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APPROXIMATION WITH CERTAIN SZÁSZ-MIRAKYAN OPERATORS

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Abstract. In the current article, we consider different growth conditions for studying the well known Szász-Mirakyan operators, which were introduced in the mid-twentieth century. Here, we obtain a new approach to find the moments using the concept of moment generating functions. Further, we discuss a uniform estimate and compare convergence behavior with the recently studied one.

1. Introduction

In the year 2003, King [20] modified the well-known Bernstein polynomials, which preserve constant as well as x^2 functions and he was able to achieve better approximation results. In the theory of approximation, to check the convergence of linear positive operators L_n, the most common result is due to Korovkin, which states that, if the three test functions $L_n(e_r(t), x)$, $e_r(t) = t^r$, $r = 0, 1, 2$ converge to $e_r(x)$, then $L_n(f, x)$ converges to $f(x)$ uniformly. Many applications of this well-known theorem are available in literature. In [9], the authors proved a general Korovkin-type theorem for the function e^{-kt}, $k = 0, 1, 2$. Holhoş [17] extended the work of [9] and established some quantitative estimates along with a Korovkin-type result for exponential functions as follows:

Theorem A. [17]. Let $f \in C^*[0, \infty)$ and $A_n : C^*[0, \infty) \to C^*[0, \infty)$ be a sequence of positive linear operators. If

$$||A_n 1 - 1||_\infty = a_n,$$

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$$||A_n(e^{-t},x)-e^{-x}||_\infty = b_n,$$
$$||A_n(e^{-2t},x)-e^{-2x}||_\infty = c_n,$$

where a_n, b_n and c_n tend to zero for n sufficiently large, then we have

$$||A_n f - f||_\infty \leqslant ||f||_\infty \; a_n + (2+a_n)\,.\,\omega^*\left(f,(a_n+2b_n+c_n)^{1/2}\right),$$

where $\omega^*(f,\delta) = \sup\limits_{\substack{x,t\geqslant 0\\ |e^{-x}-e^{-t}|\leqslant\delta}} |f(x)-f(t)|$ for every $\delta \geqslant 0$ and every function $f \in C^*[0,\infty)$, $C^*[0,\infty)$ denoting the Banach space of all real-valued continuous functions on $[0,\infty)$ with the property that $\lim\limits_{x\to\infty} f(x)$ exists and is finite, endowed with the uniform norm.

The well known Szász-Mirakyan operators are defined by

$$S_n(f,x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right).$$

Varied researchers have discussed different generalizations of similar type of operators (cf. [2]-[8], [10]-[14], [16], [18], [19], [21], [22]).

Recently, in [1], a modification of the Szász-Mirakyan operators reproducing exponential function e^{2ax}, $a > 0$ was discussed. For such modification, some approximation results were established. We may point out here that such modification does not provide better approximation and even if, the operators preserve e^{ax}, $a > 0$, there is no difference and one may not achieve better estimates, as far as Theorem A is concerned. This motivated us to study in this direction and so, we provide another modification of Szász-Mirakyan operators for $x \geqslant 0$ and $n \in \mathbb{N}$ as

$$\hat{S}_n(f,x) = e^{-n\alpha_n(x)}\sum_{k=0}^{\infty}\frac{\left(n\alpha_n(x)\right)^k}{k!}f\left(\frac{k}{n}\right), \quad f \in C[0,\infty), \tag{1.1}$$

such that these operators preserve constant as well as e^{-2x} functions, i.e., $\hat{S}_n(e^{-2t},x) = e^{-2x}$. Considering this condition and substituting in (1.1), we get the value of $\alpha_n(x)$ as

$$\alpha_n(x) = \frac{2x\,e^{2/n}}{n(e^{2/n}-1)}.$$

Therefore, for $x \in [0,\infty)$, our operators (1.1) take the following form:

$$\hat{S}_n(f,x) = e^{\frac{2x\,e^{2/n}}{(1-e^{2/n})}}\sum_{k=0}^{\infty}\frac{(2x\,e^{2/n})^k}{k!\,(e^{2/n}-1)^k}\; f\left(\frac{k}{n}\right). \tag{1.2}$$

After simple computation, the moment generating function of the operators (1.2) may be given as

$$\hat{S}_n(e^{At},x) = e^{\frac{2e^{2/n}\left(1-e^{A/n}\right)x}{1-e^{2/n}}}.$$

Since the moments are related with the moment generating function, the r-th moment $\hat{S}_n(e_r, x)$, $e_r(t) = t^r$ $(r \in \mathbb{N} \cup \{0\})$ may be obtained by the following relation:

$$\hat{S}_n(e_r, x) = \left[\frac{\partial^r}{\partial A^r}\hat{S}_n(e^{At}, x)\right]_{A=0} = \left[\frac{\partial^r}{\partial A^r}\ \left(e^{\frac{2e^{2/n}\left(1-e^{A/n}\right)x}{1-e^{2/n}}}\right)\right]_{A=0}.$$

Also, by change of scale property of moment generating functions, if we expand $e^{-Ax}\ \hat{S}_n(e^{At}, x)$ in powers of A, the central moment of r-th order $\mu_{n,r}(x) = \hat{S}_n((t-x)^r, x)$ can be obtained by collecting the coefficient of $A^r/r!$.

$$\begin{aligned} e^{-Ax}\ \hat{S}_n(e^{At}, x) &= e^{-Ax+\frac{2e^{2/n}\left(1-e^{A/n}\right)x}{1-e^{2/n}}} \\ &= 1 + \left(-x + \frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n}\right)A \\ &\quad + \frac{1}{2}\left\{\frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n^2} + \left(-x + \frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n}\right)^2\right\}A^2 \end{aligned}$$

$$+\frac{1}{3}\left\{\begin{array}{c} \frac{e^{2/n}x}{\left(-1+e^{2/n}\right)n^3} + \frac{2e^{2/n}x\left(-x+\frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n}\right)}{\left(-1+e^{2/n}\right)n^2} \\ +\frac{1}{2}\left(-x + \frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n}\right)\left(\frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n^2} + \left(-x + \frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n}\right)^2\right) \end{array}\right\}A^3$$

$$+\frac{1}{4}\left\{\begin{array}{c} \frac{e^{2/n}x}{3\left(-1+e^{2/n}\right)n^4} + \frac{e^{2/n}x\left(-x+\frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n}\right)}{\left(-1+e^{2/n}\right)n^3} \\ +\frac{e^{2/n}x\left(\frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n^2}+\left(-x+\frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n}\right)^2\right)}{\left(-1+e^{2/n}\right)n^2} \\ +\frac{1}{3}\left(-x + \frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n}\right)\left(\frac{e^{2/n}x}{\left(-1+e^{2/n}\right)n^3} + \frac{2e^{2/n}x\left(-x+\frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n}\right)}{\left(-1+e^{2/n}\right)n^2}\right. \\ \left.+\frac{1}{2}\left(-x + \frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n}\right)\left(\frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n^2} + \left(-x + \frac{2e^{2/n}x}{\left(-1+e^{2/n}\right)n}\right)^2\right)\right) \end{array}\right\}A^4$$

$$+\mathcal{O}\left(A^5\right).$$

Lemma 1.1. *The central moments may be obtained by*

$$\begin{aligned} \mu_{n,r}(x) := \hat{S}_n((t-x)^r, x) &= \left[\frac{\partial^r}{\partial A^r}\ \left(e^{-Ax}\ \hat{S}_n(e^{At}, x)\right)\right]_{A=0} \\ &= \left[\frac{\partial^r}{\partial A^r}\ \left(e^{\frac{2e^{2/n}\left(1-e^{A/n}\right)x}{1-e^{2/n}}-Ax}\right)\right]_{A=0}. \end{aligned}$$

In addition, from the above expansion, first few central moments are given by:

$$
\begin{aligned}
\mu_{n,0}(x) &= 1, \\
\mu_{n,1}(x) &= \frac{2x\, e^{2/n}}{n(e^{2/n}-1)} - x, \\
\mu_{n,2}(x) &= \left(\frac{2x\, e^{2/n}}{n(e^{2/n}-1)} - x\right)^2 + \frac{2x\, e^{2/n}}{n^2(e^{2/n}-1)}, \\
\mu_{n,3}(x) &= \left(\frac{2x\, e^{2/n}}{n(e^{2/n}-1)} - x\right)^3 + \frac{1}{n^2}\left(\frac{2x\, e^{2/n}}{n(e^{2/n}-1)}\right) + \frac{3}{n}\left(\frac{2x\, e^{2/n}}{n(e^{2/n}-1)}\right)^2 \\
&\quad -\frac{3x}{n}\left(\frac{2x\, e^{2/n}}{n(e^{2/n}-1)}\right), \\
\mu_{n,4}(x) &= \left(\frac{2x\, e^{2/n}}{n(e^{2/n}-1)} - x\right)^4 + \frac{6}{n}\left(\frac{2x\, e^{2/n}}{n(e^{2/n}-1)}\right)^3 + \frac{7}{n^2}\left(\frac{2x\, e^{2/n}}{n(e^{2/n}-1)}\right)^2 \\
&\quad -\frac{12x}{n}\left(\frac{2x\, e^{2/n}}{n(e^{2/n}-1)}\right)^2 + \frac{1}{n^3}\left(\frac{2x\, e^{2/n}}{n(e^{2/n}-1)}\right) - \frac{4x}{n^2}\left(\frac{2x\, e^{2/n}}{n(e^{2/n}-1)}\right) \\
&\quad +\frac{6x^2}{n}\left(\frac{2x\, e^{2/n}}{n(e^{2/n}-1)}\right).
\end{aligned}
$$

Furthermore,

$$\lim_{n\to\infty} n\ \mu_{n,1}(x) = x \ \textit{and}\ \lim_{n\to\infty} n\ \mu_{n,2}(x) = x.$$

In the present note, we provide quantitative estimates for the operators (1.2). It may be observed that by considering this form, one may get better approximation.

2. Main Results

In this section, we present the application of Theorem A for the operators (1.2). We may point out here that the following theorem gives better approximation than the Theorem 5 of [1].

Theorem 2.1. *For $f \in C^*[0,\infty)$, we have*

$$||\hat{S}_n f - f||_{[0,\infty)} \leqslant 2\,\omega^*\left(f, \sqrt{2\,b_n}\right),$$

where $b_n \to 0$ as $n \to \infty$.

Proof. As the operators (1.2) reproduce constant and e^{-2x} functions, one may observe that $a_n = 0$ and $c_n = 0$. Now, we find the estimate of b_n.

Using the software Mathematica, for $x \geqslant 0$,

$$
\begin{aligned}
g_n(x) &:= \hat{S}_n(e^{-t}, x) - e^{-x} \\
&= e^{\left(\frac{2x\, e^{2/n}}{e^{2/n}-1}\;\frac{1-e^{1/n}}{e^{1/n}}\right)} - e^{-x} \\
&= e^{\left(\frac{-2x}{1+e^{\frac{-1}{n}}}\right)} - e^{-x} \\
&= e^{-x} - \frac{1}{2n}(xe^{-x}) + \frac{1}{8\,n^2}(x^2 e^{-x}) - \frac{1}{48n^3}\left(x(x^2-2)e^{-x}\right) + \mathcal{O}\left(\frac{1}{n^4}\right) - e^{-x} \\
&= -\frac{1}{2n}(xe^{-x}) + \frac{1}{8\,n^2}(x^2 e^{-x}) - \frac{1}{48n^3}\left(x(x^2-2)e^{-x}\right) + \mathcal{O}\left(\frac{1}{n^4}\right),
\end{aligned}
$$

which is a positive function with $g_n(0) = 0$ and $\lim_{x\to+\infty} g_n(x) = 0$. Also, $g_n(x) \to 0$ implying that $b_n := ||g_n||_{[0,\infty)} \to 0$ for n sufficiently large and hence the desired result follows. □

Remark 2.2. It is to be noted that Theorem 2.1 can also be proved along the lines of [1], but we preferred a direct proof.

Remark 2.3. As an application of Theorem A, under the conditions of Theorem 2.1, for the usual Szász-Mirakyan operators, Holhoş [17] obtained:

$$||S_n f - f||_{[0,\infty)} \leqslant 2\,\omega^*\left(f, \sqrt{2\,b_n + c_n}\right).$$

Also, for the operators R_n^* studied in [1], (preserving the function $e^{2ax}, a > 0$), Theorem A takes the following form:

$$||R_n^* f - f||_{[0,\infty)} \leqslant 2\,\omega^*\left(f, \sqrt{2\,b_n + c_n}\right).$$

And, keeping in view Theorem 2.1, we observe that the choice here gives better approximation results.

Next, we prove the quantitative asymptotic formula.

Theorem 2.4. *Let $f, f'' \in C^*[0,\infty)$, then, for $x \in [0,\infty)$, the following inequality holds:*

$$
\begin{aligned}
\left| n\,[\hat{S}_n(f,x) - f(x)] - x\left[f'(x) + \frac{f''(x)}{2}\right]\right| &\leqslant |p_n(x)|\;|f'| + |q_n(x)|\;|f''| \\
&+\frac{1}{2}\left(2\,q_n(x) + x + r_n(x)\right)\;\omega^*(f'', n^{-1/2}),
\end{aligned}
$$

where $p_n(x) = n\;\mu_{n,1}(x) - x$, $q_n(x) = \frac{1}{2}\left(n\;\mu_{n,2}(x) - x\right)$ and
$r_n(x) = n^2\left[\hat{S}_n\left((e^{-x} - e^{-t})^4, x\right)\;\mu_{n,4}(x)\right]^{1/2}$.

Proof. By the Taylor's formula, there exists ξ lying between x and t such that

$$f(t) = f(x) + (t-x)\,f'(x)\; + (t-x)^2\,\frac{f''(x)}{2} + h(\xi, x)\;(t-x)^2,$$

where

$$h(t,x) := \frac{f''(t) - f''(x)}{2}$$

is a continuous function and ξ is between x and t. Applying the operator $\hat{S}_n$ to above equality and using Lemma 1.1, we can write that

$$\left|\hat{S}_n\left(f,x\right)-f\left(x\right)-\mu_{n,1}(x)\ f'(x)-\tfrac{f''(x)}{2}\ \mu_{n,2}(x)\right|$$
$$\leqslant \hat{S}_n\left(\left|h\left(\xi,x\right)\right|\ (t-x)^2,x\right).$$

Again using Lemma 1.1, we get

$$\left|n\ [\hat{S}_n(f,x)-f(x)]-x\left[f'(x)+\frac{f''(x)}{2}\right]\right|$$
$$\leqslant \left|n\ \mu_{n,1}(x)-x\right|\left|f'(x)\right|+\frac{1}{2}\left|n\ \mu_{n,2}(x)-x\right|\left|f''(x)\right|+\left|n\ \hat{S}_n\left(h\left(\xi,x\right)\ (t-x)^2,x\right)\right|.$$

Let $p_n(x) := n\ \mu_{n,1}(x)-x$ and $q_n(x) := \dfrac{1}{2}\left(n\ \mu_{n,2}(x)-x\right)$.
Then

$$\left|n\ [\hat{S}_n(f,x)-f(x)]-x\left[f'(x)+\frac{f''(x)}{2}\right]\right|$$
$$\leqslant\quad \left|p_n(x)\right|\left|f'(x)\right|+\left|q_n(x)\right|\left|f''(x)\right|+\left|n\ \hat{S}_n\left(h\left(\xi,x\right)\ (t-x)^2,x\right)\right|.$$

Also, from Lemma 1.1, we have $p_n(x)\to 0$ and $q_n(x)\to 0$ for n sufficiently large. Now, we just have to compute the last estimate: $n\ \hat{S}_n\left(h\left(\xi,x\right)\ (t-x)^2,x\right)$. Using the property of $\omega^*(.,\delta)$: $|f(t)-f(x)|\leqslant\left(1+\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\omega^*(f,\delta)$, $\delta>0$, we get that

$$\left|h\left(\xi,x\right)\right|\leqslant\frac{1}{2}\left(1+\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\omega^*(f'',\delta).$$

Hence, we get

$$\begin{aligned} n\ \hat{S}_n\left(\left|h\left(\xi,x\right)\right|\ (t-x)^2,x\right)\ &\leqslant\ \frac{1}{2}\ n\ \omega^*(f'',\delta)\ \mu_{n,2}(x)\\ &\quad+\frac{n}{2\delta^2}\ \omega^*(f'',\delta)\ \hat{S}_n\left(\left(e^{-x}-e^{-t}\right)^2(t-x)^2,x\right).\end{aligned}$$

Applying Cauchy-Schwarz inequality, we obtain

$$n\ \hat{S}_n\left(\left|h\left(\xi,x\right)\right|\ (t-x)^2,x\right)\leqslant\frac{1}{2}\ n\ \omega^*(f'',\delta)\ \mu_{n,2}(x)$$
$$+\frac{n}{2\delta^2}\ \omega^*(f'',\delta)\ \left[\hat{S}_n\left(\left(e^{-x}-e^{-t}\right)^4,x\right)\cdot\mu_{n,4}(x)\right]^{1/2}.$$

Considering

$$r_n(x) := \left[n^2\ \hat{S}_n\left(\left(e^{-x}-e^{-t}\right)^4,x\right)\right]^{1/2}.\left[n^2\ \mu_{n,4}(x)\right]^{1/2}=$$
$$\left[n^2\left(e^{-4x}+e^{\frac{-8x}{1+e^{-4/n}}}-4e^{-3x}e^{\frac{-2x}{1+e^{-1/n}}}+6e^{-2x}e^{\frac{-4x}{1+e^{-2/n}}}-4e^{-x}e^{\frac{-6x}{1+e^{-3/n}}}\right)\right]^{1/2}$$
$$.\left[n^2\ \mu_{n,4}(x)\right]^{1/2}$$

and choosing $\delta=n^{-1/2}$, we finally get the desired result. $\square$

Remark 2.5. The convergence of the modified Szász-Mirakyan operators (1.2) in the above theorem takes place for n sufficiently large.

Using the software Mathematica, we find that

$$\begin{aligned}
&\lim_{n\to\infty} n^2\ \mu_{n,4}(x)\\
=\ & \lim_{n\to\infty} n^2 \left[\left(\frac{2x\,e^{2/n}}{n(e^{2/n}-1)}-x\right)^4+\frac{6}{n}\left(\frac{2x\,e^{2/n}}{n(e^{2/n}-1)}\right)^3+\frac{7}{n^2}\left(\frac{2x\,e^{2/n}}{n(e^{2/n}-1)}\right)^2\right.\\
&-\frac{12x}{n}\left(\frac{2x\,e^{2/n}}{n(e^{2/n}-1)}\right)^2+\frac{1}{n^3}\left(\frac{2x\,e^{2/n}}{n(e^{2/n}-1)}\right)-\frac{4x}{n^2}\left(\frac{2x\,e^{2/n}}{n(e^{2/n}-1)}\right)\\
&\left.+\frac{6x^2}{n}\left(\frac{2x\,e^{2/n}}{n(e^{2/n}-1)}\right)\right]\\
=\ & 3x^2
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} n^2\ \hat{S}_n\left(\left(e^{-x}-e^{-t}\right)^4,x\right)\\
=\ & \lim_{n\to\infty} n^2\ \left(e^{-4x}+e^{\frac{-8x}{1+e^{-4/n}}}-4e^{-3x}e^{\frac{-2x}{1+e^{-1/n}}}+6e^{-2x}e^{\frac{-4x}{1+e^{-2/n}}}-4e^{-x}e^{\frac{-6x}{1+e^{-3/n}}}\right)\\
=\ & 3x^2e^{-4x}.
\end{aligned}$$

Remark 2.6. Lately, approximation for certain combinations have been extensively studied in [15]. One may consider linear combinations for $\hat{S}_n(f,x)$. As the analysis is different, we may discuss elsewhere.

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