Khayyam J. Math. 4 (2018), no. 1, 28–38

DOI: 10.22034/kjm.2017.53655

Khayyam Journal of Mathematics

emis.de/journals/KJM
kjm-math.org

A SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY MEANS OF DIFFERENTIAL SUBORDINATION

SERKAN ÇAKMAK, SIBEL YALÇIN AND ŞAHSENE ALTINKAYA*

Communicated by H.M. Srivastava

Abstract. The aim of this paper is to introduce a new class of harmonic functions defined by use of a subordination. We find necessary and sufficient conditions, radii of starlikeness and convexity and compactness for this class of functions. Moreover, by using extreme points theory we also obtain coefficients estimates, distortion theorems for this class of functions. On the other hand, some results (corollaries) on the paper are pointed out.

1. Introduction and preliminaries

Let H denote the family of continuous complex valued harmonic functions which are harmonic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in U. A function harmonic in U may be written as $f = h + \overline{g}$, where h and g are members of A. We call h the analytic part and g co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in U is that $|h'(z)| > |g'(z)|$ (see Clunie and Sheil-Small [1]). To this end, without loss of generality, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=2}^{\infty} b_k z^k. \tag{1.1}$$

Date: Received: 30 October 2017; Revised: 26 November 2017; Accepted: 27 November 2017.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C80.

Key words and phrases. Harmonic functions, univalent functions, modified Sălăgean operator, subordination.

Let SH denote the family of functions $f = h+\overline{g}$ which are harmonic, univalent, and sense-preserving in U for which $f(0) = f_z(0)-1 = 0$. The subclass SH^0 of SH consists of all functions in SH which have the additional property $f_{\bar{z}}(0) = b_1 = 0$.

In 1984 Clunie and Sheil-Small [1] and (more recently) by Jahangiri and Silverman [5] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on SH and its subclasses. In the present sequel to these earlier investigations, Jahangiri et al. [6] applied the Alexander integral transforms of certain analytic functions (which are starlike or convex of positive order) with a view to investigating the construction of sense-preserving, univalent, and close-to-convex harmonic functions.

Note that SH reduces to the class S of normalized analytic univalent functions in U, if the co-analytic part of f is identically zero.

For $f \in S$, the differential operator D^n $(n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\})$ of f was introduced by Sălăgean [9]. For $f = h + \overline{g}$ given by (1.1), Jahangiri et al. [8] defined the modified Sălăgean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}, \tag{1.2}$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad D^n g(z) = \sum_{k=2}^{\infty} k^n b_k z^k.$$

We say that a function $f : U \to \mathbb{C}$ is subordinate to a function $g : U \to \mathbb{C}$, and write $f(z) \prec g(z)$, if there exists a complex valued function w which maps U into itself with $w(0) = 0$, such that

$$f(z) = g(w(z)) \quad (z \in U).$$

Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

The Hadamard product (or convolution) of functions f_1 and f_2 of the form

$$f_k(z) = z + \sum_{k=2}^{\infty} a_{t,k} z^k + \sum_{k=2}^{\infty} \overline{b_{t,k} z^k} \qquad (z \in U,\ t \in \{1,2\})$$

is defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{1,k}\ a_{2,k} z^k + \sum_{k=2}^{\infty} \overline{b_{1,k}\ b_{2,k} z^k} \qquad (z \in U).$$

Denote by $SH^0_\delta(n, A, B)$ the subclass of SH^0 consisting of functions f of the form (1.1) that satisfy the condition

$$\frac{\delta D^{n+2} f(z) + (1-\delta)\, D^{n+1} f(z)}{\delta D^{n+1} f(z) + (1-\delta)\, D^n f(z)} \prec \frac{1+Az}{1+Bz}, \tag{1.3}$$

$$(0 \le \delta \le 1, -B \le A < B \le 1)$$

where $D^n f(z)$ is defined by (1.2).

By suitably specializing the parameters, the class $SH_{\delta}^{0}(n,A,B)$ reduces to the various subclasses of harmonic univalent functions, such as,
(i) $SH_{0}^{0}(\lambda,A,B)=H^{\lambda}(A,B),\ \lambda\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ ([4]),
(ii) $SH_{0}^{0}(0,A,B)=S_{H}^{*}(A,B)\cap SH^{0}$ ([2]),
(iii) $SH_{0}^{0}(n,2\alpha-1,1)=H^{0}(n,\alpha)$ ([8]),
(iv) $SH_{0}^{0}(0,2\alpha-1,1)=S_{H^0}^{*}(\alpha)$ ([7], [10], [11]),
(v) $SH_{0}^{0}(1,2\alpha-1,1)=SH_{1}^{0}(0,2\alpha-1,1)=S_{H^0}^{c}(\alpha)$ ([7]).
Making use of the techniques and methodology used by Dziok (see [2], [3]), Dziok et al. [4], in this paper we find necessary and sufficient conditions, distortion bounds, radii of starlikeness and convexity, compactness and extreme points for the above defined class $SH_{\delta}^{0}(n,A,B)$.

2. Main results

First theorem provides a necessary and sufficient convolution condition for the harmonic functions in $SH_{\delta}^{0}(n,A,B)$.

Theorem 2.1. *A function f belongs to the class $SH_{\delta}^{0}(n,A,B)$ if and only if $f\in SH^{0}$ and*

$$D^{n}f(z)*\Phi(z;\zeta)\neq 0 \qquad (\zeta\in\mathbb{C},\ |\zeta|=1,\ z\in U),$$

where

$$\begin{aligned}\Phi(z;\zeta) &= \tfrac{(1+A\zeta)(\delta-1)z^{3}+(\delta+1)z^{2}+[B(2\delta-1)+A(2-\delta)]\zeta z^{2}+(B-A)\zeta z}{(1-z)^{3}}\\ &\quad -(-1)^{n}\tfrac{(1-\delta)(1+A\zeta)\bar{z}^{3}-3(1-\delta)\bar{z}^{2}-[B+(2-3\delta)A]\zeta\bar{z}^{2}+(1-2\delta)[2+(B+A)\zeta]\bar{z}}{(1-\bar{z})^{3}}.\end{aligned}$$

Proof. Let $f\in SH^{0}$. Then $f\in SH_{\delta}^{0}(n,A,B)$ if and only if (1.3) holds or equivalently

$$\frac{\delta D^{n+2}f(z)+(1-\delta)\,D^{n+1}f(z)}{\delta D^{n+1}f(z)+(1-\delta)\,D^{n}f(z)}\neq\frac{1+A\zeta}{1+B\zeta}\ (\zeta\in\mathbb{C},\ |\zeta|=1,\ z\in U). \qquad (2.1)$$

Now for

$$D^{n}h(z)=D^{n}h(z)*\frac{z}{1-z},\ D^{n+1}h(z)=D^{n}h(z)*\frac{z}{(1-z)^{2}}\ \text{and}\ D^{n+2}h(z)=D^{n}h(z)*\frac{z+z^{2}}{(1-z)^{3}},$$

the inequality (2.1) yields

$$\begin{aligned}&(1+B\zeta)[\delta D^{n+2}f(z)+(1-\delta)\,D^{n+1}f(z)]-(1+A\zeta)[\delta D^{n+1}f(z)+(1-\delta)\,D^{n}f(z)]\\ =\ & D^{n}h(z)*\left\{(1+B\zeta)\left[\frac{\delta z+\delta z^{2}}{(1-z)^{3}}+\frac{(1-\delta)z}{(1-z)^{2}}\right]-(1+A\zeta)\left[\frac{\delta z}{(1-z)^{2}}+\frac{(1-\delta)z}{1-z}\right]\right\}\\ &+(-1)^{n}\overline{D^{n}g(z)}*\left\{(1+B\zeta)\left[\frac{\delta\bar{z}+\delta\bar{z}^{2}}{(1-\bar{z})^{3}}-\frac{(1-\delta)\bar{z}}{(1-\bar{z})^{2}}\right]+(1+A\zeta)\left[\frac{\delta\bar{z}}{(1-\bar{z})^{2}}-\frac{(1-\delta)\bar{z}}{1-\bar{z}}\right]\right\}\\ =\ & D^{n}f(z)*\Phi(z;\zeta)\neq 0.\end{aligned}$$

□

Next we give the sufficient coefficient bound for functions in $SH_{\delta}^{0}(n,A,B)$.

Theorem 2.2. *Let $f = h + \overline{g}$ be so that h and g are given by (1.1). Then $f \in SH_{\delta}^{0}(n, A, B)$, if*

$$\sum_{k=2}^{\infty} \left(\phi_k \left|a_k\right| + \psi_k \left|b_k\right|\right) \leq B - A, \tag{2.2}$$

where

$$\phi_k = k^n [(B+1)\,k - (A+1)]\,[1 + \delta(k-1)] \tag{2.3}$$

and

$$\psi_k = k^n [(B+1)\,k + (A+1)]\,\left|1 - \delta\,(k+1)\right|. \tag{2.4}$$

Proof. It is easy to see that the theorem is true for $f(z) = z$. So, we assume that $a_k \neq 0$ or $b_k \neq 0$ for $k \geq 2$. Since $\phi_k \geq k\,(B-A)$ and $\psi_k \geq k\,(B-A)$ by (2.2), we obtain

$$\begin{aligned}
\left|h'(z)\right| - \left|g'(z)\right| &\geq 1 - \sum_{k=2}^{\infty} k\left|a_k\right|\left|z\right|^{k-1} - \sum_{k=2}^{\infty} k\left|b_k\right|\left|z\right|^{k-1} \\
&\geq 1 - |z| \sum_{k=2}^{\infty} \left(k\left|a_k\right| + k\left|b_k\right|\right) \\
&\geq 1 - \frac{|z|}{B-A} \sum_{k=2}^{\infty} \left(\phi_k \left|a_k\right| + \psi_k \left|b_k\right|\right) \\
&\geq 1 - |z| > 0.
\end{aligned}$$

Therefore f is sense preserving and locally univalent in U. For the univalence condition, consider $z_1, z_2 \in U$ so that $z_1 \neq z_2$. Then

$$\left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right| = \left|\sum_{m=1}^{k} z_1^{m-1} z_2^{k-m}\right| \leq \sum_{m=1}^{k} \left|z_1^{m-1}\right|\left|z_2^{k-m}\right| < k, \quad k \geq 2.$$

Hence

$$\begin{aligned}
\left|f\left(z_1\right) - f\left(z_2\right)\right| &\geq \left|h\left(z_1\right) - h\left(z_2\right)\right| - \left|g\left(z_1\right) - g\left(z_2\right)\right| \\
&\geq \left|z_1 - z_2 - \sum_{k=2}^{\infty} a_k \left(z_1^k - z_2^k\right)\right| - \left|\sum_{k=2}^{\infty} \overline{b_k \left(z_1^k - z_2^k\right)}\right| \\
&\geq \left|z_1 - z_2\right| - \sum_{k=2}^{\infty} \left|a_k\right|\left|z_1^k - z_2^k\right| - \sum_{k=2}^{\infty} \left|b_k\right|\left|z_1^k - z_2^k\right| \\
&= \left|z_1 - z_2\right| \left(1 - \sum_{k=2}^{\infty} \left|a_k\right|\left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right| - \sum_{k=2}^{\infty} \left|b_k\right|\left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right|\right) \\
&> \left|z_1 - z_2\right| \left(1 - \sum_{k=2}^{\infty} k\left|a_k\right| - \sum_{k=2}^{\infty} k\left|b_k\right|\right) \geq 0
\end{aligned}$$

which proves univalence.

On the other hand, $f \in SH_{\delta}^{0}(n,A,B)$ if and only if there exists a complex valued function w; $w(0)=0$, $|w(z)|<1$ $(z\in U)$ such that

$$\frac{\delta D^{n+2}f(z)+(1-\delta)\,D^{n+1}f\,(z)}{\delta D^{n+1}f(z)+(1-\delta)\,D^{n}f\,(z)}=\frac{1+Aw(z)}{1+Bw(z)}$$

or equivalently

$$\left|\frac{\delta D^{n+2}f(z)+(1-2\delta)\,D^{n+1}f(z)-(1-\delta)\,D^{n}f\,(z)}{-B\delta D^{n+2}f\,(z)+[\delta\,(A+B)-B]\,D^{n+1}f(z)+A\,(1-\delta)\,D^{n}f(z)}\right|<1. \tag{2.5}$$

The above inequality (2.5) holds, since for $|z|=r$ $(0<r<1)$ we obtain

$$\begin{aligned}
&\left|\delta D^{n+2}f(z)+(1-2\delta)\,D^{n+1}f(z)-(1-\delta)\,D^{n}f\,(z)\right| \\
&-\left|B\delta D^{n+2}f\,(z)-[\delta\,(A+B)-B]\,D^{n+1}f(z)-A\,(1-\delta)\,D^{n}f(z)\right| \\
=\;&\left|\sum_{k=2}^{\infty}k^{n}\left\{\delta k^{2}+(1-2\delta)\,k-(1-\delta)\right\}a_{k}z^{k}+(-1)^{n}\sum_{k=2}^{\infty}k^{n}\left[\delta k^{2}-(1-2\delta)\,k-(1-\delta)\right]\overline{b_{k}z^{k}}\right| \\
&-\left|(B-A)\,z+\sum_{k=2}^{\infty}k^{n}\left\{B\delta k^{2}+[B-\delta\,(A+B)]\,k-A\,(1-\delta)\right\}a_{k}z^{k}\right. \\
&\left.+(-1)^{n}\sum_{k=2}^{\infty}k^{n}\left\{B\delta k^{2}-[B-\delta\,(A+B)]\,k-A\,(1-\delta)\right\}\overline{b_{k}z^{k}}\right| \\
\leq\;&\sum_{k=2}^{\infty}k^{n}\,(k-1)\,[\delta(k-1)+1]\,|a_{k}|\,r^{k} \\
&+\sum_{k=2}^{\infty}k^{n}\,(k+1)\,|\delta(k+1)-1|\,|b_{k}|\,r^{k}-(B-A)\,r \\
&+\sum_{k=2}^{\infty}k^{n}\,(Bk-A)\,[\delta(k-1)+1]\,|a_{k}|\,r^{k} \\
&+\sum_{k=2}^{\infty}k^{n}\,(Bk+A)\,|\delta(k+1)-1|\,|b_{k}|\,r^{k} \\
\leq\;&r\left\{\sum_{k=2}^{\infty}\phi_{k}\,|a_{k}|\,r^{k-1}+\sum_{k=2}^{\infty}\psi_{k}\,|b_{k}|\,r^{k-1}-(B-A)\right\}<0.
\end{aligned}$$

Therefore $f\in SH_{\delta}^{0}(n,A,B)$, and so the proof is complete. □

Next we show that the condition (2.2) is also necessary for the functions $f\in SH$ to be in the class $SHT_{\delta}^{0}(n,A,B)=T^{n}\cap SH_{\delta}^{0}(n,A,B)$, where T^{n} is the class of functions $f=h+\overline{g}\in SH^{0}$ so that

$$f=h+\overline{g}=z-\sum_{k=2}^{\infty}|a_{k}|\,z^{k}+(-1)^{n}\sum_{k=2}^{\infty}|b_{k}|\,\overline{z}^{k}\quad(z\in U). \tag{2.6}$$

Let ψ_{k} be defined by (2.4). For $\frac{1}{3}<\delta\leq 1$, we have

$$\psi_{k}=k^{n}[(B+1)\,k+(A+1)]\,[\delta\,(k+1)-1]\,.$$

Theorem 2.3. *Let $f = h + \overline{g}$ be defined by (2.6). Then $f \in SHT^0_\delta(n, A, B)$ if and only if the condition (2.2) holds where $\frac{1}{3} < \delta \leq 1$.*

Proof. The 'if' part follows from Theorem 2.2. For the 'only-if' part, assume that $f \in SHT^0_\delta(n, A, B)$, then by (2.5) we have

$$\left| \frac{\sum\limits_{k=2}^{\infty} k^n \left\{ \left[\delta k^2 + (1-2\delta)k - (1-\delta)\right] |a_k| z^k + \left[\delta k^2 - (1-2\delta)k - (1-\delta)\right] |b_k| \bar{z}^k \right\}}{(B-A)z - \sum\limits_{k=2}^{\infty} k^n \left\{ \left[B\delta k^2 + [B - \delta(A+B)]k - A(1-\delta)\right] |a_k| z^k + \left[B\delta k^2 - [B-\delta(A+B)]k - A(1-\delta)\right] |b_k| \bar{z}^k \right\}} \right| < 1.$$

For $z = r < 1$ we obtain

$$\frac{\sum\limits_{k=2}^{\infty} k^n \left\{ \left[\delta k^2 + (1-2\delta)k - (1-\delta)\right] |a_k| + \left[\delta k^2 - (1-2\delta)k - (1-\delta)\right] |b_k| \right\} r^{k-1}}{(B-A)z - \sum\limits_{k=2}^{\infty} k^n \left\{ \left[B\delta k^2 + [B - \delta(A+B)]k - A(1-\delta)\right] |a_k| + \left[B\delta k^2 - [B-\delta(A+B)]k - A(1-\delta)\right] |b_k| \right\} r^{k-1}} < 1.$$

Thus, for ϕ_k and ψ_k as defined by (2.3) and (2.4), we have

$$\sum_{k=2}^{\infty} \left[\phi_k |a_k| + \psi_k |b_k|\right] r^{k-1} < B - A \quad (0 \leq r < 1). \tag{2.7}$$

Let $\{\sigma_k\}$ be the sequence of partial sums of the series

$$\sum_{k=2}^{\infty} \left[\phi_k |a_k| + \psi_k |b_k|\right].$$

Then $\{\sigma_k\}$ is a nondecreasing sequence and by (2.7) it is bounded above by $B - A$. Thus, it is convergent and

$$\sum_{k=2}^{\infty} \left[\phi_k |a_k| + \psi_k |b_k|\right] = \lim_{k \to \infty} \sigma_k \leq B - A.$$

This gives the condition (2.2). □

In the following we show that the class of functions of the form (2.6) is convex and compact.

Theorem 2.4. *The class $SHT^0_\delta(n, A, B)$ is a convex and compact subset of SH.*

Proof. Let $f_t \in SHT^0_\delta(n, A, B)$, where

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}| z^k + (-1)^n \sum_{k=2}^{\infty} |b_{t,k}| \overline{z^k} \quad (z \in U, \ t \in \mathbb{N}). \tag{2.8}$$

Then $0 \leq \eta \leq 1$. Let $f_1, f_2 \in SHT^0_\delta(n, A, B)$ be defined by (2.8). Then

$$\begin{aligned} \kappa(z) &= \eta f_1(z) + (1-\eta) f_2(z) \\ &= z - \sum_{k=2}^{\infty} \left(\eta |a_{1,k}| + (1-\eta) |a_{2,k}|\right) z^k \\ &\quad + (-1)^n \sum_{k=2}^{\infty} \left(\eta |b_{1,k}| + (1-\eta) |b_{2,k}|\right) \overline{z^k} \end{aligned}$$

and

$$\begin{aligned}
& \sum_{k=2}^{\infty} \{\phi_k [\eta |a_{1,k}| + (1-\eta) |a_{2,k}|] + \psi_k [\eta |b_{1,k}| + (1-\eta) |b_{2,k}|]\} \\
= \;& \eta \sum_{k=2}^{\infty} \{\phi_k |a_{1,k}| + \psi_k |b_{1,k}|\} + (1-\eta) \sum_{k=2}^{\infty} \{\phi_k |a_{2,k}| + \psi_k |b_{2,k}|\} \\
\leq \;& \eta (B-A) + (1-\eta)(B-A) = B - A.
\end{aligned}$$

Thus, the function $\kappa = \eta f_1 + (1-\eta) f_2$ belongs to the class $SHT_\delta^0(n, A, B)$. This means that the class $SHT_\delta^0(n, A, B)$ is convex.

On the other hand, for $f_t \in SHT_\delta^0(n, A, B)$, $t \in \mathbb{N}$ and $|z| \leq r$ $(0 < r < 1)$, we get

$$\begin{aligned}
|f_t(z)| &\leq r + \sum_{k=2}^{\infty} \{|a_{t,k}| + |b_{t,k}|\} r^k \\
&\leq r + \sum_{k=2}^{\infty} \{\phi_k |a_{t,k}| + \psi_k |b_{t,k}|\} r^k \\
&\leq r + (B-A) r^2.
\end{aligned}$$

Therefore, $SHT_\delta^0(n, A, B)$ is locally uniformly bounded. Let

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}| z^k + (-1)^n \sum_{k=2}^{\infty} |b_{t,k}| \overline{z^k} \quad (z \in U,\ t \in \mathbb{N})$$

and let $f = h + \overline{g}$ be so that h and g are given by (1.1). Using Theorem 2.3 we obtain

$$\sum_{k=2}^{\infty} \{\phi_k |a_{t,k}| + \psi_k |b_{t,k}|\} \leq B - A. \tag{2.9}$$

If we assume that $f_t \to f$, then we conclude that $|a_{t,k}| \to |a_k|$ and $|b_{t,k}| \to |b_k|$ as $k \to \infty$ $(t \in \mathbb{N})$. Let $\{\sigma_k\}$ be the sequence of partial sums of the series $\sum_{k=2}^{\infty} \{\phi_k |a_k| + \psi_k |b_k|\}$. Then $\{\sigma_k\}$ is a nondecreasing sequence and by (2.9) it is bounded above by $B - A$. Thus, it is convergent and

$$\sum_{k=2}^{\infty} \{\phi_k |a_k| + \psi_k |b_k|\} = \lim_{k \to \infty} \sigma_k \leq B - A.$$

Therefore $f \in SHT_\delta^0(n, A, B)$ and therefore the class $SHT_\delta^0(n, A, B)$ is closed. In consequence, the class $SHT_\delta^0(n, A, B)$ is compact subset of SH, which completes the proof. □

We continue with the following lemma due to Jahangiri [7].

Lemma 2.5. *Let $f = h + \overline{g}$ be so that h and g are given by (1.1). Furthermore, let*

$$\sum_{k=2}^{\infty} \left\{ \frac{k-\alpha}{1-\alpha} |a_k| + \frac{k+\alpha}{1-\alpha} |b_k| \right\} \leq 1, \quad (z \in U)$$

where $0 \leq \alpha < 1$. Then f is harmonic, orientation preserving, univalent in U and f is starlike of order α.

In the following theorems we obtain the radii of starlikeness and convexity for functions in the class $SHT_\delta^0(n, A, B)$.

Theorem 2.6. *Let $0 \leq \alpha < 1$, ϕ_k and ψ_k be defined by (2.3) and (2.4). Then*

$$r_\alpha^*(SHT_\delta^0(n, A, B)) = \inf_{k \geq 2} \left[\frac{1-\alpha}{B-A} \min \left\{ \frac{\phi_k}{k-\alpha}, \frac{\psi_k}{k+\alpha} \right\} \right]^{\frac{1}{k-1}}. \tag{2.10}$$

Proof. Let $f \in SHT_\delta^0(n, A, B)$ be of the form (2.6). Then, for $|z| = r < 1$, we get

$$\left| \frac{Df(z) - (1+\alpha) f(z)}{Df(z) + (1-\alpha) f(z)} \right|$$

$$= \left| \frac{-\alpha z - \sum_{k=2}^{\infty} (k-1-\alpha) |a_k| z^k - (-1)^n \sum_{k=2}^{\infty} (k+1+\alpha) |b_k| \overline{z}^k}{(2-\alpha) z - \sum_{k=2}^{\infty} (k+1-\alpha) |a_k| z^k - (-1)^n \sum_{k=2}^{\infty} (k-1+\alpha) |b_k| \overline{z}^k} \right|$$

$$\leq \frac{\alpha + \sum_{k=2}^{\infty} \{(k-1-\alpha) |a_k| + (k+1+\alpha) |b_k|\} r^{k-1}}{2 - \alpha - \sum_{k=2}^{\infty} \{(k+1-\alpha) |a_k| + (k-1+\alpha) |b_k|\} r^{k-1}}.$$

Note (see Lemma 2.5) that f is starlike of order α in U_r if and only if

$$\left| \frac{Df(z) - (1+\alpha) f(z)}{Df(z) + (1-\alpha) f(z)} \right| < 1, \ z \in U_r$$

or

$$\sum_{k=2}^{\infty} \left\{ \frac{k-\alpha}{1-\alpha} |a_k| + \frac{k+\alpha}{1-\alpha} |b_k| \right\} r^{k-1} \leq 1. \tag{2.11}$$

Moreover, by Theorem 2.3, we have

$$\sum_{k=2}^{\infty} \left\{ \frac{\phi_k}{B-A} |a_k| + \frac{\psi_k}{B-A} |b_k| \right\} r^{k-1} \leq 1.$$

Since ϕ_k and ψ_k be defined by (2.3) and (2.4).

The condition (2.11) is true if

$$\frac{k-\alpha}{1-\alpha} r^{k-1} \leq \frac{\phi_k}{B-A} r^{k-1}$$

and

$$\frac{k+\alpha}{1-\alpha} r^{k-1} \leq \frac{\psi_k}{B-A} r^{k-1} \quad (k = 2, 3, \ldots)$$

or if

$$r \le \frac{1-\alpha}{B-A} \min\left\{\frac{\phi_k}{k-\alpha}, \frac{\psi_k}{k+\alpha}\right\}^{\frac{1}{k-1}} \quad (k = 2, 3, \ldots).$$

It follows that the function f is starlike of order α in the disk $U_{r_\alpha^*}$ where

$$r_\alpha^* := \inf_{k\ge 2}\left[\frac{1-\alpha}{B-A} \min\left\{\frac{\phi_k}{k-\alpha}, \frac{\psi_k}{k+\alpha}\right\}\right]^{\frac{1}{k-1}}.$$

The function

$$f_k(z) = h_k(z) + \overline{g_k(z)} = z - \frac{B-A}{\phi_k} z^k + (-1)^n \frac{B-A}{\psi_k} \overline{z}^k$$

proves that the radius r_α^* cannot be any larger. Thus we have (2.10). $\square$

Using a similar argument as above we obtain the following.

Theorem 2.7. *Let $0 \le \alpha < 1$ and ϕ_k and ψ_k be defined by (2.3) and (2.4). Then*

$$r_\alpha^c(SHT_\delta^0(n, A, B)) = \inf_{k\ge 2}\left[\frac{1-\alpha}{B-A} \min\left\{\frac{\phi_k}{k(k-\alpha)}, \frac{\psi_k}{k(k+\alpha)}\right\}\right]^{\frac{1}{k-1}}.$$

Our next theorem is on the extreme points of $SHT_\delta^0(n, A, B)$.

Theorem 2.8. *Extreme points of the class $SHT_\delta^0(n, A, B)$ are the functions f of the form (1.1) where $h = h_k$ and $g = g_k$ are of the form*

$$\begin{aligned} h_1(z) = z, \quad h_k(z) = z - \tfrac{B-A}{\phi_k} z^k, \\ g_k(z) = (-1)^n \tfrac{B-A}{\psi_k} \overline{z^k}, \quad (z \in U,\ k \ge 2), \end{aligned} \tag{2.12}$$

and $\frac{1}{3} < \delta \le 1$.

Proof. Let $g_k = \eta f_1 + (1-\eta) f_2$ where $0 < \eta < 1$ and $f_1, f_2 \in SHT_\delta^0(n, A, B)$ are functions of the form

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}|\, z^k + (-1)^n \sum_{k=2}^{\infty} |b_{t,k}|\, \overline{z^k} \quad (z \in U,\ t \in \{1, 2\}).$$

Then, by (2.2), we have

$$|b_{1,k}| = |b_{2,k}| = \frac{B-A}{\psi_k},$$

and therefore $a_{1,t} = a_{2,t} = 0$ for $t \in \{2, 3, \ldots\}$ and $b_{1,t} = b_{2,t} = 0$ for $t \in \{2, 3, \ldots\} \setminus \{k\}$. It follows that $g_k(z) = f_1(z) = f_2(z)$ and g_k are in the class of extreme points of the function class $SHT_\delta^0(n, A, B)$. Similarly, we can verify that the functions $h_k(z)$ are the extreme points of the class $SHT_\delta^0(n, A, B)$. Now, suppose that a function f of the form (1.1) is in the family of extreme points of the class $SHT_\delta^0(n, A, B)$ and f is not of the form (2.12). Then there exists $m \in \{2, 3, \ldots\}$ such that

$$0 < |a_m| < \frac{B-A}{m^n[(B+1)\,m - (A+1)]\,[1 + \delta(m-1)]}$$

or
$$0<|b_m|<\frac{B-A}{m^n[(B+1)\,m+(A+1)]\,[\delta(m+1)-1]}.$$
If
$$0<|a_m|<\frac{B-A}{m^n\left\{m^n[(B+1)\,m-(A+1)]\,[1+\delta(m-1)]\right\}},$$
then putting
$$\eta=\frac{|a_m|\,m^n\left\{m^n[(B+1)\,m-(A+1)]\,[1+\delta(m-1)]\right\}}{B-A}$$
and
$$\varphi=\frac{f-\eta h_m}{1-\eta},$$
we have $0<\eta<1,\ h_m\neq\varphi$, and
$$f=\eta h_m+(1-\eta)\,\varphi.$$

Therefore, f is not in the family of extreme points of the class $SHT_\delta^0(n,A,B)$. Similarly, if
$$0<|b_m|<\frac{B-A}{m^n[(B+1)\,m+(A+1)]\,[\delta(m+1)-1]},$$
then putting
$$\eta=\frac{|b_m|\,m^n[(B+1)\,m+(A+1)]\,[\delta(m+1)-1]}{B-A}$$
and
$$\varphi=\frac{f-\eta g_m}{1-\eta},$$
we have $0<\eta<1,\ g_m\neq\varphi$, and
$$f=\eta g_m+(1-\eta)\,\varphi.$$

It follows that f is not in the family of extreme points of the class $SHT_\delta^0(n,A,B)$ and so the proof is completed. □

Therefore, by Theorem 2.8, we have the following corollary.

Corollary 2.9. *Let $f\in SHT_\delta^0(n,A,B)$, be a function of the form (2.6). Then*
$$|a_k|\leq\frac{B-A}{k^n[(B+1)\,k-(A+1)]\,[\delta(k-1)+1]}$$
and
$$|b_k|\leq\frac{B-A}{k^n[(B+1)\,k+(A+1)]\,[\delta(k+1)-1]}.$$
The result is sharp for the extremal functions h_n,g_n of the form (2.12).

Corollary 2.10. *Let $f\in SHT_\delta^0(n,A,B)$ and $|z|=r<1$. Then*
$$r-\frac{B-A}{2^n\left[(\delta+1)\,(2B-A+1)\right]}r^2\leq|f(z)|\leq r+\frac{B-A}{2^n\left[(\delta+1)\,(2B-A+1)\right]}r^2.$$

The following covering result follows from Corollary 2.10.

Corollary 2.11. *If $f \in SHT_{\delta}^{0}(n, A, B)$, then $U_r \subset f(U)$ where*

$$r = 1 - \frac{B-A}{2^n \left[(\delta+1)(2B-A+1) \right]}.$$

Acknowledgement. The authors wish to acknowledge and thank both the reviewers and editors for job well-done reviewing this article.

References

1. J. Clunie, T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math., **9** (1984) 3-25.
2. J. Dziok, *Classes of harmonic functions defined by subordination*, Abstr. Appl. Anal., **2015** Article ID 756928 (2015) 9 pages.
3. J. Dziok, *On Janowski harmonic functions*, J. Appl. Anal., **21** (2015) 99-107 .
4. J. Dziok, J.M. Jahangiri, H. Silverman, *Harmonic functions with varying coefficients*, J. Inequal. Appl., **139** (2016) 12 pages.
5. J.M. Jahangiri and H. Silverman, *Harmonic close-to-convex mappings*, J. Appl. Math. Stochast. Anal., **15** (2002) 23–28.
6. J.M. Jahangiri , Y.C. Kim and H.M. Srivastava, *Construction of a certain class of harmonic close-to-convex functions associated with the Alexander integral transform*, Integral Transforms Spec. Funct., **14** (2003) 237-242.
7. J.M. Jahangiri, *Harmonic functions starlike in the unit disk*, J. Math. Anal. Appl., **235** (1999) 470-477.
8. J.M. Jahangiri, G. Murugusundaramoorthy, K. Vijaya, *Salagean-type harmonic univalent functions*, South J. Pure Appl. Math., **2** (2002) 77-82.
9. G.S. Sălăgean, *Subclasses of univalent functions*, (Lecture Notes in Math. Springer- Verlag Heidelberg, 1983) **1013**, 362-372.
10. H. Silverman, *Harmonic univalent functions with negative coefficients*, J. Math. Anal. Appl., **220** (1998) 283-289.
11. H. Silverman, E.M. Silvia, *Subclasses of harmonic univalent functions*, N. Z. J. Math., **28** (1999) 275-284.

[1] Department of Mathematics, Faculty of Arts and Science, Uludag University, 16059, Görükle, Bursa, Turkey.

E-mail address: serkan.cakmak64@gmail.com; syalcin@uludag.edu.tr; sahsenealtinkaya@gmail.com