather and Shah [17] pointed an error in the proof of (1.12), they not only provided a correct proof but also extended it for $0 \leq p < 1$ as well. They proved:

Theorem A. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish for $|z| < 1$, then for $0 \leq p < \infty$ and $R > 1$,*

$$\|B[P \circ \sigma](z)\|_p \leq \frac{\|R^n\Lambda_n z + \lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p\,, \tag{1.13}$$

$B \in \mathcal{B}_n$, $\sigma(z) = Rz$ and Λ_n is defined by (1.11). The result is sharp as shown by $P(z) = az^n + b$, $|a| = |b| = 1$.

Recently, Rather and Suhail Gulzar [16] obtained the following result which is a generalization of Theorem A.

Theorem B. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish for $|z| < 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $0 \leq p < \infty$ and $R > 1$,*

$$\|B[P \circ \sigma](z) - \alpha B[P](z)\|_p \leq \frac{\|(R^n - \alpha)\Lambda_n z + (1-\alpha)\lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p\,, \tag{1.14}$$

where $B \in \mathcal{B}_n$, $\sigma(z) = Rz$ and Λ_n is defined by (1.11). The result is best possible and equality in (1.14) holds for $P(z) = az^n + b$, $|a| = |b| = 1$.

If we take $\alpha = 0$ in Theorem B, we obtain Theorem A.

In this paper, we investigate the dependence of

$$\|B[P \circ \sigma](z) + \phi_n\,(R, r, \alpha, \beta)\, B[P \circ \rho](z)\|_p$$

on $\|P(z)\|_p$ for $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$, $0 \leq p < \infty$, $\sigma(z) := Rz$, $\rho(z) := rz$ and $\phi_n\,(R, r, \alpha, \beta)$ is given by (1.7), and establish certain generalized L_p-mean extensions of the inequality (1.10) for $0 \leq p < \infty$ and also a generalization of (1.5). In this direction, we first present the following result which is a compact generalization of the inequalities (1.3), (1.4), (1.5) and (1.10) for $0 \leq p < 1$ as well.

Theorem 1.1. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $0 \leq p < \infty$,*

$$\begin{aligned} &\|B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)\|_p \\ &\leq \frac{\big\|\big(R^n + \phi_n(R, r, \alpha, \beta) r^n\big)\Lambda_n z + \big(1 + \phi_n(R, r, \alpha, \beta)\big)\lambda_0\big\|_p}{\|1+z\|_p} \|P(z)\|_p \end{aligned} \tag{1.15}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, Λ_n and $\phi_n(R, r, \alpha, \beta)$ are defined by(1.7) *and* (1.11) *respectively. The result is best possible and equality in* (1.15) *holds for* $P(z) = az^n + b, |a| = |b| \neq 0$

Remark 1.2. If we take $\lambda_1 = \lambda_2 = 0$ in (1.15), we obtain inequality (1.5).

For $\beta = 0$, inequality (1.15) reduces the following result.

Corollary 1.3. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $0 \leq p < \infty$,*

$$\|B[P \circ \sigma](z) - \alpha B[P \circ \rho](z)\|_p \leq \frac{\|(R^n - \alpha r^n)\Lambda_n z + (1-\alpha)\lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p \tag{1.16}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, $\rho(z) := rz$ and Λ_n is defined by (1.11)*. The result is best possible and equality in* (1.16) *holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$.*

Remark 1.4. For taking $\alpha = 0$ in (1.16), we obtain Theorem A and for $r = 1$ in (1.16), we get Theorem B.

Instead of proving Theorem 1.1, we prove the following more general result which includes Theorem 1.1 as a special case.

Theorem 1.5. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $\alpha, \beta, \delta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $|\delta| \leq 1$, $R > r \geq 1$ and $0 \leq p < \infty$,*

$$\left\| B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z) + \delta \frac{\Big(|R^n + \phi_n(R, r, \alpha, \beta) r^n| \, |\Lambda_n| - |1 + \phi_n(R, r, \alpha, \beta)| \, |\lambda_0|\Big) m}{2} \right\|_p \leq \frac{\|(R^n + \phi_n(R, r, \alpha, \beta) r^n) \Lambda_n z + (1 + \phi_n(R, r, \alpha, \beta)) \lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p \tag{1.17}$$

where $B \in B_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, $m = \min_{|z|=1} |P(z)|$ and $\phi_n(R, r, \alpha, \beta)$, Λ_n are defined by(1.7) *and* (1.11)*, respectively. The result is best possible and equality in* (1.15) *holds for $P(z) = az^n + b, |a| = |b| \neq 0$.*

Remark 1.6. For $\delta = 0$ in (1.17), we get Theorem 1.1.

The next corollary which is a generalization of (1.5) follows by taking $\lambda_1 = \lambda_2 = 0$ in (1.17).

Corollary 1.7. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $\alpha, \beta, \delta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $|\delta| \leq 1$, $R > r \geq 1$ and $0 \leq p < \infty$,*

$$\left\| P(Rz) + \phi_n(R, r, \alpha, \beta) P(rz) + \delta \frac{\Big(|R^n + \phi_n(R, r, \alpha, \beta) r^n| - |1 + \phi_n(R, r, \alpha, \beta)|\Big) m}{2} \right\|_p$$

$$\leq \frac{\left\|\left(R^n+\phi_n(R,r,\alpha,\beta)r^n\right)z+\left(1+\phi_n(R,r,\alpha,\beta)\right)\right\|_p}{\|1+z\|_p}\|P(z)\|_p \tag{1.18}$$

where $m=\min_{|z|=1}|P(z)|$ *and* $\phi_n(R,r,\alpha,\beta)$ *is defined by*(1.7). *The result is best possible and equality in* (1.18) *holds for* $P(z)=az^n+b, |a|=|b|\neq 0$.

2. Lemmas

For the proofs of these theorems, we need the following lemmas. The first Lemma is easy to prove.

Lemma 2.1. *If* $P\in\mathscr{P}_n$ *and* $P(z)$ *has all its zeros in* $|z|\leq 1$, *then for every* $R\geq r\geq 1$ *and* $|z|=1$,

$$|P(Rz)|\geq\left(\frac{R+1}{r+1}\right)^n|P(rz)|.$$

The following Lemma follows from [10, Corollary 18.3, p. 65].

Lemma 2.2. *If all the zeros of polynomial* $P\in\mathscr{P}_n$ *lie in* $|z|\leq 1$, *then all the zeros of the polynomial* $B[P](z)$ *also lie in* $|z|\leq 1$.

Lemma 2.3. *If* $F\in\mathscr{P}_n$ *has all its zeros in* $|z|\leq 1$ *and* $P(z)$ *is a polynomial of degree at most* n *such that*

$$|P(z)|\leq|F(z)| \text{ for } |z|=1,$$

then for every $\alpha,\beta\in\mathbb{C}$ *with* $|\alpha|\leq 1$, $|\beta|\leq 1$, $R>r\geq 1$, *and* $|z|\geq 1$,

$$\begin{aligned}&|B[P\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](z)|\\&\qquad\leq|B[F\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)B[F\circ\rho](z)|\end{aligned} \tag{2.1}$$

where $B\in\mathcal{B}_n$, $\sigma(z):=Rz$, $\rho(z):=rz$, Λ_n *and* $\phi_n(R,r,\alpha,\beta)$ *are defined by* (1.11) *and* (1.7) *respectively.*

Proof. Since the polynomial $F(z)$ of degree n has all its zeros in $|z|\leq 1$ and $P(z)$ is a polynomial of degree at most n such that

$$|P(z)|\leq|F(z)| \quad\text{for}\quad |z|=1, \tag{2.2}$$

therefore, if $F(z)$ has a zero of multiplicity s at $z=e^{i\theta_0}$, then $P(z)$ has a zero of multiplicity at least s at $z=e^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then the inequality (2.1) is obvious. We now assume that $P(z)/F(z)$ is not a constant, so that by the maximum modulus principle, it follows that

$$|P(z)|<|F(z)| \text{ for } |z|<1\ .$$

Suppose $F(z)$ has m zeros on $|z|=1$ where $0\leq m\leq n$, so that we can write

$$F(z)=F_1(z)F_2(z)$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z|=1$ and $F_2(z)$ is a polynomial of degree exactly $n-m$ having all its zeros in $|z|<1$. This implies with the help of inequality (2.2) that

$$P(z)=P_1(z)F_1(z)$$

where $P_1(z)$ is a polynomial of degree at most $n-m$. Now, from inequality (2.2), we get

$$|P_1(z)| \leq |F_2(z)| \text{ for } |z| = 1$$

where $F_2(z) \neq 0$ for $|z| = 1$. Therefore for every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, a direct application of Rouche's theorem shows that the zeros of the polynomial $P_1(z) - \lambda F_2(z)$ of degree $n - m \geq 1$ lie in $|z| < 1$. Hence the polynomial

$$f(z) = F_1(z)\big(P_1(z) - \lambda F_2(z)\big) = P(z) - \lambda F(z)$$

has all its zeros in $|z| \leq 1$ with at least one zero in $|z| < 1$, so that we can write

$$f(z) = (z - te^{i\delta})H(z)$$

where $t < 1$ and $H(z)$ is a polynomial of degree $n-1$ having all its zeros in $|z| \leq 1$. Applying Lemma 2.1 to the polynomial $f(z)$ with $k = 1$, we obtain for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned}
|f(Re^{i\theta})| =& |Re^{i\theta} - te^{i\delta}||H(Re^{i\theta})| \\
\geq& |Re^{i\theta} - te^{i\delta}| \left(\frac{R+1}{r+1}\right)^{n-1} |H(re^{i\theta})| \\
=& \left(\frac{R+1}{r+1}\right)^{n-1} \frac{|Re^{i\theta} - te^{i\delta}|}{|re^{i\theta} - te^{i\delta}|} |(re^{i\theta} - te^{i\delta})H(re^{i\theta})| \\
\geq& \left(\frac{R+1}{r+1}\right)^{n-1} \left(\frac{R+t}{r+t}\right) |f(re^{i\theta})|.
\end{aligned}$$

This implies for $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\left(\frac{r+t}{R+t}\right) |f(Re^{i\theta})| \geq \left(\frac{R+1}{r+1}\right)^{n-1} |f(re^{i\theta})|. \tag{2.3}$$

Since $R > r \geq 1 > t$ so that $f(Re^{i\theta}) \neq 0$ for $0 \leq \theta < 2\pi$ and $\frac{1+r}{1+R} > \frac{r+t}{R+t}$, from inequality (2.3), we obtain $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$|f(Re^{i\theta})| > \left(\frac{R+1}{r+1}\right)^{n} |f(re^{i\theta})|. \tag{2.4}$$

Equivalently,

$$|f(Rz)| > \left(\frac{R+1}{r+1}\right)^{n} |f(rz)|$$

for $|z| = 1$ and $R > r \geq 1$. Hence for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $R > r \geq 1$, we have

$$\begin{aligned}
|f(Rz) - \alpha f(rz)| &\geq |f(Rz)| - |\alpha||f(rz)| \\
&> \left\{\left(\frac{R+1}{r+1}\right)^{n} - |\alpha|\right\} |f(rz)|, \quad |z| = 1.
\end{aligned}$$

Also, inequality (2.4) can be written in the form

$$|f(re^{i\theta})| < \left(\frac{r+1}{R+1}\right)^{n} |f(Re^{i\theta})| \tag{2.5}$$

for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$. Since $f(Re^{i\theta}) \neq 0$ and $\left(\frac{r+1}{R+1}\right)^n < 1$, from inequality (2.5), we obtain for $0 \leq \theta < 2\pi$ and $R > r \geq 1$,

$$|f(re^{i\theta})| < |f(Re^{i\theta})|.$$

Equivalently,

$$|f(rz)| < |f(Rz)| \text{ for } |z| = 1.$$

Since all the zeros of $f(Rz)$ lie in $|z| \leq (1/R) < 1$, a direct application of Rouche's theorem shows that the polynomial $f(Rz) - \alpha f(rz)$ has all its zeros in $|z| < 1$ for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. Applying Rouche's theorem again, it follows from (2.4) that for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$, all the zeros of the polynomial

$$\begin{aligned}
T(z) =& f(Rz) - \alpha f(rz) + \beta \left\{ \left(\frac{R+1}{r+1} \right)^n - |\alpha| \right\} f(rz) \\
&= f(Rz) + \phi_n(R, r, \alpha, \beta) f(rz) \\
&= \big(P(Rz) - \lambda F(Rz)\big) + \phi_n(R, r, \alpha, \beta) \big(P(rz) - \lambda F(rz)\big) \\
&= \big(P(Rz) + \phi_n(R, r, \alpha, \beta) P(rz)\big) - \lambda \big(F(Rz) + \phi_n(R, r, \alpha, \beta) F(rz)\big)
\end{aligned}$$

lie in $|z| < 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Using Lemma 2.2 and the fact that B is a linear operator, we conclude that all the zeros of polynomial

$$\begin{aligned}
W(z) &= B[T](z) \\
&= \big(B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)\big) \\
&\qquad - \lambda \big(B[F \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z)\big)
\end{aligned}$$

also lie in $|z| < 1$ for every λ with $|\lambda| > 1$. This implies

$$\begin{aligned}
&|B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)| \\
&\qquad \leq |B[F \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z)|
\end{aligned} \tag{2.6}$$

for $|z| \geq 1$ and $R > r \geq 1$. If inequality (2.6) is not true, then there exists a point $z = z_0$ with $|z_0| \geq 1$ such that

$$|B[P \circ \sigma](z_0) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z_0)| > |B[F \circ \sigma](z_0) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z_0)|.$$

But all the zeros of $F(Rz)$ lie in $|z| < 1$, therefore, it follows (as in case of $f(z)$) that all the zeros of $F(Rz) + \phi_n(R, r, \alpha, \beta) F(rz)$ lie in $|z| < 1$. Hence by Lemma 2.2, all the zeros of $B[F \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z)$ also lie in $|z| < 1$, which shows that

$$B[F \circ \sigma](z_0) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z_0) \neq 0.$$

We take

$$\lambda = \frac{B[P \circ \sigma](z_0) + \phi_n(R, r, \alpha, \beta) B[P \circ \rho](z_0)}{B[F \circ \sigma](z_0) + \phi_n(R, r, \alpha, \beta) B[F \circ \rho](z_0)},$$

then λ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of λ, we obtain $W(z_0) = 0$. This contradicts the fact that all the zeros of $W(z)$ lie in $|z| < 1$. Thus (2.6) holds and this completes the proof of Lemma 2.3. □

Lemma 2.4. *If $P \in \mathscr{P}_n$ and $P(z)$ has all its zeros in $|z| \le 1$, then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \le 1, |\beta| \le 1$ and $|z| \ge 1$,*

$$\begin{aligned} &|B[P\circ\sigma](z) + \phi_n(R,r,\alpha,\beta)\, B[P\circ\rho](z)| \\ &\qquad\qquad \ge |R^n + \phi_n(R,r,\alpha,\beta)\, r^n|\, |\Lambda_n|\, |z|^n m \end{aligned} \tag{2.7}$$

where $m = \min_{|z|=1}|P(z)|$, $B \in \mathcal{B}_n$, $\sigma(z) = Rz$, $\rho(z) = rz$, Λ_n and $\phi_n(R,r,\alpha,\beta)$ are defined by (1.11) and (1.7), respectively.

Proof. By hypothesis, all the zeros of $P(z)$ lie in $|z| \le 1$ and

$$m|z|^n \le |P(z)| \quad \text{for} \quad |z| = 1.$$

We first show that the polynomial $g(z) = P(z) - \lambda m z^n$ has all its zeros in $|z| \le 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. This is obvious if $m = 0$, that is if $P(z)$ has a zero on $|z| = 1$. Henceforth, we assume $P(z)$ has all its zeros in $|z| < 1$, then $m > 0$ and it follows by Rouche's theorem that the polynomial $g(z)$ has all its zeros in $|z| < 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Proceeding similarly as in the proof of Lemma 2.3, we obtain that for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \le 1, |\beta| \le 1$ and $R > r \ge 1$, all the zeros of the polynomial

$$\begin{aligned} H(z) =& g(Rz) - \alpha g(rz) + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\} g(rz) \\ =& g(Rz) + \phi_n(R,r,\alpha,\beta)\, g(rz) \\ =& \big(P(Rz) - \lambda R^n z^n m\big) + \phi_n(R,r,\alpha,\beta)\big(P(rz) - \lambda r^n z^n m\big) \\ =& \big(P(Rz) + \phi_n(R,r,\alpha,\beta)\, P(rz)\big) - \lambda\big(R^n + \phi_n(R,r,\alpha,\beta)\, r^n\big) m z^n \end{aligned}$$

lie in $|z| < 1$. Applying Lemma 2.1 to $H(z)$ and noting that B is a linear operator, it follows that all the zeros of polynomial

$$\begin{aligned} B[H](z) =& \{B[P\circ\sigma](z) + \phi_n(R,r,\alpha,\beta)\, B[P\circ\rho](z)\} \\ &\qquad - \lambda\big(R^n + \phi_n(R,r,\alpha,\beta)\, r^n\big) m B[z^n] \end{aligned} \tag{2.8}$$

lie in $|z| < 1$. This gives

$$\begin{aligned} &|B[P\circ\sigma](z) + \phi_n(R,r,\alpha,\beta)\, B[P\circ\rho](z)| \\ &\qquad \ge |R^n + \phi_n(R,r,\alpha,\beta)\, r^n|\, |\Lambda_n|\, |z|^n m \quad \text{for} \quad |z| \ge 1. \end{aligned} \tag{2.9}$$

If (2.9) is not true, then there is point w with $|w| \ge 1$ such that

$$|B[P\circ\sigma](w) + \phi_n(R,r,\alpha,\beta)\, B[P\circ\rho](w)| < |R^n + \phi_n(R,r,\alpha,\beta)\, r^n|\, |\Lambda_n|\, |w|^n m.$$

We choose

$$\lambda = \frac{B[P\circ\sigma](w) + \phi_n(R,r,\alpha,\beta)\, B[P\circ\rho](w)}{R^n + \phi_n(R,r,\alpha,\beta)\, r^n|\, |\Lambda_n|\, |w|^n m},$$

then clearly $|\lambda| < 1$ and with this choice of λ, from (2.8), we get $B[H](w) = 0$ with $|w| \ge 1$. This is clearly a contradiction to the fact that all the zeros of $H(z)$ lie in $|z| < 1$. Thus for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \le 1$, $|\beta| \le 1$,

$$|B[P\circ\sigma](z) + \phi_n(R,r,\alpha,\beta)\, B[P\circ\rho](z)| \ge |R^n + \phi_n(R,r,\alpha,\beta)\, r^n|\, |\Lambda_n|\, |z|^n m$$

for $|z| \ge 1$ and $R > r \ge 1$. $\square$

Lemma 2.5. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned}|B[P \circ \sigma](z) &+ \phi_n\left(R, r, \alpha, \beta\right) B[P \circ \rho](z)| \\ &\leq |B[P^* \circ \sigma](z) + \phi_n\left(R, r, \alpha, \beta\right) B[P^* \circ \rho](z)|\end{aligned}$$

where $P^*(z) := z^n \overline{P(1/\overline{z})}$, $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, and $\phi_n\left(R, r, \alpha, \beta\right)$ is defined by (1.7).

Proof. By hypothesis the polynomial $P(z)$ of degree n does not vanish in $|z| < 1$, therefore, all the zeros of the polynomial $P^*(z) = z^n \overline{P(1/\overline{z})}$ of degree n lie in $|z| \leq 1$. Applying Lemma 2.3 with $F(z)$ replaced by $P^*(z)$, it follows that

$$\begin{aligned}&|B[P \circ \sigma](z) + \phi_n\left(R, r, \alpha, \beta\right) B[P \circ \rho](z)| \\ &\qquad\qquad \leq |B[P^* \circ \sigma](z) + \phi_n\left(R, r, \alpha, \beta\right) B[P^* \circ \rho](z)|\end{aligned}$$

for $|z| \geq 1, |\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$. This proves the Lemma 2.5. ◻

Lemma 2.6. *If $P \in \mathscr{P}_n$ and $P(z)$ has no zeros in $|z| < 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned}|B[P \circ \sigma](z) &+ \phi_n\left(R, r, \alpha, \beta\right) B[P \circ \rho](z)| \\ &\leq |B[P^* \circ \sigma](z) + \phi_n\left(R, r, \alpha, \beta\right) B[P^* \circ \rho](z)| \\ &\quad - \Big(|R^n + \phi_n\left(R, r, \alpha, \beta\right) r^n|\, |\Lambda_n| - |1 + \phi_n\left(R, r, \alpha, \beta\right)|\, |\lambda_0|\Big) m, \qquad (2.10)\end{aligned}$$

where $P^*(z) = z^n \overline{P(1/\overline{z})}$, $m = \min_{|z|=1} |P(z)|$, $B \in \mathcal{B}_n$, $\sigma(z) = Rz$, $\rho(z) = rz$, Λ_n and $\phi_n\left(R, r, \alpha, \beta\right)$ are given by (1.11) and (1.7), respectively.

Proof. By hypothesis $P(z)$ has all its zeros in $|z| \geq 1$ and

$$m \leq |P(z)| \ \text{ for } \ |z| = 1. \qquad (2.11)$$

We show $F(z) = P(z) + \lambda m$ does not vanish in $|z| < 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. This is obvious if $m = 0$ that is, if $P(z)$ has a zero on $|z| = 1$. So we assume all the zeros of $P(z)$ lie in $|z| > 1$, then $m > 0$ and by the maximum modulus principle, it follows from (2.11) that

$$m < |P(z)| \ \text{ for } \ |z| < 1. \qquad (2.12)$$

Now if $F(z) = P(z) + \lambda m = 0$ for some z_0 with $|z_0| < 1$, then

$$P(z_0) + \lambda m = 0.$$

This implies

$$|P(z_0)| = |\lambda| m \leq m, \ \text{ for } \ |z_0| < 1$$

which is clearly contradiction to (2.12). Thus the polynomial $F(z)$ does not vanish in $|z| < 1$ for every λ with $|\lambda| < 1$. Applying Lemma 2.3 to the polynomial $F(z)$, we get

$$\begin{aligned}|B[F \circ \sigma](z) &+ \phi_n\left(R, r, \alpha, \beta\right) B[F \circ \rho](z)| \\ &\leq |B[F^* \circ \sigma](z) + \phi_n\left(R, r, \alpha, \beta\right) B[F^* \circ \rho](z)|\end{aligned}$$

for $|z|=1$ and $R>r\geq 1$. Replacing $F(z)$ by $P(z)+\lambda m$, we obtain

$$\begin{aligned}|B[P\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](z)+\lambda\big(1+\phi_n(R,r,\alpha,\beta)\big)\lambda_0 m|\\ \leq |B[P^*\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](z)\\ +\bar{\lambda}\big(R^n+\phi_n(R,r,\alpha,\beta)r^n\big)\Lambda_n z^n m| \qquad (2.13)\end{aligned}$$

Now choosing the argument of λ in the right hand side of (2.13) such that

$$\begin{aligned}|B[P^*\circ\sigma](z)+\phi_n\left(R,r,\alpha,\beta\right)B[P^*\circ\rho](z)+\bar{\lambda}\big(R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\big)\Lambda_n z^n m|\\ =|B[P^*\circ\sigma](z)+\phi_n\big(R,r,\alpha,\beta\big)B[P^*\circ\rho](z)|\\ -|\bar{\lambda}|\,|R^n+\phi_n(R,r,\alpha,\beta)r^n|\,|\Lambda_n||z|^n m.\end{aligned}$$

for $|z|=1$,which is possible by Lemma 2.4,we get

$$\begin{aligned}|B[P\circ\sigma](z)+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](z)|-|\lambda|\,|1+\phi_n(R,r,\alpha,\beta)|\,|\lambda_0|m\\ \leq|B[P^*\circ\sigma](z)+\phi_n\left(R,r,\alpha,\beta\right)B[P^*\circ\rho](z)|\\ -|\lambda|\,|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n|\,|\Lambda_n|\,|z|^n m.\end{aligned}$$

Equivalently,

$$\begin{aligned}|B[P\circ\sigma](z)+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](z)|\\ \leq|B[P^*\circ\sigma](z)+\phi_n\left(R,r,\alpha,\beta\right)B[P^*\circ\rho](z)|\\ -|\lambda|\Big(|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n|\,|\Lambda_n|-|1+\phi_n\left(R,r,\alpha,\beta\right)|\,|\lambda_0|\Big)m. \qquad (2.14)\end{aligned}$$

Letting $|\lambda|\to 1$ in (2.14) we obtain inequality (2.10) and this completes the proof of Lemma 2.6. $\square$

Next we describe a result of Arestov [2].
For $\gamma=(\gamma_0,\gamma_1,\cdots,\gamma_n)\in\mathbb{C}^{n+1}$ and $P(z)=\sum_{j=0}^n a_j z^j$, we define

$$C_\gamma P(z)=\sum_{j=0}^n\gamma_j a_j z^j.$$

The operator C_γ is said to be admissible if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z\in\mathbb{C}:|z|\leq 1\}$,
(ii) $P(z)$ has all its zeros in $\{z\in\mathbb{C}:|z|\geq 1\}$.

The result of Arestov may now be stated as follows.

Lemma 2.7. [2, Theorem 2] *Let $\phi(x)=\psi(\log x)$ where ψ is a convex non-decreasing function on $\mathbb{R}$. Then for all $P\in\mathscr{P}_n$ and each admissible operator C_γ,*

$$\int_0^{2\pi}\phi\left(|C_\gamma P(e^{i\theta})|\right)d\theta\leq\int_0^{2\pi}\phi\left(c(\gamma,n)|P(e^{i\theta})|\right)d\theta,$$

where $c(\gamma,n)=\max\left(|\gamma_0|,|\gamma_n|\right)$.

In particular Lemma 2.7 applies with $\phi : x \to x^p$ for every $p \in (0,\infty)$ and $\phi : x \to \log x$ as well. Therefore, we have for $0 \le p < \infty$,

$$\left\{\int_0^{2\pi} \phi\left(|C_\gamma P(e^{i\theta})|^p\right) d\theta\right\}^{1/p} \le c(\gamma,n)\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}. \tag{2.15}$$

From Lemma 2.7, we deduce the following result.

Lemma 2.8. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for each $p > 0$, $R > 1$ and η real, $0 \le \eta < 2\pi$,*

$$\begin{aligned}\int_0^{2\pi} &|\big(B[P\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\big)e^{i\eta} \\ &\qquad + \big(B[P^*\circ\sigma]^*(e^{i\theta}) + \phi_n(R,r,\bar\alpha,\bar\beta)B[P^*\circ\rho]^*(e^{i\theta})\big)|^p d\theta \\ &\le |\big(R^n + \phi_n(R,r,\alpha,\beta)r^n\big)\Lambda_n e^{i\eta} + \big(1+\phi_n(R,r,\bar\alpha,\bar\beta)\big)\bar\lambda_0|^p \int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\end{aligned}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, $B[P^*\circ\sigma]^*(z) := \big(B[P^*\circ\sigma](z)\big)^*$, Λ_n and $\phi_n(R,r,\alpha,\beta)$ are defined by (1.11) and (1.7), respectively.

Proof. Since $P(z)$ does not vanish in $|z| < 1$ and $P^*(z) = z^n\overline{P(1/\bar z)}$, by Lemma 2.5, we have for $R > r \ge 1$,

$$\begin{aligned}&|B[P\circ\sigma](z) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](z)| \\ &\qquad\le |B[P^*\circ\sigma](z) + \phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](z)|\end{aligned} \tag{2.16}$$

Also, since
$P^*(Rz) + \phi_n(R,r,\alpha,\beta)P^*(rz) = R^n z^n\overline{P(1/R\bar z)} + \phi_n(R,r,\alpha,\beta)r^n z^n\overline{P(1/r\bar z)}$,
therefore,

$$\begin{aligned}&B[P^*\circ\sigma](z) + \phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](z) \\ &= \lambda_0\big(R^n z^n\overline{P(1/R\bar z)} + \phi_n(R,r,\alpha,\beta)r^n z^n\overline{P(1/r\bar z)}\big) + \lambda_1\left(\frac{nz}{2}\right)\Big(nR^n z^{n-1}\overline{P(1/R\bar z)} \\ &\quad - R^{n-1}z^{n-2}\overline{P'(1/R\bar z)} + \phi_n(R,r,\alpha,\beta)\big(nr^n z^{n-1}\overline{P(1/r\bar z)} - r^{n-1}z^{n-2}\overline{P'(1/r\bar z)}\big)\Big) \\ &\quad + \frac{\lambda_2}{2!}\left(\frac{nz}{2}\right)^2\Big(n(n-1)R^n z^{n-2}\overline{P(1/R\bar z)} - 2(n-1)R^{n-1}z^{n-3}\overline{P'(1/R\bar z)} \\ &\quad + R^{n-2}z^{n-4}\overline{P''(1/R\bar z)} + \phi_n(R,r,\alpha,\beta)\big(n(n-1)r^n z^{n-2}\overline{P(1/r\bar z)} \\ &\quad - 2(n-1)r^{n-1}z^{n-3}\overline{P'(1/r\bar z)} + r^{n-2}z^{n-4}\overline{P''(1/r\bar z)}\big)\Big),\end{aligned}$$

and hence

$$\begin{aligned}&B[P^*\circ\sigma]^*(z) + \phi\left(R,r,\bar\alpha,\bar\beta\right)B[P^*\circ\rho]^*(z) \\ &\qquad= \big(B[P^*\circ\sigma](z) + \phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](z)\big)^* \\ &\qquad= \left(\bar\lambda_0 + \bar\lambda_1\frac{n^2}{2} + \bar\lambda_2\frac{n^3(n-1)}{8}\right)\left(R^nP(z/R) + \phi\left(R,r,\bar\alpha,\bar\beta\right)r^nP(z/r)\right) \\ &\qquad\quad - \left(\bar\lambda_1\frac{n}{2} + \bar\lambda_2\frac{n^2(n-1)}{4}\right)\Big(R^{n-1}zP'(z/R) + \phi\left(R,r,\bar\alpha,\bar\beta\right)r^{n-1}zP'(z/r)\Big) \\ &\qquad\quad + \bar\lambda_2\frac{n^2}{8}\Big(R^{n-2}z^2P''(z/R) + \phi\left(R,r,\bar\alpha,\bar\beta\right)r^{n-2}z^2P''(z/r)\Big).\end{aligned} \tag{2.17}$$

Also, for $|z|=1$

$$\begin{aligned}|B[P^*\circ\sigma](z)+&\phi_n(R,r,\alpha,\beta)\,B[P^*\circ\rho](z)|\\&=|B[P^*\circ\sigma]^*(z)+\phi\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(z)|.\end{aligned}$$

Using this in (2.16), we get for $|z|=1$ and $R>r\geq 1$,

$$\begin{aligned}|B[P\circ\sigma](z)+&\phi_n(R,r,\alpha,\beta)\,B[P\circ\rho](z)|\\&\leq|B[P^*\circ\sigma]^*(z)+\phi\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(z)|.\end{aligned}$$

Since all the zeros of $P^*(z)$ lie in $|z|\leq 1$, as before, all the zeros of $P^*(Rz)+\phi_n(R,r,\alpha,\beta)P^*(rz)$ lie in $|z|<1$ for all real or complex numbers α,β with $|\alpha|\leq 1$, $|\beta|\leq 1$ and $R>r\geq 1$. Hence by Lemma 2.2, all the zeros of $B[P^*\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](z)$ lie in $|z|<1$, therefore, all the zeros of $B[P^*\circ\sigma]^*(z)+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[P^*\circ\rho]^*(z)$ lie in $|z|>1$. Hence by the maximum modulus principle,

$$\begin{aligned}|B[P\circ\sigma](z)+&\phi_n(R,r,\alpha,\beta)\,B[P^*\circ\rho](z)|\\&<|B[P^*\circ\sigma]^*(z)+\phi\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(z)|\end{aligned}\tag{2.18}$$

for $|z|<1$. A direct application of Rouche's theorem shows that

$$\begin{aligned}C_\gamma P(z)=&\big(B[P\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](z)\big)e^{i\eta}\\&+\big(B[P^*\circ\sigma]^*(z)+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[P^*\circ\rho]^*(z)\big)\\=&\left\{\big(R^n+\phi_n(R,r,\alpha,\beta)r^n\big)\Lambda_n e^{i\eta}+\big(1+\phi_n(R,r,\bar{\alpha},\bar{\beta})\big)\bar{\lambda_0}\right\}a_nz^n\\&+\cdots+\left\{\big(R^n+\phi_n(R,r,\bar{\alpha},\bar{\beta})r^n\big)\bar{\Lambda_n}+e^{i\eta}\big(1+\phi_n(R,r,\alpha,\beta)\big)\lambda_0\right\}a_0\end{aligned}$$

does not vanish in $|z|<1$. Therefore, C_γ is an admissible operator. Applying (2.15) of Lemma 2.7, the desired result follows immediately for each $p>0$. □

We also need the following lemma [4].

Lemma 2.9. *If A,B,C are non-negative real numbers such that $B+C\leq A$, then for each real number γ,*

$$|(A-C)e^{i\gamma}+(B+C)|\leq|Ae^{i\gamma}+B|.$$

3. Proof of the Theorems

Proof of Theorem 1.5. By hypothesis $P(z)$ does not vanish in $|z|<1$, therefore by Lemma 2.6, we have

$$\begin{aligned}|B[P\circ\sigma](z)&+\phi_n(R,r,\alpha,\beta)\,B[P\circ\rho](z)|\\&\leq|B[P^*\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)\,B[P^*\circ\rho](z)|\\&\quad-\Big(|R^n+\phi_n(R,r,\alpha,\beta)\,r^n|\,|\Lambda_n|-|1+\phi_n(R,r,\alpha,\beta)\,|\,|\lambda_0|\Big)m,\end{aligned}\tag{3.1}$$

for $|z|=1$, $|\alpha|\leq 1$ and $R>r\geq 1$ where $P^*(z)=z^n\overline{P(1/\bar{z})}$. Since $B[P^*\circ\sigma]^*(z)+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(z)$ is the conjugate of $B[P^*\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)\,B[P^*\circ\rho](z)$ and

$$\begin{aligned}|B[P^*\circ\sigma]^*(z)+&\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(z)|\\&=|B[P^*\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)\,B[P^*\circ\rho](z)|\end{aligned}$$

Thus (3.1) can be written as

$$\begin{aligned}
&\left|B[P\circ\sigma](z)+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](z)\right| \\
&\qquad+\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2} \\
&\leq\left|B[P^*\circ\sigma]^*(z)+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(z)\right| \\
&\qquad-\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}
\end{aligned}\tag{3.2}$$

for $|z|=1$. Taking

$$A=\left|B[P^*\circ\sigma]^*(z)+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(z)\right|$$

$$B=\left|B[P\circ\sigma](z)+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](z)\right|,$$

and

$$C=\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}$$

in Lemma 2.9 and noting by (3.2) that

$$B+C\leq A-C\leq A,$$

we get for every real γ,

$$\begin{aligned}
&\Bigg|\Bigg\{\left|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(e^{i\theta})\right| \\
&\qquad-\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}\Bigg\}e^{i\gamma} \\
&\quad+\Bigg\{\left|B[P\circ\sigma](e^{i\theta})+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](e^{i\theta})\right| \\
&\qquad+\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}\Bigg\}\Bigg| \\
&\leq\Big|\left|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(e^{i\theta})\right|e^{i\gamma} \\
&\qquad+\left|B[P\circ\sigma](e^{i\theta})+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](e^{i\theta})\right|\Big|.
\end{aligned}$$

This implies for each $p>0$,

$$\begin{aligned}
&\int_0^{2\pi}\Bigg|\Bigg\{\left|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(e^{i\theta})\right| \\
&\qquad-\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}\Bigg\}e^{i\gamma}
\end{aligned}$$

$$+\left\{\left|B[P\circ\sigma](e^{i\theta})+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](e^{i\theta})\right|\right.$$
$$\left.+\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}\right\}\Bigg|^p d\theta$$
$$\leq\int\limits_0^{2\pi}\Big|\left|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(e^{i\theta})\right|e^{i\gamma}$$
$$+\left|B[P\circ\sigma](e^{i\theta})+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](e^{i\theta})\right|\Big|^p d\theta. \tag{3.3}$$

Integrating both sides of (3.3) with respect to γ from 0 to 2π, we get with the help of Lemma 2.8 for each $p>0$,

$$\int\limits_0^{2\pi}\int\limits_0^{2\pi}\Bigg|\left\{\left|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(e^{i\theta})\right|\right.$$
$$\left.-\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}\right\}e^{i\gamma}$$
$$+\left\{\left|B[P\circ\sigma](e^{i\theta})+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](e^{i\theta})\right|\right.$$
$$\left.+\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}\right\}\Bigg|^p d\theta d\gamma$$
$$\leq\int\limits_0^{2\pi}\int\limits_0^{2\pi}\Big|\left|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(e^{i\theta})\right|e^{i\gamma}$$
$$+\left|B[P\circ\sigma](e^{i\theta})+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](e^{i\theta})\right|\Big|^p d\theta d\gamma$$
$$\leq\int\limits_0^{2\pi}\left\{\int\limits_0^{2\pi}\Big|\left|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(e^{i\theta})\right|e^{i\gamma}\right.$$
$$\left.+\left|B[P\circ\sigma](e^{i\theta})+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](e^{i\theta})\right|\Big|^p d\gamma\right\}d\theta$$
$$\leq\int\limits_0^{2\pi}\left\{\int\limits_0^{2\pi}\Bigg|\Big(B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(e^{i\theta})\Big)e^{i\gamma}\right.$$
$$\left.+\Big(B[P\circ\sigma](e^{i\theta})+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](e^{i\theta})\Big)\Bigg|^p d\gamma\right\}d\theta$$
$$\leq\int\limits_0^{2\pi}\left\{\int\limits_0^{2\pi}\Bigg|\Big(B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(e^{i\theta})\Big)e^{i\gamma}\right.$$

$$+\Big(B[P\circ\sigma](e^{i\theta})+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](e^{i\theta})\Big)\Big|^p d\theta\Bigg\}d\gamma$$

$$\leq \int_0^{2\pi}\Big|(R^n+\phi_n(R,r,\alpha,\beta)r^n)\Lambda_n e^{i\gamma}+(1+\phi_n(R,r,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}\Big|^p d\gamma$$

$$\times\int_0^{2\pi}\Big|P(e^{i\theta})\Big|^p d\theta \tag{3.4}$$

Now it can be easily verified that for every real number γ and $s\geq 1$,

$$\left|s+e^{i\alpha}\right|\geq\left|1+e^{i\alpha}\right|.$$

This implies for each $p>0$,

$$\int_0^{2\pi}\left|s+e^{i\gamma}\right|^p d\gamma\geq\int_0^{2\pi}\left|1+e^{i\gamma}\right|^p d\gamma. \tag{3.5}$$

If

$$\left|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\right|$$
$$+\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}\neq 0,$$

we take

$$s=\frac{\left|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(e^{i\theta})\right|+\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}}{\left|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\right|+\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}},$$

then by (3.2), $s\geq 1$ and we get with the help of (3.5),

$$\int_0^{2\pi}\Bigg|\Bigg\{\left|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar{\alpha},\bar{\beta}\right)B[P^*\circ\rho]^*(e^{i\theta})\right|$$
$$-\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}\Bigg\}e^{i\gamma}$$
$$+\Bigg\{\left|B[P\circ\sigma](e^{i\theta})+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](e^{i\theta})\right|$$
$$+\frac{\Big(\left|R^n+\phi_n\left(R,r,\alpha,\beta\right)r^n\right|\left|\Lambda_n\right|-\left|1+\phi_n\left(R,r,\alpha,\beta\right)\right|\left|\lambda_0\right|\Big)m}{2}\Bigg\}\Bigg|^p d\gamma$$

$$=\Bigg|\left|B[P\circ\sigma](e^{i\theta})+\phi_n\left(R,r,\alpha,\beta\right)B[P\circ\rho](e^{i\theta})\right|$$

$$
\begin{aligned}
&+\frac{\Big(|R^n+\phi_n(R,r,\alpha,\beta)r^n|\,|\Lambda_n|-|1+\phi_n(R,r,\alpha,\beta)|\,|\lambda_0|\Big)m}{2}\Bigg|^p\\
&\times\int_0^{2\pi}\left|e^{i\gamma}+\frac{\begin{array}{c}\left|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar\alpha,\bar\beta\right)B[P^*\circ\rho]^*(e^{i\theta})\right|\\ -\dfrac{\Big(|R^n+\phi_n(R,r,\alpha,\beta)r^n|\,|\Lambda_n|-|1+\phi_n(R,r,\alpha,\beta)|\,|\lambda_0|\Big)m}{2}\end{array}}{\begin{array}{c}\left|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\right|\\ +\dfrac{\Big(|R^n+\phi_n(R,r,\alpha,\beta)r^n|\,|\Lambda_n|-|1+\phi_n(R,r,\alpha,\beta)|\,|\lambda_0|\Big)m}{2}\end{array}}\right|^p d\gamma\\
&=\Bigg|\left|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\right|\\
&\qquad+\frac{\Big(|R^n+\phi_n(R,r,\alpha,\beta)r^n|\,|\Lambda_n|-|1+\phi_n(R,r,\alpha,\beta)|\,|\lambda_0|\Big)m}{2}\Bigg|^p\\
&\times\int_0^{2\pi}\left|e^{i\gamma}+\left|\frac{\begin{array}{c}\left|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n\left(R,r,\bar\alpha,\bar\beta\right)B[P^*\circ\rho]^*(e^{i\theta})\right|\\ -\dfrac{\Big(|R^n+\phi_n(R,r,\alpha,\beta)r^n|\,|\Lambda_n|-|1+\phi_n(R,r,\alpha,\beta)|\,|\lambda_0|\Big)m}{2}\end{array}}{\begin{array}{c}\left|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\right|\\ +\dfrac{\Big(|R^n+\phi_n(R,r,\alpha,\beta)r^n|\,|\Lambda_n|-|1+\phi_n(R,r,\alpha,\beta)|\,|\lambda_0|\Big)m}{2}\end{array}}\right|^p\right| d\gamma\\
&\geq\Bigg|\left|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\right|\\
&\quad+\frac{\Big(|R^n+\phi_n(R,r,\alpha,\beta)r^n|\,|\Lambda_n|-|1+\phi_n(R,r,\alpha,\beta)|\,|\lambda_0|\Big)m}{2}\Bigg|^p\int_0^{2\pi}|1+e^{i\gamma}|^p d\gamma.
\end{aligned}
\tag{3.6}
$$

For

$$
\begin{aligned}
&\left|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\right|\\
&\qquad+\frac{\Big(|R^n+\phi_n(R,r,\alpha,\beta)r^n|\,|\Lambda_n|-|1+\phi_n(R,r,\alpha,\beta)|\,|\lambda_0|\Big)m}{2}\neq 0,
\end{aligned}
$$

then (3.6) is trivially true. Using this in (3.4), we conclude that for every $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\leq 1,|\beta|\leq 1$ $R>r\geq 1$ and $p>0$,

$$
\begin{aligned}
&\int_0^{2\pi}\Bigg|\left|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\right|\\
&\quad+\frac{\Big(|R^n+\phi_n(R,r,\alpha,\beta)r^n|\,|\Lambda_n|-|1+\phi_n(R,r,\alpha,\beta)|\,|\lambda_0|\Big)m}{2}\Bigg|^p d\theta\int_0^{2\pi}|1+e^{i\gamma}|^p d\gamma
\end{aligned}
$$

$$\leq \int_0^{2\pi} \left|(R^n + \phi_n(R,r,\alpha,\beta)r^n)\Lambda_n e^{i\gamma} + (1+\phi_n(R,r,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}\right|^p d\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta.$$

This gives for every δ, α, β with $|\delta| \leq 1$, $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and γ real

$$\int_0^{2\pi} \Bigg| B[P\circ\sigma](e^{i\theta}) + \phi_n\left(R,r,\alpha,\beta\right) B[P\circ\rho](e^{i\theta})$$
$$+\delta\frac{\Big(|R^n + \phi_n\left(R,r,\alpha,\beta\right) r^n|\,|\Lambda_n| - |1+\phi_n\left(R,r,\alpha,\beta\right)|\,|\lambda_0|\Big)m}{2}\Bigg|^p d\theta \int_0^{2\pi} |1+e^{i\gamma}|^p d\gamma$$
$$\leq \int_0^{2\pi} \left|\big(R^n + \phi_n(R,r,\alpha,\beta)r^n\big)\Lambda_n e^{i\gamma} + \big(1+\phi_n(R,r,\bar{\alpha},\bar{\beta})\big)\bar{\lambda_0}\right|^p d\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta \tag{3.7}$$

Since

$$\int_0^{2\pi} \left|\big(R^n + \phi_n(R,r,\alpha,\beta)r^n\big)\Lambda_n e^{i\gamma} + (1+\phi_n(R,r,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}\right|^p d\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta$$
$$= \int_0^{2\pi} \left||\big(R^n + \phi_n(R,r,\alpha,\beta)r^n\big)\Lambda_n|e^{i\gamma} + |\big(1+\phi_n(R,r,\bar{\alpha},\bar{\beta})\big)\bar{\lambda_0}|\right|^p d\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta$$
$$= \int_0^{2\pi} \left||\big(R^n + \phi_n(R,r,\alpha,\beta)r^n\big)\Lambda_n|e^{i\gamma} + |\big(1+\phi_n(R,r,\alpha,\beta)\big)\lambda_0|\right|^p d\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta$$
$$= \int_0^{2\pi} \left|\big(R^n + \phi_n(R,r,\alpha,\beta)r^n\big)\Lambda_n e^{i\gamma} + \big(1+\phi_n(R,r,\alpha,\beta)\big)\lambda_0\right|^p d\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta, \tag{3.8}$$

the desired result follows immediately by combining (3.7) and (3.8). This completes the proof of Theorem 1.5 for $p > 0$. To establish this result for $p = 0$, we simply let $p \to 0+$. □

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References

1. N.C. Ankeny and T.J. Rivlin, *On a theorm of S. Bernstein*, Pacific J. Math., **5** (1955), 849–852.
2. V.V. Arestov, *On integral inequalities for trigonometric polynimials and their derivatives*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), 3–22, (in Russian). English translation: Math. USSR–Izv. **18** (1982), 1–17.
3. A. Aziz, *A new proof and a generalization of a theorem of De Bruijn*, Proc. Amer. Math. Soc., **106**(1989), 345–350.
4. A. Aziz and N.A. Rather, *L_p inequalities for polynomials*, Glas. Mat. Ser. III, **32** (1997), 39–43.

5. A. Aziz and N.A. Rather, *Some new generalizations of Zygmund-type inequalities for polynomials*, Math. Ineq. Appl. **15** (2012), 469–486.
6. R.P. Boas Jr. and Q.I. Rahman, *L^p inequalities for polynomials and entire functions*, Arch. Ration. Mech. Anal. **11**(1962), 34–39.
7. N.G. Bruijn, *Inequalities concerning polynomials in the complex domain*, Nederal. Akad. Wetensch. Proc. **50** (1947), 1265–1272.
8. G.H. Hardy, *The mean value of the modulus of an analytic functions*, Proc. Lond. Math. Soc. **14** (1915), 269–277.
9. P.D. Lax, *Proof of a conjecture of P.Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513.
10. M. Marden, *Geometry of Polynomials*, Math. Surveys Monogr. **3**, Amer. Math. Soc. Providence, RI, 1949.
11. G.V. Milovanovic, D.S. Mitrinovic and Th.M. Rassias, *Topics in Polynomials: Extremal Properties, Inequalities, Zeros*, World scientific Publishing Co., Singapore, 1994.
12. G. Polya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Springer-Verlag, Berlin, 1925.
13. Q.I. Rahman, *Functions of exponential type,* Trans. Amer. Math. Soc. **135** (1969), 295–309.
14. Q.I. Rahman and G. Schmeisser, *L^p inequalities for polynomials*, J. Approx. Theory, **53** (1988), 26–32.
15. Q.I. Rahman and G. Schmisser, *Analytic Theory of Polynomials*, Oxford University Press, New York, 2002.
16. N.A. Rather and S. Gulzar, *Integral mean estimates for an operator preserving inequalities between polynomials*, J. Inequal. Spec. Funct. **3**(2012), 24–41.
17. N.A. Rather and M.A. Shah, *On an operator preserving L_p inequalities between polynomials*, J. Math. Anal. Appl. **399** (2013), 422–432.
18. A.C. Schaffer, *Inequalities of A. Markov and S. Bernstein for polynomials and related functions*, Bull. Amer. Math. Soc. **47** (1941), 565–579.
19. W.M. Shah and A. Liman, *Integral estimates for the family of B-operators*, Oper. Matrices, **5** (2011), 79-87.
20. A. Zygmund, *A remark on conjugate series*, Proc. Lond. Math. Soc. **34** (1932), 292–400.

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