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ACCURATE NUMERICAL METHOD FOR SINGULARLY PERTURBED DIFFERENTIAL-DIFFERENCE EQUATIONS WITH MIXED SHIFTS

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Abstract. This paper is concerned with the numerical solution of the singularly perturbed differential-difference equations with small shifts called delay and advanced parameters. A fourth order finite difference method with a fitting factor is proposed for the solution of the singularly perturbed differential-difference equations with mixed shifts. The delay and advanced shifts are managed by Taylor series, and an asymptotically equivalent singularly perturbed two-point boundary value problem is obtained. A fitting factor is introduced in the fourth order finite difference scheme for the problem which takes care of the small values of the perturbation parameter. This fitting factor is obtained from the asymptotic solution of singular perturbations. The Thomas algorithm is used to solve the discrete system of the difference scheme. Convergence of the proposed method is analyzed. Maximum absolute errors in comparison with the several numerical experiments are tabulated to illustrate the proposed method.

1. Introduction

Singularly perturbed differential-difference equations (SPDDEs) arise very frequently in the mathematical modeling of real life situations in science and engineering (see [3, 6, 12]). Mathematically, any ordinary differential equation in which the highest derivative is multiplied by a small positive parameter and

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containing at least one delay/advance parameter, is known as a singularly perturbed differential-difference equation. Lange and Miura [14, 15, 16, 17, 18] developed a series of papers to obtain an approximate solution of singularly perturbed differential-difference equations. Numerical analysis of SPDDE turning point problems is initiated by Kadalbajoo and Sharma. In a series of papers, see [7, 8, 9, 10, 11], they gave many robust numerical techniques for the solution of such type of problems. Kadalbajoo and Sharma [8] elucidated a numerical method to solve boundary value problems for SPDDEs. Kadalbajoo and Sharma [9] proposed a numerical method to solve a SPDDE of a mixed type with the case in which the solution of the problem exhibits rapid oscillations. Kadalbajoo and Sharma [10] described a numerical approach based on finite difference method to solve a mathematical model arising from a model of neuronal variability. Patidar and Sharma [20] combined fitted-operator methods with Micken's nonstandard finite difference techniques for the numerical approximations of singularly perturbed linear delay differential equations. Kadalbajoo et al. [11] derived ϵ-uniformly convergent fitted methods for the solution of SPDDE. Kumar and Sharma [13] presented a numerical scheme based on B-spline collocation to approximate the solution of boundary value problems for SPDDEs with delay as well as advance. Amiraliyev and Cimen [1] derived a numerical method for singularly perturbed boundary value problem for a linear second order delay differential equation with a large delay in the reaction term. The authors constructed an exponentially fitted differential scheme on a uniform mesh accomplished by the method of integral identities with the use of exponential basis functions and interpolation quadrature rules with weight and the remainder term in the integral form. The paper is organized as follows:

In Section 2, the description of the problem is given. In Section 3, numerical scheme for the solution of the problem is derived, and Section 4 deals with convergence analysis of the proposed scheme. To demonstrate the efficiency of the proposed method, numerical experiments are carried out for several test problems, and the results are given in Section 5. Finally the conclusion is given in the last section.

2. Description of the problem

Consider a linear singularly perturbed differential-difference equation with mixed shifts of the following form:

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x-\delta) + r(x)y(x) + s(x)y(x+\eta) = f(x) \tag{2.1}$$

on (0, 1), under the boundary conditions

$$\begin{aligned} y(x) &= \varphi(x) \quad \text{on} \ -\delta \le x \le 0, \\ y(\mathrm{x}) &= \gamma(x) \quad \text{on} \ 1 \le x \le 1+\eta, \end{aligned} \tag{2.2}$$

where ε is a small parameter, $0 < \varepsilon << 1$, $p(x), q(x), r(x), s(x), f(x), \varphi(x)$, and $\gamma(x)$ are smooth functions, and $0 < \delta = o(\varepsilon)$ and $0 < \eta = o(\varepsilon)$ are, respectively, the delay (negative shift) and the advance (positive shift) parameters. The solution of Eqs. (2.1) and (2.2) exhibits; the layer is at the left end of the interval if $p(x) - \delta q(x) + \eta s(x) > 0$ and the layer is at the right end of the interval if

$p(x) - \delta q(x) + \eta s(x) < 0$. If $p(x) = 0$, then the solution of the given problem may have oscillatory solution or two layers (one at each end) depending upon the cases whether $q(x) + r(x) + s(x)$ is positive or negative. Since the solution $y(x)$ of the boundary value problem Eqs. (2.1) and (2.2) is sufficiently differentiable, using Taylor series, the terms $y(x - \delta)$ and $y(x + \eta)$ are expanded as follows[5]:

$$y(x - \delta) \approx y(x) - \delta y'(x), \tag{2.3}$$

$$y(x + \eta) \approx y(x) + \eta y'(x). \tag{2.4}$$

Using Eqs. (2.3) and (2.4) in Eq. (2.1), we get

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \tag{2.5}$$

with boundary conditions

$$y(0) = \varphi(0) = \varphi_0$$
$$y(1) = \gamma(1) = \gamma_1.$$

Here

$$a(x) = p(x) - \delta q(x) + \eta s(x)$$
$$b(x) = q(x) + r(x) + s(x).$$

Eq. (2.5) is a second order singularly perturbed two point boundary value problem.

The zeroth order approximation to the solution of (2.5) [19] is

$$\lim_{h\to 0} y(ih) \approx y_0(0) + (\varphi(0) - y_0(0) \exp\{-a(0)i\rho\} \quad \text{for } i = 0,1,2,\ldots,N,$$

where $\rho = \frac{h}{\varepsilon}$ and y_0 represents the zeroth order approximate outer solution (i.e., the solution of the reduced problem).

3. Numerical method

Consider the uniform grid ω_h=$\{x_i = ih \quad \text{for} \quad i = 0, 1, 2, \ldots, N, h = 1/N\}$ on [0, 1]. At the grid point $x = x_i$, Eq. (2.5) becomes

$$\varepsilon y_i'' + a_i y_i' + b_i y_i = f_i. \tag{3.1}$$

We now consider higher order central difference formulas for y_i' and y_i'' as given below

$$y_i'' \cong \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - \frac{h^2}{12} y_i^{(4)} + R_1 \tag{3.2}$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i''' + R_2, \tag{3.3}$$

where

$$R_1 = -\frac{2h^4 y^{(6)}(\xi)}{6!}$$

$$R_2 = -\frac{h^4 y^{(5)}(\eta)}{5!}$$

for $\xi, \eta \in [x_{i-1},\ x_{i+1}]$. Differentiating Eq. (2.5), we obtain y_i''' and $y_i^{(4)}$ as follows:

$$y_i''' = \left[-\frac{a_i}{\varepsilon} y_i'' - \frac{(a_i' + b_i)}{\varepsilon} y_i' - \frac{b_i'}{\varepsilon} y + \frac{f'}{\varepsilon} \right] \tag{3.4}$$

$$y_i^{(4)} = \left[\frac{a_i^2}{\varepsilon^2} - \frac{(2a_i' + b_i)}{\varepsilon}\right] y_i'' + \left[\frac{a_i\,(a_i' + b_i)}{\varepsilon^2} - \frac{(a_i'' + 2b_i')}{\varepsilon}\right] y_i' + \left[\frac{ab_i'}{\varepsilon^2} - \frac{b_i''}{\varepsilon}\right] y_i + \frac{1}{\varepsilon} f_i''. \tag{3.5}$$

Using Eqs. (3.4) and (3.5) in Eqs. (3.2) and (3.3) and then substituting y_i' and y_i'' in Eq. (3.1), we get

$$\varepsilon\left[A_1(i)\left(\tfrac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right) + B_1(i)\tfrac{(y_{i+1}-y_{i-1})}{2h} - C_1(i)y_i - \tfrac{h^2}{12\varepsilon}f''_i\right]$$

$$+a_i\left[A_2(i)\left(\tfrac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right) + B_2(i)\tfrac{(y_{i+1}-y_{i-1})}{2h} + C_2(i)y_i - \tfrac{h^2 f'_i}{6\varepsilon}\right] + b_i y_i = f_i$$

Here $A(i) = 1 - \frac{h^2 {a_i}^2}{12\varepsilon^2} + \frac{h^2(2a'_i+b_i)}{12\varepsilon}$, $B(i) = \frac{h^2(a''_i+2b'_i)}{12\varepsilon} - \frac{h^2 a_i(a'_i+b_i)}{12\varepsilon}$,

$C(i) = \frac{h^2 b''_i}{12\varepsilon} - \frac{a_i b'_i h^2}{12\varepsilon^2}$, $A_2(i) = \frac{a_i h^2}{6\varepsilon}$, $B_2(i) = 1 + \frac{h^2}{6\varepsilon}(a'_i + b_i)$, and $C_2(i) = \frac{h^2}{6\varepsilon}b'_i$.

Now introducing a fitting factor σ in the above finite difference scheme, we get

$$\sigma\varepsilon\left[A_1(i)\left(\tfrac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right) + B_1(i)\tfrac{(y_{i+1}-y_{i-1})}{2h} - C_1(i)y_i - \tfrac{h^2}{12\varepsilon}f''_i\right]$$

$$+a_i\left[A_2(i)\left(\tfrac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right) + B_2(i)\tfrac{(y_{i+1}-y_{i-1})}{2h} + C_2(i)y_i - \tfrac{h^2 f'_i}{6\varepsilon}\right] + b_i y_i = f_i \tag{3.6}$$

Multiplying the above equation by h, taking the limit as $h \to 0$ and using Lemma 3 in [4], we get the fitting factor as

$$\sigma = \frac{a(0)}{2}\left(\frac{\coth\left(\frac{a(0)\rho}{2}\right) - \frac{\rho a^2(0)}{3}}{\left(\frac{1}{\rho} - \frac{\rho a^2(0)}{12}\right)}\right).$$

The tridiagonal system Eq. (3.6) is given by

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \ldots, N-1, \tag{3.7}$$

where

$$E_i = \tfrac{\sigma\varepsilon}{h^2} - \tfrac{\sigma {a_i}^2}{12\varepsilon} + \tfrac{\sigma(2a'_i+b_i)}{12} + \tfrac{\sigma {a_i}^2}{6\varepsilon} - \tfrac{\sigma h}{24}(a''_i + 2b'_i) + \tfrac{\sigma h a_i(a'_i+b_i)}{24\varepsilon}$$
$$-\tfrac{a_i}{2h}\left(1 + \tfrac{h^2}{6\varepsilon}(a'_i + b_i)\right)$$

$$F_i = \tfrac{2\sigma a_i^2}{12\varepsilon} - \tfrac{2\sigma\varepsilon}{h^2} - \tfrac{2\sigma(2a_i'+b_i)}{12} - \tfrac{2\sigma a_i^2}{6} + \tfrac{\sigma h^2 b_i''}{12} - \tfrac{\sigma h^2 a_i b_i'}{12\varepsilon} + \tfrac{h^2 a_i^2 b_i'}{6\varepsilon} + b_i$$

$$G_i = \tfrac{\sigma\varepsilon}{h^2} - \tfrac{\sigma a_i^2}{12\varepsilon} + \tfrac{\sigma(2a_i'+b_i)}{12} + \tfrac{a_i^2}{6\varepsilon} + \tfrac{\sigma h}{24}(a_i'' + 2b_i') - \tfrac{\sigma h a_i(a_i'+b_i)}{24\varepsilon}$$
$$+\tfrac{a_i}{2h}\left(1 + \tfrac{h^2}{6\varepsilon}(a_i' + b_i)\right)$$

$$H_i = \frac{\sigma\varepsilon h^2}{12\varepsilon} f_i'' + \frac{a_i h^2}{6\varepsilon} f_i' + f_i.$$

Eq. (3.7) is solved by using the Thomas algorithm [2].

4. Convergence analysis

The matrix-vector form of the tridiagonal system of Eq. (3.7) is

$$AY = C, \tag{4.1}$$

where $A = (m_{i\,j})\,,\ \ 1 \le i, j \le N\text{-}1,$ is a tridiagonal matrix of order N-1 with

$$m_{ii+1} = \tfrac{\sigma\varepsilon}{h^2} - \tfrac{\sigma a_i^2}{12\varepsilon} + \tfrac{\sigma(2a_i'+b_i)}{12} + \tfrac{a_i^2}{6\varepsilon} + \tfrac{\sigma h}{24}(a_i''+2b_i') - \tfrac{\sigma h a_i(a_i'+b_i)}{24\varepsilon} + \tfrac{a_i}{2h}\left(1+\tfrac{h^2}{6\varepsilon}(a_i'+b_i)\right),$$

$$m_{ii} = \tfrac{2\sigma a_i^2}{12\varepsilon} - \tfrac{2\sigma\varepsilon}{h^2} - \tfrac{2\sigma(2a_i'+b_i)}{12} - \tfrac{2\sigma a_i^2}{6} + \tfrac{\sigma h^2 b_i''}{12} - \tfrac{\sigma h^2 a_i b_i'}{12\varepsilon} + \tfrac{h^2 a_i^2 b_i'}{6\varepsilon} + b_i,$$

$$\begin{aligned} m_{ii-1} = {} & \tfrac{\sigma\varepsilon}{h^2} - \tfrac{\sigma a_i^2}{12\varepsilon} + \tfrac{\sigma(2a_i'+b_i)}{12} + \tfrac{\sigma a_i^2}{6\varepsilon} - \tfrac{\sigma h}{24}(a_i''+2b_i') + \tfrac{\sigma h a_i(a_i'+b_i)}{24\varepsilon} \\ & -\tfrac{a_i}{2h}\left(1+\tfrac{h^2}{6\varepsilon}(a_i'+b_i)\right), \end{aligned}$$

and $C = (d_i)$ is a column vector with $d_1 = -f_i h$ where i = 1,2,...,N-1 with local truncation error

$$\tau_i = \sigma\varepsilon\{\frac{y_{i+1}-y_i+y_{i-1}}{h^2} - \frac{h^2}{12}y_i^{4)} - y_i''\} + a(x)\{\left(\frac{y_{i+1}-y_{i-1}}{2h} - \frac{h^2}{6}y_i^{(3)}\right) - y_i'\},$$

$$|\tau_i| \le \max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{h^4 a(x)}{5!}\left|y^{(5)}(x)\right|\right\} + \max_{x_{i-1}\le x\le x_{i+1}}\left\{\frac{2h^4\sigma\varepsilon}{6!}\left|y^{(6)}(x)\right|\right\};$$

that is,

$$|\tau_i| \le O\left(h^4\right). \tag{4.2}$$

We have $Y = (y_0, y_1, y_2, \ldots, y_N)^t$. We also have

$$A\bar{Y} - T(h) = C, \tag{4.3}$$

where $\bar{Y} = (\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_N)^t$ denotes the actual solution and
$T(h) = (T_0(h), T_1(h), \ldots, T_N(h))^t$ is the local truncation error. From Eqs. (4.1) and (4.3), we get $A\left(\bar{Y} - Y\right) = T(h)$;
Thus the error equation is

$$AE = T(h), \tag{4.4}$$

where $E = \bar{Y} - Y = (e_0, e_1, \ldots, e_N)^t$. Clearly, we have

$$\begin{aligned} S_i = \sum_{j=1}^{N-1} m_{ij} = {} & -\tfrac{\sigma\varepsilon}{h^2} + \tfrac{\sigma a_i^2}{12\varepsilon} - \tfrac{\sigma(2a_i'+b_i)}{12} - \tfrac{a_i^2}{6\varepsilon} + \tfrac{a_i h^2 b_i''}{12} - \tfrac{\sigma h^2 a_i b_i^1}{12\varepsilon} + \tfrac{a_i^2 h^2 b_i^1}{6\varepsilon} + b_i + \tfrac{\sigma h a_i''}{24\varepsilon} \\ & + \tfrac{\sigma h b_i'}{12} - \tfrac{\sigma h a_i a_i'}{24\varepsilon} - \tfrac{\sigma h a_i b_i}{24\varepsilon} + \tfrac{a_i}{2h}\left(1+\tfrac{h^2}{6\varepsilon}(a_i'+b_i)\right), \qquad \text{for } \ i = 1, \end{aligned}$$

$$\begin{aligned} S_i = \sum_{j=1}^{N-1} m_{ij} & = b_i - \tfrac{\sigma h^2 b_i''}{12} - \tfrac{\sigma a_i b_i' h^2}{12\varepsilon} + \tfrac{a_i h^2 b_i'}{6\varepsilon} \\ & = b_i + O(h^2) = B_{i_0}, \qquad \text{for } i = 2, 3, \ldots, N-2, \end{aligned}$$

$$\begin{aligned} S_i = \sum_{j=1}^{N-1} m_{ij} = {} & -\tfrac{\sigma\varepsilon}{h^2} + \tfrac{\sigma a_i^2}{12\varepsilon} - \tfrac{\sigma(2a_i'+b_i)}{12} - \tfrac{a_i^2}{6\varepsilon} - \tfrac{\sigma h(a_i''+2b_i')}{24} + \tfrac{\sigma h a_i(a_i'+b_i)}{24\varepsilon} \\ & -\tfrac{a_i}{2h}\left(1+\tfrac{h^2}{6\varepsilon}(a_i'+b_i)\right) + \tfrac{a_i h^2 b_i''}{12} - \tfrac{\sigma h^2 a_i b_i^1}{12\varepsilon} + \tfrac{a_i^2 h^2 b_i^1}{6\varepsilon} \\ & + b_i \qquad \text{for } i = N-1. \end{aligned}$$

We can choose h; so that the matrix A is irreducible and monotone. It follows that A^{-1} exists and its elements are non-negative. Hence, using Eq. (4.4), we get $E = A^{-1}T(h)$ and

$$\|E\| \leq \|A^{-1}\|\|T(h)\|. \tag{4.5}$$

Also from the theory of matrices, we have

$$\sum_{i=1}^{N-1} \bar{m}_{k,i}\, S_i = 1, \quad k = 1, 2, \ldots, N-1,$$

where $\bar{m}_{k,i}$ is (k, i) element of the matrix A^{-1} for some i_0 between 1 and $N-1$. Therefore

$$\sum_{i=1}^{N-1} \bar{m}_{k,i} \leq \frac{1}{\min\limits_{1\leq i\leq N-1} S_i} = \frac{1}{B_{i_0}} \leq \frac{1}{|B_{i_0}|}. \tag{4.6}$$

We define $\|A^{-1}\| = \sum\limits_{i=1}^{N-1} |\bar{m}_{k,i}|$ and $\|T(h)\| = \sum\limits_{i=1}^{N-1} |T_i(h)|$. From Eqs. (4.2), (4.5), and (4.6), we get

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{k.i} T_i(h), \;\; j = 1, 2, \ldots, N-1,$$

which implies

$$e_j \leq \frac{O\left(h^4\right)}{|B_{i_0}|}, \quad j = 1, 2, \ldots, N-1,$$

where $B_{i_0} = b_i$. Therefore,

$$\|E\| = O\left(h^4\right);$$

that is, the proposed method reduces to a fourth order convergent on uniform mesh.

5. Numerical examples

To demonstrate the method computationally, we have considered three numerical examples. The numerical results are compared with the other method [10]. We have traced the graphs of the computed solution of the problem for different values of δ and η.

The solution of the problem

$$\epsilon y^{''} + p(x)y^{'} + q(x)y(x-\delta) + r(x)y(x) + s(x)y(x+\eta) = f(x)$$

under the boundary conditions

$$y(x) = \phi(x) \;\, on \;\, -\delta \leq x \leq 0$$
$$y(x) = \gamma(x) \;\, on \;\, 1 \leq x \leq 1+\eta$$

is

$$y_\epsilon(x) = c_1 e^{m_1(x)} + c_2 e^{m_2(x)} + \frac{f}{c},$$

where

$$c_1 = \frac{[-f + \gamma c + e^{m_2}(f - \phi c)]}{(e^{m_1} - e^{m_2})c},$$

$$c_2 = \frac{[-f + \gamma c + e^{m_1}(-f + \phi c)]}{(e^{m_1} - e^{m_2})c},$$

$$m_1 = \frac{-(p - q\delta + s\eta) + \sqrt{(p - q\delta + s\eta)^2 - 4\epsilon c}}{2\epsilon},$$

$$m_2 = \frac{-(p - q\delta + s\eta) - \sqrt{(p - q\delta + s\eta)^2 - 4\epsilon c}}{2\epsilon},$$

with $c = (q + r + s)$.

Example 1. Consider the model boundary value problem of the type given by Eqs. (2.1) and (2.2) having the boundary layer at the left-end

$$\epsilon y'' + y' - 2y(x - \delta) - 5y + y(x + \eta) = 0$$

with boundary conditions $y(x) = 1,\ -\delta \le x \le 0$, and $y(x) = 1,\ 1 \le x \le 1 + \eta$.

TABLE 1. The maximum absolute errors in solution of Example 1.

$\epsilon \downarrow$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
			$\delta = \eta = 0.5\,\epsilon$			
10^{-1}	0.0033038	0.0002201	$1.29e-005$	$7.98e-007$	$4.96e-008$	$3.10e-009$
10^{-2}	0.0235839	0.0076541	0.0031533	0.0009899	$5.76e-005$	$3.42e-006$
10^{-3}	0.0399002	0.0228969	0.0114511	0.0048135	0.0011596	0.0028588
10^{-4}	0.0418088	0.0250830	0.0137409	0.0071056	0.0035127	0.0016458
10^{-5}	0.0420041	0.0253101	0.0139842	0.0073554	0.0037635	0.0018932
10^{-6}	0.0420237	0.0253329	0.0140087	0.0073808	0.0037892	0.0019190
			Results in [10]			
10^{-1}	0.1201156	0.0711396	0.0448298	0.0269461	0.0151609	0.0077503
10^{-2}	0.1872710	0.1069782	0.0590411	0.3079689	0.0156796	0.0079907
10^{-3}	0.2042972	0.1191502	0.0687923	0.0365523	0.0189384	0.0096330
10^{-4}	0.2061414	0.1204841	0.0698994	0.0372137	0.0193277	0.0098423
10^{-5}	0.2063274	0.1206188	0.0700116	0.0372808	0.0136732	0.0098636
10^{-6}	0.2063460	0.1206323	0.0700229	0.0372876	0.0193712	0.0098657

The maximum absolute errors are given in Tables 1 and 2 for different values of the delay and advanced parameters with perturbation parameter. The effect of the small parameters on the boundary layer solutions is shown in Figures 1 and 2.

Example 2. Consider the nonhomogeneous boundary value problem of the type given by Eqs. (2.1) and (2.2) having the boundary layer at the left end

$$\epsilon y'' + y' - 2y(x - \delta) + y - y(x + \eta) = -1$$

with boundary conditions $y(x) = 1,\ -\delta \le x \le 0$, and $y(x) = 1,\ 1 \le x \le 1 + \eta$.

The maximum absolute errors are given in Table 3 for different values of the delay and advance parameter values . The effect of the small parameters on the boundary layer solutions is shown in Figures 3 and 4.

TABLE 2. The maximum absolute errors in solution of Example 1 with $\epsilon = 0.1$.

	$N = 8$	$N = 32$	$N = 128$	$N = 512$
$\delta \downarrow$		$\eta = 0.5\,\epsilon$		
0.00	0.00265280	$1.0220e-005$	$3.9258e-008$	$1.5344e-010$
0.05	0.00330379	$1.2961e-005$	$4.9696e-008$	$1.9390e-010$
0.09	0.003858959	$1.5323e-005$	$5.8654e-008$	$2.2897e-010$
$\eta \downarrow$		$\delta = 0.5\,\epsilon$		
0.00	0.00297213	$1.1559e-005$	$4.4365e-008$	$1.7319e-010$
0.05	0.00330379	$1.2961e-005$	$4.9696e-008$	$1.9390e-010$
0.09	0.00357767	$1.4124e-005$	$5.4113e-008$	$2.1112e-010$
$\delta \downarrow$		$\eta = 0.5\,\epsilon$(Results in [10])		
0.00	0.09190267	0.03453494	0.01164358	0.00300463
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.11018870	0.04110846	0.01400144	0.00362925
$\eta \downarrow$		$\delta = 0.5\,\epsilon$		
0.00	0.09720079	0.03640446	0.01229476	0.00317786
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.10632014	0.03965833	0.01348348	0.00349050

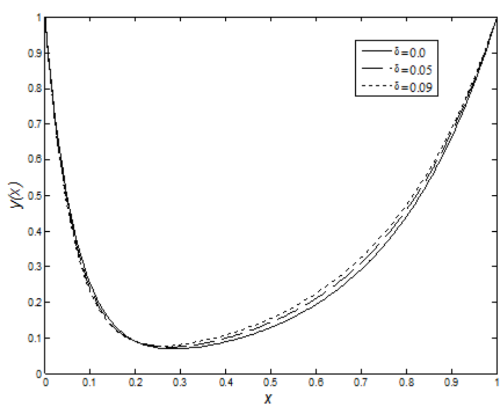

FIGURE 1. figure
Numerical solution of Example 1 for different values of δ with ϵ=0.1 and η=0.05.

Example 3. Consider the model boundary value problem of the type given by Eqs. (2.1) and (2.2) having the boundary layer at the right end

$$\epsilon y^{''} - y^{'} - 2y(x-\delta) + y - y(x+\eta) = 0$$

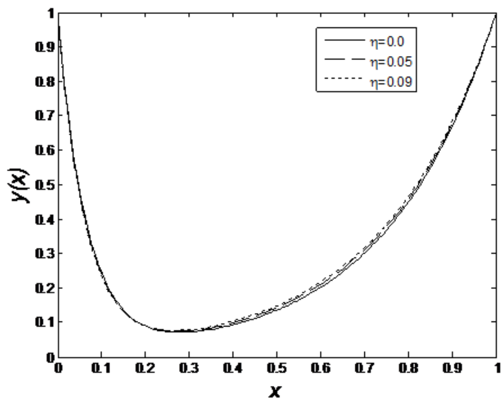

FIGURE 2. Numerical solution of Example 1 for different values of η with ϵ=0.1 and δ=0.05.

TABLE 3. The maximum absolute errors in solution of Example 2.

$\epsilon \downarrow$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
			$\delta = \eta = 0.5\epsilon$			
10^{-1}	0.0004362	$2.64e-005$	$1.58e-006$	$9.99e-008$	$6.22e-009$	$3.89e-010$
10^{-2}	0.0059028	0.0016684	0.0030736	0.0001274	$7.60e-006$	$4.51e-007$
10^{-3}	0.0075977	0.0040202	0.0019920	0.0009022	0.00030729	0.00031863
10^{-4}	0.0007726	0.0041663	0.0021546	0.0010890	0.00054083	0.00026222
10^{-5}	0.0007738	0.0041800	0.0021689	0.0011038	0.00055620	0.00027856
10^{-6}	0.0077398	0.0041813	0.0021703	0.0011052	0.00055765	0.00028002
			Results in [10]			
10^{-1}	0.0857969	0.0512956	0.0320213	0.0192472	0.0109835	0.0055359
10^{-2}	0.1337650	0.0764130	0.0421722	0.0219977	0.0111997	0.0057076
10^{-3}	0.1459266	0.0851073	0.0491373	0.0261088	0.0135274	0.0068807
10^{-4}	0.1472439	0.0860601	0.0499281	0.0265812	0.0138055	0.0070302
10^{-5}	0.1473767	0.0861563	0.0500083	0.0266292	0.0138338	0.0070454
10^{-6}	0.1473900	0.0861659	0.0500163	0.0266340	0.0138366	0.0070469

with boundary conditions $y(x) = 1, \ -\delta \leq x \leq 0$, and $y(x) = -1, \ 1 \leq x \leq 1 + \eta$.

The maximum absolute errors are given in Table 4 with $\epsilon = 0.1$ for different values of the delay and advance parameters. The effect of the small parameters on the boundary layer solutions is shown in Figures 5 and 6.

6. Discussions and conclusions

A fourth order finite difference method with fitting factor has been presented for solving singularly perturbed differential-difference equations with delay as well

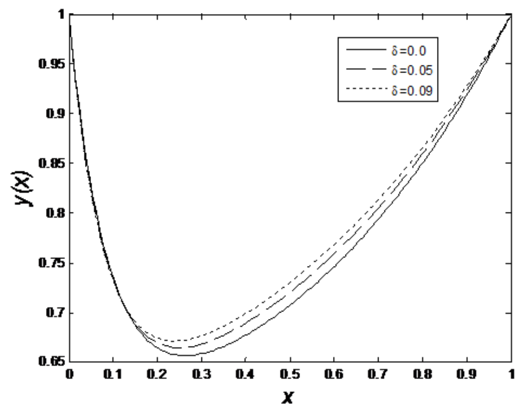

FIGURE 3. Numerical solution of Example 2 for different values of δ with ϵ=0.1 and η=0.05.

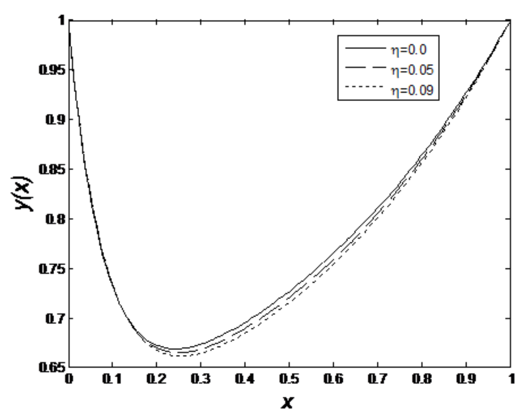

FIGURE 4. Numerical solution of Example 2 for different values of η with ϵ=0.1 and δ=0.05.

as advance parameters. A fitting factor is assigned to this scheme to control rapid behavior in the boundary layer due to the perturbation parameter when it takes small values. To demonstrate the method computationally, two examples with left-end and one with right-end boundary layer have been solved for different values of the delay, advance, and perturbation parameters. MATLAB is used for the numerical results and graphs. Maximum absolute errors in the solution of the problems are presented in tables. To support the method, numerical results taken by the proposed scheme are compared with the results of Kadalbajoo and Sharma [10]. It is observed that the present method approximates the exact solution very well for which other classical finite difference methods fail to give good results. The effect of the delay and advance parameters on the solutions has also been investigated and shown in graphs. From the numerical results, we noticed that the method gives good results even for $h > \varepsilon$. From the graphs, we observed

TABLE 4. The maximum absolute errors in solution of Example 3 with $\epsilon = 0.1$.

	$N = 8$	$N = 32$	$N = 128$	$N = 512$
$\delta \downarrow$		$\eta = 0.5\,\epsilon$		
0.00	0.002425427	$8.4802e-006$	$3.3166e-008$	$1.2946e-010$
0.05	0.001907515	$6.7239e-006$	$2.6104e-008$	$1.0189e-010$
0.09	0.001543162	$5.4589e-006$	$2.1118e-008$	$8.2514e-011$
$\eta \downarrow$		$\delta = 0.5\,\epsilon$		
0.00	0.001458758	$5.1627e-006$	$1.9978e-008$	$7.8025e-011$
0.05	0.001907515	$6.7239e-006$	$2.6104e-008$	$1.0189e-010$
0.09	0.002316112	$8.1139e-006$	$3.1667e-008$	$1.2364e-010$
$\delta \downarrow$		$\eta = 0.5\,\epsilon$(Results in [10])		
0.00	0.09930002	0.03685072	0.01331683	0.00342882
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.10044578	0.02850398	0.01038902	0.00266379
$\eta \downarrow$		$\delta = 0.5\,\epsilon$		
0.00	0.10055269	0.02759534	0.01007834	0.00258299
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.09944067	0.03591410	0.01297367	0.00334044

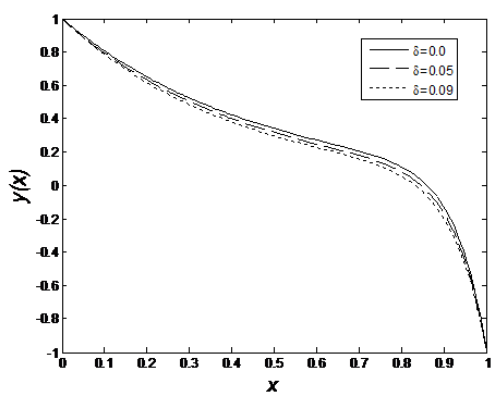

FIGURE 5. Numerical solution of Example 3 for different values of η with ϵ=0.1 and η=0.05.

that, when the solution of the boundary-value problem exhibits layer behavior on the left side, the effect of delay or advance on the solution in the boundary layer region is negligible while in the outer region, it is considerable; that is, the increase in the delay increases the width of outer region while the increase in the advance decreases the width of outer region (Figures 1–4). When the solution of the boundary-value problem exhibits layer behavior on the right side, the changes

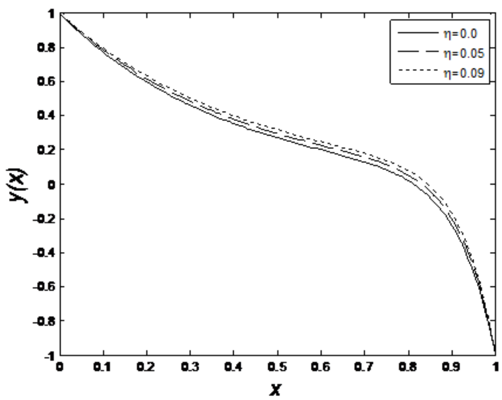

FIGURE 6. Numerical solution of Example 3 for different values of η with ϵ=0.1 and δ=0.05.

in delay or advance affect the solution in boundary layer region as well as outer region. The thickness of the layer increases as the size of the delay increases while it decreases as the size of the advance increases (Figures 5–6).

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