Khayyam J. Math. 5 (2019), no. 2, 15–29.
DOI: 10.22034/kjm.2019.84204

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

# POWER INEQUALITIES FOR THE NUMERICAL RADIUS OF OPERATORS IN HILBERT SPACES

MOHAMMAD H.M. RASHID

Communicated by M. Ito

Abstract. We generalize several inequalities involving powers of the numerical radius for the product of two operators acting on a Hilbert space. Moreover, we give a Jensen operator inequality for strongly convex functions. As a corollary, we improve the operator Hölder–McCarthy inequality under suitable conditions. In particular, we prove that if $f : J \to \mathbb{R}$ is strongly convex with modulus $c$ and differentiable on $\text{int}(J)$ whose derivative is continuous on $\text{int}(J)$ and if $T$ is a self-adjoint operator on the Hilbert space $\mathscr{H}$ with $\sigma(T) \subset \text{int}(J)$, then

$$\langle T^2x, x\rangle - \langle Tx, x\rangle^2 \leq \frac{1}{2c}(\langle f'(T)Tx, x\rangle - \langle Tx, x\rangle \langle f'(T)x, x\rangle)$$

for each $x \in \mathscr{H}$, with $\|x\| = 1$.

## 1. Introduction and preliminaries

Let $\mathscr{H}$ be complex Hilbert space and let $\mathscr{B}(\mathscr{H})$ be the Hilbert space of all bounded linear operators on $\mathscr{H}$. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *positive* if $\langle Tx, x\rangle \geq 0$ holds for all $x \in \mathscr{H}$. We write $T \geq 0$ if $T$ is positive.

The numerical radius of $T \in \mathscr{B}(\mathscr{H})$ is defined by

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle| : x \in \mathscr{H}, \|x\| = 1\}.$$

It is well known that $w(.)$ defines a norm on $\mathscr{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|.\|$. In fact, for any $T \in \mathscr{B}(\mathscr{H})$,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.1}$$

*Date*: Received: 2 August 2018; Revised: 3 November 2018; Accepted: 14 November 2018.
2010 *Mathematics Subject Classification.* Primary 47B20; Secondary 47A10, 47A11.
*Key words and phrases.* Numerical range, numerical radius, Aluthge transformation, strongly convex.

Also if $T \in \mathscr{B}(\mathscr{H})$ is normal, then $w(T) = \|T\|$.

An important inequality for $w(T)$ is the power inequality stating that $w(T^n) \leq (w(T))^n$ for every natural number $n$. Several numerical radius inequalities improving the inequalities in (1.1) has been recently given in [3, 4, 9].

Dragomir [5, 3] proved that for every $T, S \in \mathscr{B}(\mathscr{H})$,

$$w^2(T) \leq \frac{1}{2}(w(T^2) + \|T\|^2) \tag{1.2}$$

and

$$w^r(S^*T) \leq \frac{1}{2}\left\||T|^{2r} + |S|^{2r}\right\| \tag{1.3}$$

for all $r \geq 1$. Some interesting inequalities may be found in [9, 12, 13].

Every operator $T \in \mathscr{B}(\mathscr{H})$ can be decomposed into $T = U|T|$, where $U$ is a partial isometry and $|T|$ is the square root of $T^*T$. If $U$ is determined uniquely by the kernel condition $\ker(U) = \ker(|T|)$, then this decomposition is called the *polar decomposition*, which is one of the most important results in operator theory. In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $\ker(U) = \ker(|T|)$.

The Aluthge transform of an operator $T \in \mathscr{B}(\mathscr{H})$, denoted by $\widetilde{T}$, is defined by $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. For every $s, t > 0$, the generalized Aluthge transformation $\widetilde{T}_{s,t}$ is defined by $\widetilde{T}_{s,t} = |T|^s U|T|^t$, where $T = U|T|$ is the polar decomposition of $T$. If $s = t = \frac{1}{2}$, then $\widetilde{T}_{s,t}$ is the Aluthge transformation $\widetilde{T}$ of $T$. For $T \in \mathscr{B}(\mathscr{H})$, we generalize the Aluthge transformation of the operator $T$ to the form

$$\widetilde{T}_{f,g} = f(|T|)Ug(|T|),$$

in which $f$ and $g$ are nonnegative continuous functions such that $f(t)g(t) = t$ $(t \geq 0)$.

## 2. **Numerical radius inequalities**

To prove our generalized numerical radius, we need several well-known lemmas.

**Lemma 2.1.** *Let $a, b \geq 0$, $0 \leq \alpha \leq 1$, and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then for all nonnegative nondecreasing convex functions $h$ on $[0, \infty)$, we have*

(i) $h(a^\alpha b^{1-\alpha}) \leq \alpha h(a) + (1-\alpha)h(b)$.

(ii) $h(ab) \leq \dfrac{1}{p}h(a^p) + \dfrac{1}{q}h(b^q)$.

If we take $h(u) = u^r$ $(r \geq 1)$, we have the following result.

**Lemma 2.2.** *Let $a, b \geq 0$, $0 \leq \alpha \leq 1$ and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then*

(i) $a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}$;

(ii) $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q} \leq \left(\dfrac{a^{pr}}{p} + \dfrac{b^{qr}}{q}\right)^{\frac{1}{r}}$;

*for every $r \geq 1$.*

The following result that provides an operator version for Jensen's inequality is due to Mond and Pečarić [8].

**Theorem 2.3.** *Let $h(t)$ be a real valued continuous convex function, and let $T$ be a self-adjoint operator on a Hilbert space $\mathscr{H}$. Then Jensen's inequality asserts that*

$$h(\langle Tx, x\rangle) \le \langle h(T)x, x\rangle \tag{2.1}$$

*for any unit vector $x \in \mathscr{H}$.*

Notice that, if $h$ is concave, then inequality (2.1) is reversed.

The Hölder–McCarthy inequality [7] is a special case of Theorem 2.3.

**Lemma 2.4** (Hölder–McCarty inequality)**.** *Let $T \in \mathscr{B}(\mathscr{H})$, $T \ge 0$, and let $x \in \mathscr{H}$ be any unit vector. Then, we have*

(i) $\langle Tx, x\rangle^r \le \langle T^r x, x\rangle$ *for* $r \ge 1$.
(ii) $\langle T^r x, x\rangle \le \langle Tx, x\rangle^r$ *for* $0 < r \le 1$.
(iii) *If $T$ is invertible, then* $\langle Tx, x\rangle^r \le \langle T^r x, x\rangle$ *for all* $r < 0$.

The third lemma as the generalized mixed Schwarz inequality.

**Lemma 2.5.** *Let $T \in \mathscr{B}(\mathscr{H})$ and $x, y \in \mathscr{H}$ be any vectors.*

(i) *If $\alpha, \beta \ge 0$ such that $\alpha + \beta = 1$, then* $|\langle Tx, y\rangle|^2 \le \langle |T|^{2\alpha}x, x\rangle \left\langle |T|^{2\beta}y, y\right\rangle$.
(ii) *If $f$ and $g$ are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ $(t \ge 0)$, then* $|\langle Tx, y\rangle| \le \|f(|T|)x\| \, \|g(|T^*|)y\|$.

**Theorem 2.6.** *Let $T \in \mathscr{B}(\mathscr{H})$ be self-adjoint. Then for each nonnegative non-decreasing convex function $h$ on $[0, \infty)$, we have*

$$h(w^2(T)) \le \frac{1}{2}\left(h(w(T^2)) + h(\|T\|^2)\right). \tag{2.2}$$

*Proof.* We recall the following refinement of the Cauchy–Schwarz inequality obtained by Dragomir in [2]. It says that

$$\|u\|\,\|v\| \ge |\langle u, v\rangle - \langle u, z\rangle \langle z, v\rangle| + |\langle u, z\rangle \langle z, v\rangle| \ge |\langle u, v\rangle|, \tag{2.3}$$

where $u, v, z$ are vectors in $\mathscr{H}$ and $\|z\| = 1$. From inequality (2.3), we deduce that

$$|\langle u, z\rangle \langle z, v\rangle| \le \frac{1}{2}\left(\|u\|\,\|v\| + |\langle u, v\rangle|\right).$$

Put $z = x$ with $\|x\| = 1$, $u = Tx$, and $v = T^*x$ in the above inequality and use part (i) of Lemma 2.1 to get

$$|\langle Tx, x\rangle|^2 \le \frac{1}{2}\left(\|Tx\|^2 + \left\langle T^2x, x\right\rangle\right).$$

Now by convexity of $h$, we have

$$h(|\langle Tx, x\rangle|^2) \le \frac{1}{2}\left(h(\|Tx\|^2) + h(\left\langle T^2x, x\right\rangle)\right). \tag{2.4}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in inequality (2.4), we obtain the desired result. $\square$

**Theorem 2.7.** *Let $T \in \mathscr{B}(\mathscr{H})$, and let $f, g$ be nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t, (t \geq 0)$. Then for all nonnegative nondecreasing convex functions $h$ on $[0, \infty)$, we have*

$$h(w^2(T)) \leq \frac{1}{2}\left(h(\|T\|^2) + \left\|\frac{1}{p}h(f^p(|T^2|)) + \frac{1}{q}h(g^q(|T^2|))\right\|\right) \tag{2.5}$$

*for all $p \geq q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

*Proof.* Let $x \in \mathscr{H}$ be a unit vector. We have

$$\begin{aligned} h(|\langle T^2x, x\rangle|) &\leq h(\|f(|T^2|)\| \, \|g(|(T^*)^2|)\|) && \text{(by Lemma 2.5(b))} \\ &= h(\langle f^2(|T^2|)x, x\rangle^{\frac{1}{2}} \langle g^2(|(T^*)^2|)x, x\rangle^{\frac{1}{2}}) \\ &\leq \frac{1}{p}h(\langle f^2(|T^2|)x, x\rangle^{\frac{p}{2}}) + \frac{1}{q}h(\langle g^2(|(T^*)^2|)x, x\rangle^{\frac{q}{2}}) \\ & && \text{(by Lemma 2.1(b))} \\ &\leq \frac{1}{p}h(\langle f^p(|T^2|)x, x\rangle) + \frac{1}{q}h(\langle g^q(|(T^*)^2|)x, x\rangle) && \text{(by Lemma 2.3)} \\ &= \left\langle \left(\frac{1}{p}h(\langle f^p(|T^2|)x, x\rangle) + \frac{1}{q}h(\langle g^q(|(T^*)^2|)x, x\rangle)\right) x, x \right\rangle \end{aligned}$$

It follows from inequality (2.4) that

$$\begin{aligned} h(|\langle Tx, x\rangle|^2) \leq \frac{1}{2}\Bigg( & h(\|Tx\| \, \|T^*x\|) \\ & + \left\langle \left(\frac{1}{p}h(\langle f^p(|T^2|)x, x\rangle) + \frac{1}{q}h(\langle g^q(|(T^*)^2|)x, x\rangle)\right) x, x \right\rangle \Bigg). \end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in the above inequality, we obtain the desired result. □

**Theorem 2.8.** *Let $T \in \mathscr{B}(\mathscr{H})$, let $T = U|T|$ be the polar decomposition of $T$, and let $\widetilde{T}_{s,t} = |T|^s U|T|^t$ be the generalized Aluthge transformation of $T$ with $s + t = 1$. Then*

$$w^r(T) \leq \frac{1}{2}\left\||T|^{2rs} + |T^*|^{2rt}\right\|. \tag{2.6}$$

*for every $r \geq 1$.*

*Proof.* Using the Schwarz inequality in the Hilbert space, we have

$$\begin{aligned} |\langle Tx, x\rangle| = |\langle |T|^s x, |T|^t U^* x\rangle| &\leq \||T|^s x\| \, . \, \||T|^t U^* x\| \\ &= \langle |T|^{2s} x, x\rangle^{1/2} \langle |T^*|^{2r} x, x\rangle^{1/2}, \qquad x \in \mathscr{H}. \end{aligned} \tag{2.7}$$

Utilizing the arithmetic-mean geometry mean inequality and then the convexity of the function $h(u) = u^r$, $r \geq 1$, we have successively,

$$\left\langle |T|^{2s}x, x\right\rangle^{1/2} \left\langle |T^*|^{2r}x, x\right\rangle^{1/2} \leq \frac{\langle |T|^{2s}x, x\rangle + \langle |T^*|^{2t}x, x\rangle}{2} \leq \left(\frac{\langle |T|^{2s}x, x\rangle^r + \langle |T^*|^{2t}x, x\rangle^r}{2}\right)^{\frac{1}{r}} \tag{2.8}$$

for every $x \in \mathscr{H}$. It is known that if $Q$ is positive operator, then for every $r \geq 1$ and $x \in \mathscr{H}$ with $\|x\| = 1$, we have the inequality

$$\langle Qx, x\rangle^r \leq \langle Q^r x, x\rangle . \tag{2.9}$$

Applying this property to the positive operators $|T|^{2s}$ and $|T^*|^{2t}$, we deduce that

$$\left(\frac{\langle |T|^{2s}x, x\rangle^r + \langle |T^*|^{2t}x, x\rangle^r}{2}\right)^{\frac{1}{r}} \leq \left(\frac{\langle |T|^{2rs}x, x\rangle + \langle |T^*|^{2rt}x, x\rangle}{2}\right)^{\frac{1}{r}} \tag{2.10}$$

$$= \left(\frac{\langle (|T|^{2rs} + |T^*|^{2tr})x, x\rangle}{2}\right)^{\frac{1}{r}} \tag{2.11}$$

for any $x \in \mathscr{H}$, $\|x\| = 1$.

Now, on making use the inequalities (2.7), (2.8), and (2.10), we get the inequality

$$|\langle Tx, x\rangle|^r \leq \frac{1}{2}\left\langle (|T|^{2rs} + |T^*|^{2tr})x, x\right\rangle \tag{2.12}$$

for any $x \in \mathscr{H}$, $\|x\| = 1$.

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in inequality (2.12) and since the operator $|T|^{2rs} + |T^*|^{2tr}$ is self-adjoint, we deduce the desired inequality (2.6). □

**Theorem 2.9.** *Let $T \in \mathscr{B}(\mathscr{H})$, let $T = U|T|$ be the polar decomposition of $T$, and let $\widetilde{T}_{s,t} = |T|^s U|T|^t$ be the generalized Aluthge transformation of $T$ with $s + t = 1$. Then for each $\alpha \in (0, 1)$ and $r \geq 1$, we have*

$$w^{2r}(T) \leq \left\| \alpha |T|^{\frac{2rs}{\alpha}} + (1-\alpha)|T^*|^{\frac{2rt}{1-\alpha}} \right\|. \tag{2.13}$$

*Proof.* Using the Schwarz inequality, we have

$$|\langle Tx, x\rangle|^2 \leq \left\langle |T|^{2s}x, x\right\rangle \left\langle |T^*|^{2t}x, x\right\rangle \leq \left\langle (|T|^{\frac{2s}{\alpha}})^{\alpha}x, x\right\rangle \left\langle (|T^*|^{\frac{2t}{1-\alpha}})^{1-\alpha}x, x\right\rangle \tag{2.14}$$

for any $x \in \mathscr{H}$.

It is well-known that if $Q$ is a positive operator and $k \in (0, 1]$, then for any $u \in \mathscr{H}$, $\|u\| = 1$, we have

$$\left\langle Q^k u, u\right\rangle \leq \langle Qu, u\rangle^k . \tag{2.15}$$

Applying this property to the positive operators $|T|^{\frac{2s}{\alpha}}$ and $|T^*|^{\frac{2t}{1-\alpha}}$ $(\alpha \in (0, 1))$, we have

$$\left\langle (|T|^{\frac{2s}{\alpha}})^{\alpha}x, x\right\rangle . \left\langle (|T^*|^{\frac{2t}{1-\alpha}})^{1-\alpha}x, x\right\rangle \leq \left\langle |T|^{\frac{2s}{\alpha}}x, x\right\rangle^{\alpha} . \left\langle |T^*|^{\frac{2t}{1-\alpha}}x, x\right\rangle^{1-\alpha} \tag{2.16}$$

for every $x \in \mathscr{H}$, $\|x\|=1$.

Now, utilizing the weighted arithmetic mean-geometric mean inequality, that is, $a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b$, $\alpha \in (0,1), a, b \geq 0$, we obtain

$$\left\langle (|T|^{\frac{2s}{\alpha}})^{\alpha}x, x\right\rangle . \left\langle (|T^*|^{\frac{2t}{1-\alpha}})^{1-\alpha}x, x\right\rangle \leq \alpha \left\langle |T|^{\frac{2s}{\alpha}}x, x\right\rangle + (1-\alpha)\left\langle |T^*|^{\frac{2t}{1-\alpha}}x, x\right\rangle \tag{2.17}$$

for every $x \in \mathscr{H}$, $\|x\|=1$.

Moreover, by the following elementary inequality from the convexity of $h(v) = v^r$, $r \geq 1$, namely,

$$\alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}, \qquad \alpha \in (0,1), a, b \geq 0,$$

we deduce that

$$\begin{aligned}\alpha \left\langle |T|^{\frac{2s}{\alpha}}x, x\right\rangle + (1-\alpha)\left\langle |T^*|^{\frac{2t}{1-\alpha}}x, x\right\rangle &\leq \left(\alpha \left\langle |T|^{\frac{2s}{\alpha}}x, x\right\rangle^r + (1-\alpha)\left\langle |T^*|^{\frac{2t}{1-\alpha}}x, x\right\rangle^r\right)^{\frac{1}{r}} \\ &\leq \left(\alpha \left\langle |T|^{\frac{2rs}{\alpha}}x, x\right\rangle + (1-\alpha)\left\langle |T^*|^{\frac{2rt}{1-\alpha}}x, x\right\rangle\right)^{\frac{1}{r}}\end{aligned} \tag{2.18}$$

for any $x \in \mathscr{H}$, $\|x\|=1$.

Now, by making use of the inequalities (2.14), (2.16), (2.17), and (2.18), we obtain

$$|\langle Tx, x\rangle|^{2r} \leq \left\langle \left(\alpha |T|^{\frac{2rs}{\alpha}} + (1-\alpha)|T^*|^{\frac{2rt}{1-\alpha}}\right)x, x\right\rangle \tag{2.19}$$

for any $x \in \mathscr{H}$ with $\|x\|=1$.

Taking the supremum over $x \in \mathscr{H}$ with $\|x\|=1$ in inequality (2.19), we obtain the desired inequality. □

**Corollary 2.10.** *Let $T \in \mathscr{B}(\mathscr{H})$, let $T = U|T|$ be the polar decomposition of $T$, and let $\widetilde{T}_{s,t} = |T|^sU|T|^t$ be the generalized Aluthge transformation of $T$ with $s+t=1$. Then for every $r \geq 1$ and $p, q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$w^{2r}(T) \leq \left\| \frac{1}{p}|T|^{2prs} + \frac{1}{q}|T^*|^{2rqt}\right\|. \tag{2.20}$$

**Theorem 2.11.** *Let $T \in \mathscr{B}(\mathscr{H})$, let $T = U|T|$ be the polar decomposition of $T$, and let $\widetilde{T}_{s,t} = |T|^sU|T|^t$ be the generalized Aluthge transformation of $T$ with $s+t=1$. Then for every $\alpha \in (0,1)$ and $r \geq 1$, we have*

$$w^{2r}(T) \leq \frac{1}{2}\left(\|T\|^{2r} + \left\|\alpha|T^2|^{\frac{sr}{\alpha}} + (1-\alpha)|(T^2)^*|^{\frac{tr}{1-\alpha}}\right\|\right). \tag{2.21}$$

*Proof.* Let $x \in \mathscr{H}$ be a unit vector. We get

$$\begin{aligned}
|\langle T^2x,x\rangle|^r &\le \left\||T^2|^s x\right\|^r \left\||(T^2)^*|^s x\right\|^r && \text{(by Lemma 2.5(i))}\\
&= \left\langle |T^2|^{2s}x,x\right\rangle \left\langle |(T^2)^*|^{2t}x,x\right\rangle^{\frac{r}{2}}\\
&\le \alpha \left\langle |T^2|^{2s}x,x\right\rangle^{\frac{r}{2\alpha}} + (1-\alpha)\left\langle |(T^2)^*|^{2t}x,x\right\rangle^{\frac{r}{2(1-\alpha)}}\\
& && \text{(by Lemma 2.1(i))}\\
&\le \alpha \left\langle |T^2|^{\frac{rs}{\alpha}}x,x\right\rangle + (1-\alpha)\left\langle |(T^2)^*|^{\frac{rt}{1-\alpha}}x,x\right\rangle && \text{(by Lemma 2.4)}\\
&\le \left\langle \left(\alpha|T^2|^{\frac{rs}{\alpha}} + (1-\alpha)|(T^2)^*|^{\frac{rt}{1-\alpha}}\right)x,x\right\rangle.
\end{aligned}$$

It follows from inequality (2.4) with the convex function $h(u) = u^r$, $(r \ge 1)$ that

$$|\langle Tx,x\rangle|^{2r} \le \frac{1}{2}\left(\|Tx\|^r\,\|T^*x\|^r + \left\langle \left(\alpha|T^2|^{\frac{rs}{\alpha}} + (1-\alpha)|(T^2)^*|^{\frac{rt}{1-\alpha}}\right)x,x\right\rangle\right).$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in the above inequality, we deduce the desired inequality (2.21). □

Inequality (2.21) induces several numerical radius inequalities as special cases. For example, the following result may be stated as well.

**Corollary 2.12.** *If we take $\alpha = 1/2$ in inequality (2.21), then*

$$w^{2r}(T) \le \frac{1}{2}\left(\|T\|^{2r} + \frac{1}{2}\left\||T|^{4sr} + |T^*|^{4tr}\right\|\right)$$

*for any $r \ge 1$ and $s + t = 1$.*

In addition, by choosing $t = s = 1/2$, we obtain $w^{2r}(T) \le \|T\|^{2r}$ for any $r \ge 1$.

**Theorem 2.13.** *Let $T, S, X \in \mathscr{B}(\mathscr{H})$, and let $f, g$ be nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t, (t \ge 0)$ and $\alpha \in (0,1)$. Then for all nonnegative nondecreasing convex functions $h$ on $[0,\infty)$, we have*

$$h(w(TXS)) \le \left\|\alpha[h(Tf^2(|X^*|)T^*)]^{\frac{1}{2\alpha}} + (1-\alpha)[h(S^*g^2(|X|)S)]^{\frac{1}{2(1-\alpha)}}\right\|. \tag{2.22}$$

**Corollary 2.14.** *Let $T, S \in \mathscr{B}(\mathscr{H})$. Then*

$$h(w(S^*T)) \le \left\|\alpha h(|T|^{\frac{1}{\alpha}}) + (1-\alpha)h(|S|^{\frac{1}{1-\alpha}})\right\|$$

*holds for any $\alpha \in (0,1)$.*

Corollary 2.14 is deduced by putting $X = I$ and $f(t) = g(t) = \sqrt{t}$ in Theorem 2.13. In fact, Corollary 2.14 and Theorem 2.13 are equivalent.

*Proof of Theorem 2.13.* Let $X = U|X|$ be the polar decomposition of $X$. Put $B = f(|X|)U^*T$ and $A = g(|X|)S$. Then by Corollary 2.14, we have

$$\begin{aligned}
&h(w(B^*A)) \le \left\|\alpha h(|T|^{\frac{1}{\alpha}}) + (1-\alpha)h(|S|^{\frac{1}{1-\alpha}})\right\|\\
&\qquad \Longleftrightarrow h(w(TXS)) \le \left\|\alpha[h(Tf^2(|X^*|)T^*)]^{\frac{1}{2\alpha}} + (1-\alpha)[h(S^*g^2(|X|)S)]^{\frac{1}{2(1-\alpha)}}\right\|.
\end{aligned}$$

□

The following theorem gives an upper bound for $w(S^*T)$.

**Theorem 2.15.** *Let $T, S \in \mathscr{B}(\mathscr{H})$. Then for all nonnegative nondecreasing convex functions $h$ on $[0,\infty)$, we have*

$$h(w(S^*T)) \leq \frac{1}{4}\left\|h(|T^*|^2)+h(|S^*|^2)\right\| + \frac{1}{2}h(w(TS^*)).$$

*Proof.* First of all, we note that

$$w(A) = \sup_{\theta\in\mathbb{R}} \left\|\mathrm{Re}(e^{i\theta}A)\right\|, \tag{2.23}$$

where $\mathrm{Re}(Y)$ denotes the real part of the operator $Y$.

For every unit vector $x \in \mathscr{H}$, we have

$$\begin{aligned}
Re\left\langle e^{i\theta}S^*Tx, x\right\rangle &= Re\left\langle e^{i\theta}Tx, Sx\right\rangle \\
&= \frac{1}{4}\left\|(e^{i\theta}T+S)x\right\|^2 - \frac{1}{4}\left\|(e^{i\theta}T-S)x\right\|^2 \\
&\qquad\qquad\qquad\qquad \text{(by Polarization Identity)} \\
&\leq \frac{1}{4}\left\|(e^{i\theta}T+S)x\right\|^2 \leq \frac{1}{4}\left\|(e^{i\theta}T+S)\right\|^2 \\
&= \frac{1}{4}\left\|e^{-i\theta}T^*+S^*\right\|^2 \qquad \text{(Since } \|Y\| = \|Y^*\|\text{)} \\
&= \frac{1}{4}\left\|(e^{-i\theta}T^*+S^*)^*(e^{-i\theta}T^*+S^*)\right\| \qquad \text{(Since } \|Y\|^2 = \|Y^*Y\|\text{)} \\
&= \frac{1}{4}\left\|TT^*+SS^*+e^{i\theta}TS^*+e^{-i\theta}ST^*\right\| \\
&\leq \frac{1}{4}\|TT^*+SS^*\| + \frac{1}{2}\left\|Re(e^{i\theta}TS^*)\right\| \\
&\leq \frac{1}{4}\|TT^*+SS^*\| + \frac{1}{2}w(ST^*). \qquad \text{(by (2.23))}
\end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in the above inequality, we obtain

$$w(S^*T) \leq \frac{1}{4}\left\||T^*|^2+|S^*|^2\right\| + \frac{1}{2}w(TS^*).$$

Now since $h(\cdot)$ is a nondecreasing convex function, we have

$$\begin{aligned}
h(w(S^*T)) &\leq h\left(\frac{1}{4}\left\||T^*|^2+|S^*|^2\right\| + \frac{1}{2}w(TS^*)\right) \\
&\leq \frac{1}{2}h\left(\frac{\left\||T^*|^2+|S^*|^2\right\|}{2}\right) + \frac{1}{2}h(w(TS^*)) \\
&\leq \frac{1}{2}\left\|\frac{h(|T^*|^2)+h(|S^*|^2)}{2}\right\| + \frac{1}{2}h(w(TS^*)) \\
&= \frac{1}{4}\left\|h(|T^*|^2)+h(|S^*|^2)\right\| + \frac{1}{2}h(w(TS^*)).
\end{aligned}$$

This completes the proof. □

The next corollary is an extension of [12, Corollary 2.11].

**Corollary 2.16.** *Let $A \in \mathscr{B}(\mathscr{H})$, let $A = U|A|$ be the polar decomposition of $T$, let $f, g$ be nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ $(t \geq 0)$, and let $\widetilde{A}_{f,g} = f(|A|)Ug(|A|)$ be the generalized Aluthge transformation of $T$. Then we have*

$$w^r(A) \leq \frac{1}{4}\left\|f^{2r}(|A|) + g^{2r}(|A|)\right\| + \frac{1}{2}w^r(\widetilde{A}_{f,g}).$$

*Proof.* Put $T = f(|A|)$, $S = g(|A|)U^*$, and $h(u) = u^r$ $(r \geq 1)$ in Theorem 2.15. Then, we have

$$\begin{aligned} h(w(S^*T)) &\leq \frac{1}{4}\left\|h(|T^*|^2) + h(|S^*|^2)\right\| + \frac{1}{2}h(w(TS^*)) \Longleftrightarrow \\ w^r(A) &\leq \frac{1}{4}\left\|f^{2r}(|A|) + g^{2r}(|A|)\right\| + w^r(\widetilde{A}_{f,g}). \end{aligned}$$

□

Our next result is to find an upper bound for power of the numerical radius of $T^{\frac{1}{p}}XS^{\frac{1}{q}}$ under assumption $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

**Theorem 2.17.** *Suppose that $T, S, X \in \mathscr{B}(\mathscr{H})$ and that $T$ and $S$ are positive. Then*

$$w^r(T^{\frac{1}{p}}XS^{\frac{1}{q}}) \leq \|X\|^r \left\|\frac{1}{p}T^r + \frac{1}{q}S^r\right\|$$

*for every $r \geq 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

*Proof.* Let $x \in \mathscr{H}$ be a unit vector. Then

$$\begin{aligned} \left|\left\langle T^{\frac{1}{p}}XS^{\frac{1}{q}}x, x\right\rangle\right|^r &= \left|\left\langle XS^{\frac{1}{q}}x, T^{\frac{1}{p}}x\right\rangle\right|^r \\ &\leq \|X\|^r \left\|S^{\frac{1}{q}}x\right\|^r \left\|T^{\frac{1}{p}}x\right\|^r \\ &\leq \|X\|^r \left\langle S^{\frac{2}{q}}x, x\right\rangle^{\frac{r}{2}} \left\langle T^{\frac{2}{p}}x, x\right\rangle^{\frac{r}{2}} \\ &\leq \|X\|^r \langle S^r x, x\rangle^{\frac{1}{q}} \langle T^r x, x\rangle^{\frac{1}{p}} && \text{(by Lemma 2.4)} \\ &\leq \|X\|^r \left\langle \left(\frac{1}{p}S^r + \frac{1}{q}T^r\right)x, x\right\rangle. && \text{(by Lemma 2.2)} \end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in the above inequality, we obtain the desired inequality. □

**Corollary 2.18.** *Let $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ be the Aluthge transformation of $A$ such that $U$ is a partial isometry. Then*

$$w(\widetilde{A}) \leq \|A\|.$$

*Proof.* If we take $r = 1$, $p = q = 2$, $T = S = |A|$, and $X = U$ in Theorem 2.17, then

$$w(\widetilde{A}) \leq \left\|\frac{1}{2}|A| + \frac{1}{2}|A|\right\| = \||A|\| = \|A\|.$$

□

**Theorem 2.19.** *If $T \in \mathscr{B}(\mathscr{H})$, then*

$$w^{2r}(T) \leq \frac{\alpha}{2} w^r(T^2) + \left(\frac{1-\alpha}{2}\right) \|T\|^{2r}$$

*for every $r \geq 1$ and $0 < \alpha \leq 1$.*

*Proof.* We recall the following refinement of the Cauchy–Schwarz inequality obtained by Dragomir in [2]. It says that

$$|\langle a, e\rangle \langle e, b\rangle| \leq \alpha |\langle a, b\rangle| + (1-\alpha)\|a\| \|b\|,$$

where $a, b, e$ are vectors in $\mathscr{H}$ and $\|e\| = 1$.

Put $e = x$ with $\|x\| = 1$, $a = Tx$ and $b = T^*x$ in the above inequality and use Lemma 2.1(i) with $h(u) = u^r$ $(r \geq 1)$ to get

$$\begin{aligned} |\langle Tx, x\rangle|^2 &\leq \frac{\alpha}{2} |\langle T^2x, x\rangle| + \left(\frac{1-\alpha}{2}\right) \|Tx\| \|T^*x\| \\ &\leq \left(\frac{\alpha}{2} |\langle T^2x, x\rangle|^r + \left(\frac{1-\alpha}{2}\right) \|Tx\|^r \|T^*x\|^r\right)^{\frac{1}{r}}. \end{aligned}$$

Hence

$$|\langle Tx, x\rangle|^{2r} \leq \frac{\alpha}{2} |\langle T^2x, x\rangle|^r + \left(\frac{1-\alpha}{2}\right) \|Tx\|^r \|T^*x\|^r. \tag{2.24}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in inequality (2.24), we get the desired inequality. □

**Corollary 2.20.** *Let $T \in \mathscr{B}(\mathscr{H})$, and let $f, g$ be nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, $(t \geq 0)$. Then*

$$w^{2r}(T) \leq \left(\frac{1-\alpha}{2}\right) \|T\|^{2r} + \left(\frac{\alpha}{2}\right) \left\|\alpha f^{\frac{r}{\alpha}}(|T^2|) + (1-\alpha) g^{\frac{r}{1-\alpha}}(|(T^2)^*|)\right\| \tag{2.25}$$

*for all $r \geq 1$ and $0 < \alpha \leq 1$.*

*Proof.* Let $x \in \mathscr{H}$ be a unit vector. We have

$$\begin{aligned} |\langle T^2x, x\rangle|^r &\leq \left\|f(|T^2|)x\right\|^r \left\|g(|(T^2)^*|)x\right\|^r && \text{(by Lemma 2.5(ii))} \\ &= \left\langle f^2(|T^2|)x, x\right\rangle^{\frac{r}{2}} \left\langle g^2(|(T^2)^*|)x, x\right\rangle^{\frac{r}{2}} \\ &\leq \alpha \left\langle f^2(|T^2|)x, x\right\rangle^{\frac{r}{2\alpha}} + (1-\alpha) \left\langle g^2(|(T^2)^*|)x, x\right\rangle^{\frac{r}{2(1-\alpha)}} \\ & && \text{(by Lemma 2.2)} \\ &\leq \alpha \left\langle f^{\frac{r}{\alpha}}(|T^2|)x, x\right\rangle + (1-\alpha) \left\langle g^{\frac{r}{1-\alpha}}(|(T^2)^*|)x, x\right\rangle && \text{(by Lemma 2.4)} \\ &= \left\langle \left(\alpha f^{\frac{r}{\alpha}}(|T^2|) + (1-\alpha) g^{\frac{r}{1-\alpha}}(|(T^2)^*|)\right) x, x\right\rangle. \end{aligned}$$

It follows from (2.24) that

$$|\langle Tx, x\rangle|^{2r} \leq \frac{\alpha}{2} \left\langle \left(\alpha f^{\frac{r}{\alpha}}(|T^2|) + (1-\alpha) g^{\frac{r}{1-\alpha}}(|(T^2)^*|)\right) x, x\right\rangle + \left(\frac{1-\alpha}{2}\right) \|Tx\|^r \|T^*x\|^r.$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in the above inequality, we deduce the desired inequality. □

Inequality (2.25) induces several radius inequalities as special cases. For example, the following result reads as follows.

**Corollary 2.21.** *If we take $f(t) = t^p, g(t) = t^q$ with $p + q = 1$ and $\alpha = 1/2$ in inequality (2.25), then*

$$w^{2r}(T) \leq \frac{1}{4}\left(\|T\|^{2r} + \left\||T|^{4rq} + |T^*|^{4rq}\right\|\right)$$

*for every $r \geq 1$.*

## 3. Numerical radius and strongly convex function

Let $J \subset \mathbb{R}$ be an interval and let $c$ be a positive number. By following Polyak [11], a function $f : J \to \mathbb{R}$ is called strongly convex with modulus $c$ if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - c\lambda(1-\lambda)(x-y)^2 \tag{3.1}$$

for all $x, y \in J$ and $\lambda \in [0,1]$. The function $f$ is called strongly concave with modulus $c$ if $-f$ is strongly convex with modulus $c$.

Since strong convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just stronger versions of known properties of convex functions. For instance, a function $f : J \to \mathbb{R}$ is strongly convex with modulus $c$ if and only if for every $x_0 \in \operatorname{int}(J)$ (the interior of $J$) there exists a number $l \in \mathbb{R}$ such that

$$c(x-x_0)^2 + l(x-x_0) + f(x_0) \leq f(x), \quad x \in I. \tag{3.2}$$

In other word, $f$ has a quadratic support at $x_0$.

The differentiable function $f$ is strongly convex with modulus $c$ if and only if

$$(f'(x) - f'(y))(x-y) \geq 2c(x-y)^2 \tag{3.3}$$

for each $x, y \in J$. For more properties of this class of functions, see [6].

**Theorem 3.1.** *Let $f : J \to \mathbb{R}$ be strongly convex with modulus c and differentiable on $\operatorname{int}(J)$. If $T$ is a self-adjoint operator on the Hilbert space $\mathscr{H}$ with $\sigma(T) \subset \operatorname{int}(J)$, then*

$$f(w(T)) + cw(T^2) \leq w(f(T)) + cw^2(T) \tag{3.4}$$

*for each $x \in \mathscr{H}$, with $\|x\| = 1$.*

*Proof.* It follows from (3.2) by utilizing functional calculus that

$$c(T^2 + x_0^2 I - 2x_0 T) + lT - lx_0 I + f(x_0)I \leq f(T), \tag{3.5}$$

which is equivalent to

$$c(\left\langle T^2x, x\right\rangle + x_0^2 - 2x_0) + l\left\langle Tx, x\right\rangle - lx_0 + f(x_0) \leq \left\langle f(T)x, x\right\rangle \tag{3.6}$$

for each $x \in \mathscr{H}$, with $\|x\| = 1$.

Now, applying (3.6) for $x_0 = \langle Tx, x\rangle$, we obtain

$$f(\langle Tx, x\rangle) \leq \langle f(T)x, x\rangle - c(\left\langle T^2x, x\right\rangle - \langle Tx, x\rangle^2), \tag{3.7}$$

and so

$$f(\langle Tx, x\rangle) + c\left\langle T^2x, x\right\rangle \leq \langle f(T)x, x\rangle + c\left\langle Tx, x\right\rangle^2. \tag{3.8}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in inequality (3.8), we deduce the desired inequality. $\square$

*Remark* 3.2. Notice that the quantity $w(T^2) - w^2(T)$ is positive, therefore we have

$$f(w(T)) \leq w(f(T)) - c\left(w(T^2) - w^2(T)\right) \leq w(f(T))$$

for each $x \in \mathscr{H}$, with $\|x\| = 1$.

*Remark* 3.3 ([?, Proposition 1.1.2]). The function $f : J \to \mathbb{R}$ is strongly convex with modulus $c$ if and only if the function $g : J \to \mathbb{R}$ defined by $g(x) = f(x) - cx^2$ is convex. Consider the function $f : (1, \infty) \to \mathbb{R}$, given by $f(x) = x^r$ with $r \geq 2$. It can be easily verified that, this function is strongly convex with modulus $c = \dfrac{r^2 - r}{2}$ . Based on this fact, from Theorem 3.1, we obtain

$$w^r(T) \leq w(T^r) - \frac{r^2 - r}{2}(w(T^2) - w^2(T))$$

for each positive operator $T$ and $x \in \mathscr{H}$ with $\|x\| = 1$.
It is readily checked that the function $f : (0, 1) \to \mathbb{R}$ given by $f(x) = -x^r$ with $0 < r < 1$ is a strongly convex function with modulus $c = \dfrac{r - r^2}{2}$ . Similar to the above argument, by using Theorem 3.1, we get

$$w(T^r) \leq w^r(T) + \frac{r - r^2}{2}(w^2(T) - w(T^2)),$$

for each positive operator $T$ and $x \in \mathscr{H}$ with $\|x\| = 1$.

The following theorem is a generalization of Theorem 3.1.

**Theorem 3.4.** *Let all the assumptions of Theorem 3.1 be satisfied. Moreover, let $f(0) \leq 0$. Then*

$$f(w(T)) \leq w(f(T)) - c\left(w(T^2) - \frac{1}{\|x\|^2} w(T)^2\right)$$

*for each $x \in \mathscr{H}$ with $\|x\| \leq 1$.*

*Proof.* Let $y = \dfrac{x}{\|x\|}$, so that $\|y\| = 1$. We have

$$\begin{aligned} f(\langle Tx, x\rangle) &= f(\|x\|^2 f(Ty, y) + (1 - \|x\|^2 .0) \\ &\leq \|x\|^2 f(\langle Ty, y\rangle) + (1 - \|x\|^2 f(0) && \text{(by (3.1))} \\ &\leq \|x\|^2 f(\langle Ty, y\rangle) && \text{(since } f(0) \leq 0) \\ &\leq \|x\|^2 \left(\langle f(T)y, y\rangle - c(\langle T^2 y, y\rangle - \langle Ty, y\rangle^2)\right) && \text{(by inequality (3.8))} \\ &= \langle f(T)x, x\rangle - c\left(\langle T^2 x, x\rangle - \frac{1}{\|x\|^2}\langle Tx, x\rangle^2\right). \end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| \leq 1$ in the above inequality, we deduce the desired inequality. $\square$

In what follows, we make use of Lemma 2.1 from [1, Theorem 3.1.2] which we cite here.

**Lemma 3.5.** *Let $\Phi$ be a unital completely positive linear map from a $C^*$-subalgebra $\mathcal{A}$ of $M_n(\mathbb{C})$ into $M_m(\mathbb{C})$. Then there exist a Hilbert space $\mathscr{K}$, an isometry $V : \mathbb{C}^m \to \mathscr{K}$, and a unital $*$-homomorphism $\pi$ from $\mathcal{A}$ into the $C^*$-algebra $\mathscr{B}(\mathscr{K})$ such that $\Phi(A) = V^*\pi(A)V$.*

In the next theorem, we extend Theorem 3.1 to all positive linear maps.

**Theorem 3.6.** *Let all the assumptions of Theorem 3.1 be satisfied, and let $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a unital positive linear map. Then*

$$f(w(\Phi(A)) + c\,\|\Phi(A)\|^2 \leq w(\phi(f(A)) + cw^2(\Phi(A)) \tag{3.9}$$

*for every Hermitian matrix $A \in M_n(\mathbb{C})$ and every unit vector $x \in \mathbb{C}^m$.*

*Proof.* We may assume that $A$ is the unital $C^*$-algebra generated by a single positive operator $A$. Hence by a classical dilation theorem of Naimark (see [10, Theorem 3.10]), our maps $\Phi$ will be automatically completely positive. So, by Lemma 3.5, there exist a Hilbert space $\mathscr{K}$, an isometry $V : \mathbb{C}^m \to \mathscr{K}$ and a unital $*$-homomorphism $\pi$ from $\mathcal{A}$ into the $C^*$-algebra $\mathscr{B}(\mathscr{K})$ such that $\Phi(A) = V^*\pi(A)V$. Clearly,

$$f(\pi(A)) = \pi(f(A)). \tag{3.10}$$

Taking into account that $\|V\| = 1$ ($V$ is an isometry), we observe that

$$\begin{aligned}
f(\langle\Phi(A)x, x\rangle) &= f(\langle V^*\pi(A)Vx, x\rangle)\\
&= f(\langle\pi(A)Vx, Vx\rangle)\\
&\leq \langle f(\pi(A))Vx, Vx\rangle - c(\langle\pi(A)^2Vx, Vx\rangle - \langle\pi(A)Vx, Vx\rangle^2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Theorem 3.1)}\\
&= \langle\pi(f(A))Vx, Vx\rangle - c(\langle\pi(A)^2Vx, Vx\rangle - \langle\pi(A)Vx, Vx\rangle^2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (3.10))}\\
&= \langle V^*\pi(f(A))Vx, x\rangle - c(\langle(V^*\pi(A)V)^2x, x\rangle - \langle V^*\pi(A)Vx, x\rangle^2)\\
&= \langle\Phi(f(A)x, x\rangle - c(\langle\Phi(A)^2x, x\rangle - \langle\Phi(A)x, x\rangle^2).
\end{aligned}$$

Hence

$$f(\langle\Phi(A)x, x\rangle) + c\,\|\Phi(A)x\|^2 \leq \langle\Phi(f(A))x, x\rangle + c\,\langle\Phi(A)x, x\rangle^2.$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| \leq 1$ in the above inequality, we deduce the desired inequality. □

**Theorem 3.7.** *Let $f : J \to \mathbb{R}$ be strongly convex with modulus $c$ and differentiable on $\operatorname{int}(J)$, and let the derivative of $f$ be continuous on $\operatorname{int}(J)$. If $T$ is a self-adjoint operator on the Hilbert space $\mathscr{H}$ with $\sigma(T) \subset \operatorname{int}(J)$, then*

$$\langle T^2x, x\rangle - \langle Tx, x\rangle^2 \leq \frac{1}{2c}(\langle f'(T)Tx, x\rangle - \langle Tx, x\rangle\langle f'(T)x, x\rangle) \tag{3.11}$$

*for each $x \in \mathscr{H}$, with $\|x\| = 1$.*

*Proof.* It follows from (3.3) by utilizing functional calculus that

$$2c(T^2 - y^2 I + 2T^2 y) \leq f'(T)T - yf'(T) - f'(y)T + yf'(y)I$$

which is equivalent to

$$2c(\langle T^2 x, x\rangle + y^2 - 2y\langle Tx, x\rangle) \leq \langle f'(T)Tx, x\rangle - y\langle f'(T)x, x\rangle - f'(y)\langle Tx, x\rangle + yf'(y)$$

for each $x \in \mathscr{H}$, with $\|x\| = 1$.

Now, applying the last inequality with $y = \langle Tx, x\rangle$, we have

$$2c(\langle T^2 x, x\rangle - \langle Tx, x\rangle^2) \leq \langle f'(T)Tx, x\rangle - \langle Tx, x\rangle \langle f'(T)x, x\rangle .$$

Hence the desired inequality is obtained. □

By replacing $c(x-y)^2$ with a nonnegative real valued function $G(x-y)$, we can define $G$-strongly convex functions as follows:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - \lambda(1-\lambda)G(x-y) \tag{3.12}$$

for each $\lambda \in [0,1]$ and $x, y \in J$. (Very recently, this approach has been investigated by Adamek in [**?**]).

We should note that, if $G$ is $G$-strongly affine, then the function $f$ is $G$-strongly convex if and only if $g = f - G$ is convex (see [**?**, Lemma 4]).

**Theorem 3.8.** *Let $f : J \to \mathbb{R}$ be an $G$-strongly convex and differentiable function on* $\operatorname{int}(J)$*, and let $G : J \to [0,\infty)$ be a continuous function. If $T$ is a self-adjoint operator on the Hilbert space $\mathscr{H}$ with $\sigma(T) \subset int(T)$, then*

$$f(\langle Tx, x\rangle) \leq \langle f(T)x, x\rangle - \langle G(T - \langle Tx, x\rangle)x, x\rangle \tag{3.13}$$

*for each $x \in \mathscr{H}$, with $\|x\| = 1$.*

*Proof.* From (3.12), we infer that

$$f(\lambda(x-y)+y) - f(y) + \lambda(1-\lambda)G(x-y) \leq \lambda(f(x) - f(y)).$$

By dividing both sides by $\lambda$, we obtain

$$\frac{f(\lambda(x-y)+y) - f(y)}{\lambda} + (1-\lambda)F(x-y) \leq f(x) - f(y).$$

Notice that if $f$ is differentiable, then, letting $\lambda \to 0$, we have

$$f'(y)(x-y) + G(x-y) + f(y) \leq f(x), \tag{3.14}$$

for all $x, y \in J$ and $\lambda \in [0,1]$.

It follows from (3.14) by utilizing functional calculus that

$$f'(y)(T - yI) + G(T-y) + f(y)I \leq f(T)$$

which is equivalent to

$$f'(y)(\langle Tx, x\rangle - y) + \langle G(T-y)x, x\rangle + f(y) \leq \langle f(T)x, x\rangle .$$

for each $x \in \mathscr{H}$, with $\|x\| = 1$.

Now, by applying the last inequality with $y = \langle Tx, x\rangle$, we have

$$\langle G(T - \langle Tx, x\rangle)x, x\rangle + f(\langle Tx, x\rangle) \leq \langle f(T)x, x\rangle ,$$

so the result is obtained. □

## References


1. R. Bhatia, *Positive Definite Matrices*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2007.
2. S.S. Dragomir, *Some refinements of Schwarz inequality*, Simposional de Math. Si Appl. Polytechnical Inst. Timisoara, Romania, **1-2** (1985) 13–16.
3. S.S. Dragomir, *Reverse inequalities for the numerical radius of linear operators in Hilbert spaces*, Bull. Austral. Math. Soc. **73** (2006), no. 2, 255–262.
4. S.S. Dragomir, *Some inequalities for the norm and the numerical radius of linear operators in Hilbert Spaces*, Tamkang J. Math. **39** (2008), no. 1, 1–7.
5. S.S. Dragomir, *Power inequalities for the numerical radius of a product of two operators in Hilbert spaces*, Sarajevo J. Math. **5(18)** (2009), no. 2, 269–278.
6. K. Nikodem, *On strongly convex functions and related classes of functions*, in: Handbook of Functional Equations, pp. 365–405, Springer Optim. Appl. 95, Springer, New York, 2014.
7. C.A. McCarthy, $c_p$, Israel J. Math. **5** (1967), 249–271.
8. B. Mond, J. Pečarić, *Convex inequalities in Hilbert space*, Houston J. Math. **19** (1993) 405–420.
9. M. Omidvar, M. Moslehian, A. Niknam, *Some numerical radius inequalities for Hilbert space operators*, Involve **2** (2009), no. 4, 471–478.
10. V.I. Paulsen, *Completely Bounded Maps and Dilations*, Pitman Research Notes in Mathematics Series 146, Longman Scientific, Technical, John Wiley & Sons, Harlow-New York, 1986.
11. B.T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl. **7** (1966) 72–75.
12. M. Sattari, M.S. Moslehian, T. Yamazaki, *Some generalized numerical radius inequalities for Hilbert space operators*, Linear Algebra Appl. **470** (2015) 216–227
13. T. Yamazaki, *On upper and lower bounds of the numerical radius and an equality condition*, Studia Math. **178** (2007) 83–89.


Department of Mathematics and Statistics, Faculty of Science P.O.Box(7), Mu'tah University, Alkarak-Jordan

*E-mail address*: malik_okasha@yahoo.com