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TRACES OF SCHUR AND KRONECKER PRODUCTS FOR BLOCK MATRICES

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Abstract. In this paper, we define two new Schur and Kronecker-type products for block matrices. We present some equalities and inequalities involving traces of matrices generated by these products and in particular we give conditions under which the trace operator is sub-multiplicative for them. Also, versions in the block matrix framework of results of Das, Vashisht, Taskara and Gumus will be obtained.

1. Introduction and preliminaries

If we take two matrices $A = (a_{k,j})_{k,j}$ and $B = (b_{k,j})_{k,j}$ of the same size, with entries in the complex or real field, their Hadamard product is just their element-wise product, that is,

$$A * B = (a_{k,j} \cdot b_{k,j})_{k,j}.$$

Since Schur provided the initial studies about its properties, it is widely known also as the "Schur product". Horn, in 1990, gave a profound insight on this product (see [6]).

In what follows, we will denote by $\mathcal{M}_{n,m}(X)$ the space of matrices of size $n \times m$ with entries in X. If X is also a space of matrices, we will use the expression "block matrices" to refer to the elements of $\mathcal{M}_{n,m}(X)$. Consider now $A = (a_{k,j})_{k,j} \in \mathcal{M}_{n,m}(\mathbb{C})$ and $B = (b_{k,j})_{k,j} \in \mathcal{M}_{p,q}(\mathbb{C})$. Their Kronecker product,

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denoted by $A \otimes B$, is defined as follows:

$$A \otimes B := \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,m}B \end{pmatrix} \in \mathcal{M}_{np,mq}(\mathbb{C}).$$

Both the Schur product and the Kronecker product are studied and applied in fields such as matrix theory, matricial analysis or statistics. For instance, the reader is referred to [7], where Magnus and Neudecker gave some results and statistical applications regarding the Schur and Kronecker products and to [8], where Persson and Popa used the Schur product as a tool in the area of matricial harmonic analysis to develop theories of matrix spaces parallel to their scalar counterparts.

The trace of a matrix $A = (a_{k,j})_{k,j} \in \mathcal{M}_{n\times n}(\mathbb{C})$ is the sum of its diagonal elements, that is,

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{i,i}.$$

The reader is referred to the papers of Das and Vashisht (see [3]) and Taskara and Gumus (see [9]), where the authors investigated traces of Schur and Kronecker products. One of our goals in this paper will be to generalize some of those results in the context of block matrices, for certain versions of Schur and Kronecker products that we shall define now.

Definition 1.1. Let $A = (T_{k,j})_{k,j} \in \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C}))$ and $B = (S_{k,j})_{k,j} \in \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C}))$. We define the Schur product of A and B as

$$A \circledast B := (T_{k,j} * S_{k,j})_{k,j},$$

where $T_{k,j} * S_{k,j}$ denotes the classical Schur product of the matrices $T_{k,j}$ and $S_{k,j}$.

If $m, n = 1$ or $N, M = 1$, this product coincides with the classical Schur product. Although the previous definition is natural, other options to define a Schur product for block matrices exist, of course. For example, in [1, 2], we worked with a definition that involved the entry-wise composition of operators and proved some results with it in the field of matricial harmonic analysis for infinite matrices. In this paper, we will focus on the context of finite matrices.

Consider now $T \in \mathcal{M}_{n,m}(\mathbb{C})$ and $B = (B_{k,j})_{k,j} \in \mathcal{M}_{N,M}(\mathcal{M}_{n,m}(\mathbb{C}))$. We define a block Kronecker product of T and B as $T \boxtimes B = (T * B_{k,j})_{k,j}$. Taking this into account, we can define our Kronecker product of two block matrices as follows.

Definition 1.2. Let $A = (T_{k,j})_{k,j} \in \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C}))$ and $B = (S_{k,j})_{k,j} \in \mathcal{M}_{P\times Q}(\mathcal{M}_{n\times m}(\mathbb{C}))$. We define their Kronecker product, $A \boxtimes B$, as

$$A \boxtimes B := (T_{k,j} \boxtimes B)_{k,j} \in \mathcal{M}_{NP\times MQ}(\mathcal{M}_{n\times m}(\mathbb{C})).$$

Observe that this product is not commutative, but the Schur product for block matrices is. Also, note that if $m, n = 1$, then this Kronecker product becomes the Kronecker product of matrices with complex entries; and if $P, Q, N, M = 1$, then the classical Schur product is obtained. Again, we point out that other natural definitions of a block Kronecker product exist (see [5]).

Some basic properties satisfied by these products are the following ones. Let $N, M, P, Q, n, m \in \mathbb{N}$.

- **Property (1)** The products $\circledast$ and $\boxtimes$ establish bilinear maps between spaces of block matrices:

$$\circledast : \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C})) \times \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C})) \longrightarrow \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C}))$$

  and

$$\boxtimes : \mathcal{M}_{N\times M}(\mathcal{M}_{n\times m}(\mathbb{C})) \times \mathcal{M}_{P\times Q}(\mathcal{M}_{n\times m}(\mathbb{C})) \longrightarrow \mathcal{M}_{NP\times MQ}(\mathcal{M}_{n\times m}(\mathbb{C})).$$

- **Property (2)** (Associativity).
  - $\mathbf{A} \circledast (\mathbf{B} \circledast \mathbf{C}) = (\mathbf{A} \circledast \mathbf{B}) \circledast \mathbf{C}$.
  - $\mathbf{A}\boxtimes(\mathbf{B}\boxtimes\mathbf{C}) = (\mathbf{A}\boxtimes\mathbf{B})\boxtimes\mathbf{C}$.
- **Property (3)** (Distributivity with respect to the sum).
  - $(\mathbf{A} + \mathbf{B}) \circledast \mathbf{C} = (\mathbf{A} \circledast \mathbf{C}) + (\mathbf{B} \circledast \mathbf{C})$.
  - $\mathbf{A} \circledast (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \circledast \mathbf{B}) + (\mathbf{A} \circledast \mathbf{C})$.
  - $(\mathbf{A} + \mathbf{B})\boxtimes\mathbf{C} = (\mathbf{A}\boxtimes\mathbf{C}) + (\mathbf{B}\boxtimes\mathbf{C})$.
  - $\mathbf{A}\boxtimes(\mathbf{B} + \mathbf{C}) = (\mathbf{A}\boxtimes\mathbf{B}) + (\mathbf{A}\boxtimes\mathbf{C})$.
- **Property (4)** (Mixed associativity). For every $\alpha \in \mathbb{C}$,
  - $\alpha(\mathbf{A} \circledast \mathbf{B}) = (\alpha\mathbf{A}) \circledast \mathbf{B} = \mathbf{A} \circledast (\alpha\mathbf{B})$.
  - $\alpha(\mathbf{A}\boxtimes\mathbf{B}) = (\alpha\mathbf{A})\boxtimes\mathbf{B} = \mathbf{A}\boxtimes(\alpha\mathbf{B})$.
- **Property (5)** (Commutativity of $\circledast$).
  - $\mathbf{A} \circledast \mathbf{B} = \mathbf{B} \circledast \mathbf{A}$.

Since the space of block matrices with the operations "+" (sum of matrices) and "·" (product by scalar) is a vector space, it becomes an algebra when equipped with $\boxtimes$ due to properties (2)–(4), and it becomes a commutative algebra when equipped with $\circledast$ instead, due to properties (2)–(5).

2. On traces of block matrices

In what follows, we will study equalities and inequalities involving traces of block matrices and the products defined above. From now on we will work in the context of square block matrices whose entries are also square matrices, with entries in $\mathbb{R}$, and will abbreviate the notation for these matrices in the following way: $\mathcal{M}_N(\mathcal{M}_n) := \mathcal{M}_{N\times N}(\mathcal{M}_{n\times n}(\mathbb{R}))$. First of all, take into account that the trace of a block matrix is computed by summing the traces of its diagonal elements. That is, if $A = (T_{k,j})_{k,j} \in \mathcal{M}_N(\mathcal{M}_n)$, then

$$\operatorname{tr}(A) = \sum_{i=1}^{N} \operatorname{tr}(T_{i,i}) = \sum_{i=1}^{N} \sum_{l=1}^{n} T_{i,i}(l,l).$$

Proposition 2.1. *Let $A \in \mathcal{M}_N(\mathcal{M}_n)$ and $B \in \mathcal{M}_M(\mathcal{M}_n)$ with $A = (T_{k,j})_{k,j}$ and $B = (S_{k,j})_{k,j}$. Then*

(a) *If $M = N$, then* $\operatorname{tr}(A \circledast B) = \sum_{i=1}^{N} \sum_{l=1}^{n} T_{i,i}(l,l) S_{i,i}(l,l)$.

(b) $\operatorname{tr}(A\boxtimes B) = \displaystyle\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{l=1}^{n} T_{i,i}(l,l)S_{j,j}(l,l)$.

(c) *If* $M = N$, *then* $\operatorname{tr}(A\boxtimes B) = \operatorname{tr}(A \circledast B) + \displaystyle\sum_{\substack{i=1,j=1\\ i\neq j}}^{N}\sum_{l=1}^{n} T_{i,i}(l,l)S_{j,j}(l,l)$.

Proof. (a)

$$\begin{aligned}\operatorname{tr}(A \circledast B) &= \sum_{i=1}^{N} \operatorname{tr}(A \circledast B)_{i,i} = \sum_{i=1}^{N}\sum_{l=1}^{n} (T_{i,i} * S_{i,i})(l,l)\\ &= \sum_{i=1}^{N}\sum_{l=1}^{n} T_{i,i}(l,l)S_{i,i}(l,l).\end{aligned}$$

(b)

$$\begin{aligned}\operatorname{tr}(A\boxtimes B) &= \sum_{i=1}^{N} \operatorname{tr}(T_{i,i}\boxtimes B) = \sum_{i=1}^{N}\sum_{j=1}^{M} \operatorname{tr}(T_{i,i} * S_{j,j})\\ &= \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{l=1}^{n} T_{i,i}(l,l)S_{j,j}(l,l).\end{aligned}$$

(c) Follows from (a) and (b). □

Remark 2.2. Observe that although we know that given A and B square block matrices in general one has $A\boxtimes B \neq B\boxtimes A$, a glance at part (b) in Proposition 2.1 shows that $\operatorname{tr}(A\boxtimes B)$ is always equal to $\operatorname{tr}(B\boxtimes A)$. Of course, $\operatorname{tr}(A\circledast B) = \operatorname{tr}(B\circledast A)$ since the matrices coincide.

Proposition 2.3. *Let* $A, B \in \mathcal{M}_N(\mathcal{M}_n)$. *We have*

(a) $\operatorname{tr}((A + B) \circledast (A - B))) = \operatorname{tr}(A \circledast A) - \operatorname{tr}(B \circledast B)$.
(b) $\operatorname{tr}((A \pm B) \circledast (A \pm B))) = \operatorname{tr}(A \circledast A) \pm 2\operatorname{tr}(A \circledast B) + \operatorname{tr}(B \circledast B)$.
(c) $\operatorname{tr}((A + B)\boxtimes(A - B))) = \operatorname{tr}(A\boxtimes A) - \operatorname{tr}(B\boxtimes B)$.
(d) $\operatorname{tr}((A \pm B)\boxtimes(A \pm B))) = \operatorname{tr}(A\boxtimes A) \pm 2\operatorname{tr}(A\boxtimes B) + \operatorname{tr}(B\boxtimes B)$.

Proof. The four assertions are consequence of the properties of mixed associativity and distributivity with respect to the sum that the Kronecker and the Schur product have, and also the linearity of the trace and Remark 2.2. □

Remark 2.4. Recall that the arithmetic mean of a sequence $(\alpha_l)_{l=1}^{m}$ is greater or equal than its geometric mean, that is,

$$\frac{\sum_{l=1}^{m}\alpha_l}{m} \geq \left(\prod_{l=1}^{m}\alpha_l\right)^{\frac{1}{m}}.$$

Proposition 2.5. *Let* $p \in \mathbb{N}$, *and consider a finite sequence of block matrices* $(A^{s)})_{s=1}^{p} \subset \mathcal{M}_N(\mathcal{M}_n)$. *Then, we have the following inequality:*

$$\operatorname{tr}(\circledast_{s=1}^{p} A^{s)}) \leq \operatorname{tr}\left(\circledast^{p}\sum_{s=1}^{p}\frac{A^{s)}}{p}\right).$$

Proof.

$$\begin{aligned}
\operatorname{tr}\left(\circledast^p \sum_{s=1}^{p} \frac{A^{s)}}{p}\right) &= \operatorname{tr}\left(\left(\frac{A^{1)}+\cdots+A^{p)}}{p}\right)\circledast \overset{p}{\cdots} \circledast \left(\frac{A^{1)}+\cdots+A^{p)}}{p}\right)\right)\\
&= \sum_{i=1}^{N} \operatorname{tr}\left(\left(\frac{A^{1)}+\cdots+A^{p)}}{p}\right)\circledast \overset{p}{\cdots} \circledast \left(\frac{A^{1)}+\cdots+A^{p)}}{p}\right)\right)_{i,i}\\
&= \sum_{i=1}^{N}\sum_{l=1}^{n}\left(\frac{A^{1)}_{i,i}(l,l)+\cdots+A^{p)}_{i,i}(l,l)}{p}\right)^p\\
&\geq \sum_{i=1}^{N}\sum_{l=1}^{n}\prod_{s=1}^{p} A^{s)}_{i,i}(l,l) = \sum_{i=1}^{N}\operatorname{tr}\left(\circledast_{s=1}^{p} A^{s)}_{i,i}\right) \qquad \text{(by Remark 2.4)}\\
&= \sum_{i=1}^{N}\operatorname{tr}\left(\left(\circledast_{s=1}^{p} A^{s)}\right)_{i,i}\right) = \operatorname{tr}\left(\circledast_{s=1}^{p} A^{s)}\right).
\end{aligned}$$

□

3. Trace sub-multiplicativity and the spaces $\mathcal{M}_N^S(\mathcal{M}_n)$ and $\mathcal{M}_N^+(\mathcal{M}_n)$

Now, we will explore the relation between the value of the trace of Schur or Kronecker products of matrices and the product of the traces of the original matrices. First of all, recall that in the case of matrices with scalar entries, it is known that for the Kronecker product one has $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\cdot\operatorname{tr}(B)$. This is not the case for block matrices, neither for the product $\boxtimes$ nor for the product $\circledast$, as the following example shows.

Example 3.1. Consider the following matrices from $\mathcal{M}_2(\mathcal{M}_2)$:

$$A = \begin{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \end{pmatrix}, \qquad B = \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} & \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}.$$

Their block Schur product is the matrix

$$A \circledast B = \begin{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} & \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \end{pmatrix} \in \mathcal{M}_2(\mathcal{M}_2),$$

and their block Kronecker product is the matrix

$$A\boxtimes B=\begin{pmatrix}\begin{pmatrix}-1&0\\0&-5\end{pmatrix}&\begin{pmatrix}-2&0\\0&-5\end{pmatrix}&\begin{pmatrix}0&1\\1&0\end{pmatrix}&\begin{pmatrix}0&0\\0&0\end{pmatrix}\\\begin{pmatrix}0&0\\0&10\end{pmatrix}&\begin{pmatrix}3&0\\0&-5\end{pmatrix}&\begin{pmatrix}0&0\\1&0\end{pmatrix}&\begin{pmatrix}0&0\\0&0\end{pmatrix}\\\begin{pmatrix}2&2\\3&-1\end{pmatrix}&\begin{pmatrix}4&0\\0&-1\end{pmatrix}&\begin{pmatrix}1&0\\0&2\end{pmatrix}&\begin{pmatrix}2&0\\0&2\end{pmatrix}\\\begin{pmatrix}0&0\\3&2\end{pmatrix}&\begin{pmatrix}-6&0\\0&-1\end{pmatrix}&\begin{pmatrix}0&0\\0&-4\end{pmatrix}&\begin{pmatrix}-3&0\\0&2\end{pmatrix}\end{pmatrix}\in\mathcal{M}_4(\mathcal{M}_2).$$

Then, $\operatorname{tr}(A)=3$, $\operatorname{tr}(B)=-4$, but $\operatorname{tr}(A\circledast B)=-7\neq\operatorname{tr}(A)\cdot\operatorname{tr}(B)$ and $\operatorname{tr}(A\boxtimes B)=-6\neq\operatorname{tr}(A)\cdot\operatorname{tr}(B)$.

Example 3.1 also revealed that the inequalities $\operatorname{tr}(A\boxtimes B)\leq\operatorname{tr}(A)\cdot\operatorname{tr}(B)$ and $\operatorname{tr}(A\circledast B)\leq\operatorname{tr}(A)\cdot\operatorname{tr}(B)$ are not true in general. However, we ask ourselves if the trace operator might be submultiplicative for these products under certain restrictions. The following two spaces of block matrices will be relevant for that matter.

Definition 3.2. Given $N,n\in\mathbb{N}$, we define the following subsets of $\mathcal{M}_N(\mathcal{M}_n)$:

$$\mathcal{M}_N^S(\mathcal{M}_n):=\{(T_{k,j})_{k,j}\in\mathcal{M}_N(\mathcal{M}_n)\ /\ \sum_{k=1}^N T_{k,k}(l,l)\geq 0,\ \forall\, 1\leq l\leq n\},$$

$$\mathcal{M}_N^+(\mathcal{M}_n):=\{(T_{k,j})_{k,j}\in\mathcal{M}_N(\mathcal{M}_n)\ /\ T_{k,k}(l,l)\geq 0,\ \forall\, 1\leq k\leq N,\ \forall\, 1\leq l\leq n\}.$$

Of course, $\mathcal{M}_N^+(\mathcal{M}_n)\subsetneq\mathcal{M}_N^S(\mathcal{M}_n)$.

In the next theorem, we show that the trace operator is actually submultiplicative when acting on matrices that fulfill conditions related to the spaces of matrices from Definition 3.2.

Theorem 3.3. *Let $A=(T_{k,j})_{k,j}\in\mathcal{M}_N(\mathcal{M}_n)$ and $B=(S_{k,j})_{k,j}\in\mathcal{M}_M(\mathcal{M}_n)$.*

(a) *If $M=N$, $A\in\mathcal{M}_N^S(\mathcal{M}_n)$, and $B\in\mathcal{M}_N^+(\mathcal{M}_n)$, then*

$$\operatorname{tr}(A\circledast B)\leq\operatorname{tr}(A)\cdot\operatorname{tr}(B).$$

(b) *If $A\in\mathcal{M}_N^S(\mathcal{M}_n)$ and $B\in\mathcal{M}_M^S(\mathcal{M}_n)$, then*

$$\operatorname{tr}(A\boxtimes B)\leq\operatorname{tr}(A)\cdot\operatorname{tr}(B).$$

Proof. (a) First, observe that since $A\in\mathcal{M}_N^S(\mathcal{M}_n)$, then we have that

$$\sum_{k=1}^N T_{k,k}(l,l)\leq\sum_{l=1}^n\sum_{k=1}^N T_{k,k}(l,l)=\operatorname{tr}(A).$$

Also, since $B\in\mathcal{M}_N^+(\mathcal{M}_n)$, we get the following estimation:

$$\sup_i S_{i,i}(l,l)\leq\sum_{i=1}^N S_{i,i}(l,l)\leq\sum_{l=1}^n\sum_{i=1}^N S_{i,i}(l,l)\leq\operatorname{tr}(B).$$

Therefore,

$$\begin{aligned}\operatorname{tr}(A \circledast B) &= \sum_{i=1}^{N}\sum_{l=1}^{n} T_{i,i}(l,l)S_{i,i}(l,l) \qquad \text{(by Proposition 2.1(a))}\\ &\leq \sum_{l=1}^{n} \sup_i S_{i,i}(l,l) \sum_{i=1}^{N} T_{i,i}(l,l)\\ &\leq \operatorname{tr}(A)\cdot\operatorname{tr}(B).\end{aligned}$$

(b) In a similar way, we get the estimation for the trace of the block Kronecker product:

$$\begin{aligned}\operatorname{tr}(A \boxtimes B) &= \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{l=1}^{n} T_{i,i}(l,l)S_{j,j}(l,l) \qquad \text{(by Proposition 2.1(b))}\\ &= \sum_{l=1}^{n}\left(\sum_{i=1}^{N} T_{i,i}(l,l)\right)\cdot\left(\sum_{j=1}^{M} S_{j,j}(l,l)\right)\\ &\leq \left(\sup_l \sum_{j=1}^{M} S_{j,j}(l,l)\right)\cdot\sum_{l=1}^{n}\left(\sum_{i=1}^{N} T_{i,i}(l,l)\right)\\ &\leq \left(\sum_{l=1}^{n}\sum_{j=1}^{M} S_{j,j}(l,l)\right)\cdot\operatorname{tr}(A) \leq \operatorname{tr}(A)\cdot\operatorname{tr}(B).\end{aligned}$$

□

Corollary 3.4. (a) *Let $A_1, A_2, \ldots, A_m \in \mathcal{M}_N^+(\mathcal{M}_n)$. Then*

$$\operatorname{tr}(A_1 \circledast A_2 \circledast \cdots \circledast A_m) \leq \prod_{i=1}^{m} \operatorname{tr}(A_i).$$

(b) *Let $B_i \in \mathcal{M}_{N_i}^+(\mathcal{M}_n)$, $1 \leq i \leq m$. Then*

$$\operatorname{tr}(B_1 \boxtimes B_2 \boxtimes \cdots \boxtimes B_m) \leq \prod_{i=1}^{m} \operatorname{tr}(B_i).$$

Proof. (a) Note that for any $1 \leq i \leq m$, one has

$$(A_1 \circledast A_2 \circledast \cdots \circledast A_i)_{k,k}(l,l) = (A_1)_{k,k}(l,l)\cdot(A_2)_{k,k}(l,l)\cdot\ldots\cdot(A_i)_{k,k}(l,l) \geq 0$$

for all k, l such that $1 \leq k \leq N$ and $1 \leq l \leq n$, since by hypothesis all matrices A_i are in $\mathcal{M}^+$. Therefore, $A_1 \circledast A_2 \circledast \cdots \circledast A_i$ is in $\mathcal{M}^+$ for each i. Now, the inequality follows from a combined use of this observation, part (a) of Theorem 3.3 and an induction argument.

(b) Following the same line as in (a), since the matrix $B_1 \boxtimes \cdots \boxtimes B_i$ is also in $\mathcal{M}^+$ for each $1 \leq i \leq m$ because its diagonals are just Schur products of diagonals of matrices that are all of them in $\mathcal{M}^+$ by hypothesis. This allows us to apply part (b) of Theorem 3.3, and an induction argument concludes the proof. □

As a small application of the previous inequality, we will take a look now at the trace of a version of the exponential of a block matrix based on our block Schur product. Let $A \in \mathcal{M}_N(\mathcal{M}_n)$ be a block matrix. We define e^A as follows:

$$e^A := \sum_{j=0}^{\infty} \frac{\circledast_{i=1}^{j} A}{j!},$$

where $\circledast_{i=1}^{0} A := I \in \mathcal{M}_N(\mathcal{M}_n)$ is the identity matrix for the block Schur product, with $\operatorname{tr}(I) = Nn$. Observe that e^A is well defined, since taking multiplier norms, we have

$$\left\| \sum_{j=0}^{\infty} \frac{\circledast_{i=1}^{j} A}{j!} \right\| \leq \sum_{j=0}^{\infty} \left\| \frac{\circledast_{i=1}^{j} A}{j!} \right\| \leq \sum_{j=0}^{\infty} \frac{\|A\|^j}{j!} = e^{\|A\|}.$$

Notice that in the second inequality we used the fact that the product $\circledast$ endows the space of multipliers from $\mathcal{M}_N(\mathcal{M}_n)$ to $\mathcal{M}_N(\mathcal{M}_n)$ with a structure of Banach algebra. Furthermore, it can be seen that the product $\circledast$ also endows the space of bounded linear operators represented by elements of $\mathcal{M}_N(\mathcal{M}_n)$ with a structure of Banach algebra with the operator norm (see [4]).

Now, by letting $d = Nn - 1$, the trace of e^A can be bounded from above when $A \in \mathcal{M}_N^+(\mathcal{M}_n)$ as follows:

$$\begin{aligned} \operatorname{tr}(e^A) &= \operatorname{tr}\left(\sum_{j=0}^{\infty} \frac{\circledast_{i=1}^{j} A}{j!} \right) = \sum_{j=0}^{\infty} \frac{\operatorname{tr}(\circledast_{i=1}^{j} A)}{j!} && (3.1) \\ &\leq Nn + \sum_{j=1}^{\infty} \frac{\operatorname{tr}(A)^j}{j!} = d + e^{\operatorname{tr}(A)}. && \text{(by Corollary 3.4(a))} \end{aligned}$$

Proposition 3.5. *Let $A, B \in \mathcal{M}_N(\mathcal{M}_n)$ such that $A + B \in \mathcal{M}_N^+(\mathcal{M}_n)$, and let $d = Nn - 1$. Then*

$$\operatorname{tr}(e^A \circledast e^B) \leq d + e^{tr(A)} \cdot e^{tr(B)}.$$

Proof. Note that, since the product $\circledast$ is commutative, we can write

$$\begin{aligned} e^A \circledast e^B &= \left(\sum_{j=0}^{\infty} \frac{\circledast_{i=1}^{j} A}{j!} \right) \left(\sum_{j=0}^{\infty} \frac{\circledast_{i=1}^{j} B}{j!} \right) \\ &= \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{\circledast_{i=1}^{m} A \circledast_{i=1}^{j} B}{m! j!} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{\circledast_{i=1}^{m} A \circledast_{i=1}^{l-m} B}{m!(l-m)!} \\ &= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{m=0}^{l} \frac{l!}{m!(l-m)!} \circledast_{i=1}^{m} A \circledast_{i=1}^{l-m} B \\ &= \sum_{l=0}^{\infty} \frac{\circledast_{i=1}^{l} (A+B)}{l!} = e^{A+B}. \end{aligned}$$

Using that and applying inequality (3.1) to $A + B$, we have

$$\operatorname{tr}(e^A \circledast e^B) = \operatorname{tr}(e^{A+B})$$

$$\leq d + e^{\mathrm{tr}(A+B)} = d + e^{(\mathrm{tr}(A)+\mathrm{tr}(B))} \qquad \text{(by 3.1)}$$
$$= d + e^{\mathrm{tr}(A)} \cdot e^{\mathrm{tr}(B)}.$$

□

As a direct consequence of applying induction to Proposition 3.5, we obtain this corollary that gives an upper estimate for the trace of a finite Schur product of exponentials of matrices.

Corollary 3.6. *Let $\{A_i\}_{i=1}^m \subset \mathcal{M}_N(\mathcal{M}_n)$ such that $\sum_{i=1}^m A_i \in \mathcal{M}_N^+(\mathcal{M}_n)$, and let $d = Nn - 1$. Then, we have*

$$\mathrm{tr}(\circledast_{i=1}^m e^{A_i}) \leq \ d + \prod_{i=1}^m e^{tr(A_i)}.$$

4. Trace inequalities combining both products

Finally, we present some results that give upper estimates for the traces of block matrices generated by combined Kronecker and Hadamard products, in terms of the trace of matrices where only one of the products is involved. The utility of these lies in the fact that it is easier to compute the latter ones.

Theorem 4.1. *Let $A_i, B_i \in \mathcal{M}_N^+(\mathcal{M}_n)$, for $1 \leq i \leq m$. Then*

(a) $\mathrm{tr}\Big((A_1 \boxtimes A_2 \boxtimes \cdots \boxtimes A_m) \circledast (B_1 \boxtimes B_2 \boxtimes \cdots \boxtimes B_m) \Big) \leq \prod_{i=1}^m \mathrm{tr}(A_i \circledast B_i)$.

(b) $\mathrm{tr}\Big((A_1 \circledast A_2 \circledast \cdots \circledast A_m) \boxtimes (B_1 \circledast B_2 \circledast \cdots \circledast B_m) \Big) \leq \prod_{i=1}^m \mathrm{tr}(A_i \boxtimes B_i)$.

Proof. First, observe that direct computations show that the product $\circledast$ and the product $\boxtimes$ are linked by the following relation, that we shall use below:

$$(A \boxtimes B) \circledast (C \boxtimes D) = (A \circledast C) \boxtimes (B \circledast D). \tag{4.1}$$

(a) We use an induction argument. For $m = 1$, the result is obvious. Let us assume that the result is true for $m = s$, and let us prove that it is also true for $m = s + 1$. We can write

$$\begin{aligned}
&\mathrm{tr}\Big((A_1 \boxtimes A_2 \boxtimes \cdots \boxtimes A_{s+1}) \circledast (B_1 \boxtimes B_2 \boxtimes \cdots \boxtimes B_{s+1}) \Big) \\
&\quad = \mathrm{tr}\Big(\big((A_1 \boxtimes A_2 \boxtimes \cdots \boxtimes A_s) \circledast (B_1 \boxtimes B_2 \boxtimes \cdots \boxtimes B_s) \big) \boxtimes (A_{s+1} \circledast B_{s+1}) \Big). \quad \text{(by (4.1))}
\end{aligned}$$

Now, by hypothesis, $A_{s+1} \circledast B_{s+1}$ is in a space $\mathcal{M}^+$. The matrix

$$(A_1 \boxtimes A_2 \boxtimes \cdots \boxtimes A_s) \circledast (B_1 \boxtimes B_2 \boxtimes \cdots \boxtimes B_s)$$

also belongs to the same space, since all of its diagonal entries are Schur products of the diagonal entries of the matrices A_i, B_j which all of them were in $\mathcal{M}^+$ by the hypothesis. Therefore, by using part (b) of Theorem 3.3 and induction hypothesis, we conclude

$$\mathrm{tr}\Big(\big((A_1 \boxtimes A_2 \boxtimes \cdots \boxtimes A_s) \circledast (B_1 \boxtimes B_2 \boxtimes \cdots \boxtimes B_s) \big) \boxtimes (A_{s+1} \circledast B_{s+1}) \Big)$$

$$\le \operatorname{tr}\Big((A_1 \boxtimes A_2 \boxtimes \cdots \boxtimes A_s) \circledast (B_1 \boxtimes B_2 \boxtimes \cdots \boxtimes B_s) \Big) \cdot \operatorname{tr}(A_{s+1} \circledast B_{s+1})$$
(by Theorem 3.3(b))

$$\le \prod_{i=1}^{s} \operatorname{tr}(A_i \circledast B_i) \cdot \operatorname{tr}(A_{s+1} \circledast B_{s+1}) = \prod_{i=1}^{s+1} \operatorname{tr}(A_i \circledast B_i).$$

(b) By using part (a) of Theorem 3.3, the proof is analogous. □

Corollary 4.2. *Let $A_i, B_i \in \mathcal{M}_N^+(\mathcal{M}_n)$, for $1 \le i \le m$. Then*

(a) $\operatorname{tr}\Big((A_1 \circledast A_2 \circledast \cdots \circledast A_m) \boxtimes (B_1 \circledast B_2 \circledast \cdots \circledast B_m) \Big) \le \operatorname{tr}(\circledast_{i=1}^m A_i) \cdot \operatorname{tr}(\circledast_{i=1}^m B_i)$.

(b) $\operatorname{tr}\Big((A_1 \boxtimes A_2 \boxtimes \cdots \boxtimes A_m) \circledast (B_1 \boxtimes B_2 \boxtimes \cdots \boxtimes B_m) \Big) \le \operatorname{tr}(\boxtimes_{i=1}^m A_i) \cdot \operatorname{tr}(\boxtimes_{i=1}^m B_i)$.

Proof. (a)

$$\operatorname{tr}\Big((A_1 \circledast A_2 \circledast \cdots \circledast A_m) \boxtimes (B_1 \circledast B_2 \circledast \cdots \circledast B_m) \Big)$$
$$= \operatorname{tr}\Big(\big((A_1 \circledast A_2 \circledast \cdots \circledast A_{m-1}) \circledast A_m \big) \boxtimes \big((B_1 \circledast B_2 \circledast \cdots \circledast B_{m-1}) \circledast B_m \big) \Big)$$
$$= \operatorname{tr}\Big(\big((A_1 \circledast A_2 \circledast \cdots \circledast A_{m-1}) \boxtimes (B_1 \circledast B_2 \circledast \cdots \circledast B_{m-1}) \big) \circledast (A_m \boxtimes B_m) \Big)$$
(by (4.1))

$$\le \operatorname{tr}(A_1 \circledast \cdots \circledast A_{m-1} \circledast A_m) \cdot \operatorname{tr}(B_1 \circledast \cdots B_{m-1} \circledast B_m)$$
(by Theorem 4.1(a))

$$= \operatorname{tr}(\circledast_{i=1}^m A_i) \cdot \operatorname{tr}(\circledast_{i=1}^m B_i).$$

(b) Follows by the same argument, but using part (b) of Theorem 4.1 instead. □

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