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DIRECT ESTIMATES FOR STANCU VARIANT OF LUPAŞ-DURRMEYER OPERATORS BASED ON POLYA DISTRIBUTION

LAKSHMI NARAYAN MISHRA[1,2*] AND ALOK KUMAR[3]

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Abstract. In this paper, we study approximation properties of a family of linear positive operators and establish the Voronovskaja type asymptotic formula, local approximation and pointwise estimates using the Lipschitz type maximal function. In the last section, we consider the King type modification of these operators to obtain better estimates.

1. Introduction and preliminaries

In the field of approximation theory, the Bernstein polynomials discovered by Bernstein [4] in 1912, possess many remarkable properties, and new generalizations and applications are being discovered by using these polynomials. The aim of these generalizations is to provide appropriate and powerful tools to applied areas such as numerical analysis, computer-aided geometric design, solutions of differential equations and so on.

In 1968, Stancu [37] introduced a sequence of positive linear operators $P_n^{(\alpha)} : C[0,1] \to C[0,1]$, depending on a nonnegative parameter α given by

$$P_n^{(\alpha)}(f;x) = \sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

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* Corresponding author.

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where $p_{n,k}^{(\alpha)}(x)$ is the Polya distribution with the density function given by

$$p_{n,k}^{(\alpha)}(x) = \binom{n}{k} \frac{\prod_{\nu=0}^{k-1}(x+\nu\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{\prod_{\lambda=0}^{n-1}(1+\lambda\alpha)}, \quad x \in [0,1].$$

In the case $\alpha = 0$, these operators reduce to the classical Bernstein polynomials. For $\alpha = 1/n$ a special case of the operators (1.1) was considered by Lupaş and Lupaş [26], which can be represented in an alternate form as

$$P_n^{(1/n)}(f;x) = \frac{2(n!)}{(2n)!}\sum_{k=0}^{n}\binom{n}{k} f\left(\frac{k}{n}\right)(nx)_k(n-nx)_{n-k}, \tag{1.2}$$

where $f \in C(I)$, with $I = [0,1]$ and $(n)_k = n(n+1)(n+2)\dots(x+k-1)$ is the rising factorial. Recently Miclăuş [27] established some approximation results for the operators (1.1) and for the case (1.2).

Recently, Gupta and Rassias [14] introduced the Durrmeyer type integral modification of the operators (1.2), which is based on the Polya distribution as follows:

$$D_n^{(1/n)}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(x)\int_0^1 p_{n,k}(t)f(t)dt, \tag{1.3}$$

where

$$p_{n,k}^{(1/n)}(x) = \frac{2(n!)}{(2n)!}\binom{n}{k}(nx)_k(n-nx)_{n-k}$$

and

$$p_{n,k}(t) = \binom{n}{k} t^k(1-t)^{n-k}, \quad t \in I,$$

and studied asymptotic approximation, local and global results. In [3], Aral and Gupta obtained a quantitative Voronovskaja type asymptotic formula and the rate of convergence for bounded variation functions for the operators (1.3). Some approximation properties related the present paper can be found in [13] and in the recent book by Gupta and Agarwal [12].

In [37], Stancu introduced and investigated a new parameter-dependent linear positive operators of Bernstein type associated to a function $f \in C(I)$. The new construction of his operators shows that the new sequence of Bernstein polynomials present a better approach with the suitable selection of the parameters.

In the recent years, Stancu type generalization of the certain operators are introduced by several researchers and they obtained different type of approximation properties of many operators; we refer the reader to some of the important papers in this direction such as [5, 18, 20, 21, 23, 35] and so on. Various investigators such as [7–10, 14–16, 19, 20, 29–36] determined interesting results with their approximation properties.

Inspired by the above work, for $f \in C(I)$, we introduce the Stancu type generalization of the operators (1.3):

$$D_{n,\alpha,\beta}^{(1/n)}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(x)\int_0^1 p_{n,k}(t) f\left(\frac{nt+\alpha}{n+\beta}\right)dt. \tag{1.4}$$

The purpose of this paper is to study the Voronovskaja type theorem, local approximation, and pointwise estimates for the operators (1.4). We also propose and discuss the King type modification of the operators (1.4).

2. Moment estimates

We start this section with the following useful lemmas, which will be used in main results.

Lemma 2.1 ([14]). *For the operators $D_n^{(1/n)}(f;x)$, we have*

(1) $D_n^{(1/n)}(1;x)=1$,
(2) $D_n^{(1/n)}(t;x)=\frac{nx+1}{n+2}$,
(3) $D_n^{(1/n)}(t^2;x)=\frac{n^3x^2+5n^2x-n^2x^2+3nx+2n+2}{(n+1)(n+2)(n+3)}$,
(4) $$D_n^{(1/n)}(t^3;x)=\frac{1}{(n+2)(n+3)(n+4)}\Big(n^3x^3+\frac{6n^4x^2(1-x)}{(n+1)(n+2)}+\frac{6n^3x(1-x)}{(n+1)(n+2)}+6n^2x^2 +\frac{12n^2x(1-x)}{n+1}+11nx+6\Big),$$
(5) $$D_n^{(1/n)}(t^4;x)=\frac{1}{(n+2)(n+3)(n+4)(n+5)}\Big(n^4x^4+\frac{12n^4(n^2+1)x^3(1-x)}{(n+1)(n+2)(n+3)}+\frac{12n^4(3n-1)x^2(1-x)}{(n+1)(n+2)(n+3)} +\frac{2n^4(13n-1)x(1-x)}{n(n+1)(n+2)(n+3)}+10n^3x^3+\frac{60n^4x^2(1-x)}{(n+1)(n+2)} +\frac{60n^3x(1-x)}{(n+1)(n+2)}+35n^2x^2+\frac{70n^2x(1-x)}{n+1} +50nx+24\Big).$$

Lemma 2.2. *For the operators $D_{n,\alpha,\beta}^{(1/n)}(f;x)$, we have*

(1) $D_{n,\alpha,\beta}^{(1/n)}(1;x)=1$,
(2) $D_{n,\alpha,\beta}^{(1/n)}(t;x)=\frac{n^2x+n(\alpha+1)+2\alpha}{(n+\beta)(n+2)}$,
(3) $$D_{n,\alpha,\beta}^{(1/n)}(t^2;x)=\left(\frac{n^4(n-1)}{(n+\beta)^2(n+1)(n+2)(n+3)}\right)x^2+\left(\frac{n^3(5n+3)+2n^2\alpha(n+1)(n+3)}{(n+\beta)^2(n+1)(n+2)(n+3)}\right)x +\frac{2n^2+2n\alpha(n+3)+\alpha^2(n+2)(n+3)}{(n+\beta)^2(n+2)(n+3)},$$
(4) $$D_{n,\alpha,\beta}^{(1/n)}(t^3;x)=\frac{1}{(n+\beta)^3(n+1)(n+2)(n+3)(n+4)} \times\Big(n^6(n+1)x^3+\frac{6n^7x^2(1-x)}{n+2}+\frac{6n^6x(1-x)}{n+2}+6n^5x(2-x)+11n^4x +6n^3+3n^2\alpha(n+4)(n^3x^2+5n^2x-n^2x^2+3nx+2n+2)\Big) +\frac{3n\alpha^2(nx+1)+\alpha^3(n+2)}{(n+\beta)^3(n+2)},$$
(5) $$D_{n,\alpha,\beta}^{(1/n)}(t^4;x)=\frac{1}{(n+\beta)^4(n+1)(n+2)(n+3)(n+4)(n+5)} \times\Big(n^8(n+1)x^4+\frac{12n^8(n^2+1)x^3(1-x)}{(n+2)(n+3)}+\frac{12n^8(3n-1)x^2(1-x)}{(n+2)(n+3)} +\frac{2n^7(13n-1)x(1-x)}{(n+2)(n+3)}+10n^7(n+1)x^3+\frac{60n^8x^2(1-x)}{(n+2)}+\frac{60n^7x(1-x)}{(n+2)} +35n^6(n+1)x^2+70n^6x(1-x)+50n^5(n+1)x+24n^4(n+1) +4n^6(n+1)\alpha x^3+\frac{6n^7x^2(1-x)\alpha}{n+2}+\frac{6n^6x(1-x)\alpha}{n+2}+6n^5x(2-x)\alpha +11n^4(n+1)\alpha x+6n^3(n+1)\alpha x +6n^2\alpha^2(n+4)(n+5)(n^3x^2+5n^2x-n^2x^2+3nx+2n +2)\Big)+\frac{4n\alpha^3(nx+1)+\alpha^4(n+2)}{(n+\beta)^4(n+2)}.$$

Proof. For $x \in I$, in view of Lemma 2.1, we have

$$D_{n,\alpha,\beta}^{(1/n)}(1;x) = 1.$$

The first order moment is given by

$$D_{n,\alpha,\beta}^{(1/n)}(t;x) = \frac{n}{n+\beta} D_n^{(1/n)}(t;x) + \frac{\alpha}{n+\beta} = \frac{n^2x + n(\alpha+1) + 2\alpha}{(n+\beta)(n+2)}.$$

The second order moment is given by

$$\begin{aligned} D_{n,\alpha,\beta}^{(1/n)}(t^2;x) &= \left(\frac{n}{n+\beta}\right)^2 D_n^{(1/n)}(t^2;x) + \frac{2n\alpha}{(n+\beta)^2} D_n^{(1/n)}(t;x) + \left(\frac{\alpha}{n+\beta}\right)^2 \\ &= \left(\frac{n^4(n-1)}{(n+\beta)^2(n+1)(n+2)(n+3)}\right)x^2 \\ &\quad + \left(\frac{n^3(5n+3) + 2n^2\alpha(n+1)(n+3)}{(n+\beta)^2(n+1)(n+2)(n+3)}\right)x \\ &\quad + \frac{2n^2 + 2n\alpha(n+3) + \alpha^2(n+2)(n+3)}{(n+\beta)^2(n+2)(n+3)}. \end{aligned}$$

Similarly, we obtain third and fourth order moments. □

Lemma 2.3. *For $f \in C_B(I)$ (space of all real valued bounded functions on I endowed with norm $\| f \|_{C_B(I)} = \sup_{x \in I} |f(x)|$), we have*

$$\| D_{n,\alpha,\beta}^{(1/n)}(f) \| \leq \| f \| .$$

Proof. In view of (1.4) and Lemma 2.2, we get

$$\|D_{n,\alpha,\beta}^{(1/n)}(f)\| \leq \|f\| D_{n,\alpha,\beta}^{(1/n)}(1;x) = \|f\|.$$

□

Remark 2.4. By simple applications of Lemma 2.2, we have

$$\begin{aligned} D_{n,\alpha,\beta}^{(1/n)}\left((t-x);x\right) &= \frac{n(\alpha+1) + 2\alpha - (2\beta + n\beta + 2n)x}{(n+\beta)(n+2)} \\ &= \xi_{n,\alpha,\beta}^{(1/n)}(x) \end{aligned}$$

and

$$\begin{aligned} & D_{n,\alpha,\beta}^{(1/n)}\left((t-x)^2;x\right) \\ &= \left(\frac{-3n^4 + 5n^3 + n^3\beta^2 + 4n^3\beta + 6n^2\beta^2 + 11n\beta^2 + 16n^2\beta + 12n\beta}{(n+\beta)^2(n+1)(n+2)(n+3)}\right)x^2 \\ &\quad + \left(\frac{5n^4 + 3n^3 + 2n^2\alpha(n+1)(n+3) - 2(n\alpha + n + 2\alpha)(n+\beta)(n+1)(n+3)}{(n+\beta)^2(n+1)(n+2)(n+3)}\right)x \\ &\quad + \frac{2n^2 + 2n\alpha(n+3) + \alpha^2(n+2)(n+3)}{(n+\beta)^2(n+2)(n+3)} \\ &= \zeta_{n,\alpha,\beta}^{(1/n)}(x). \end{aligned}$$

Further,

$$D_{n,\alpha,\beta}^{(1/n)}\left((t-x)^4;x\right) \quad = \quad O\left(\frac{1}{n^2}\right), \ \textit{as } n\to\infty.$$

3. Main results

Let $e_i(t)=t^i,\ i=0,1,2.$

Theorem 3.1. *Let $f\in C(I)$. Then $\lim_{n\to\infty} D_{n,\alpha,\beta}^{(1/n)}(f;x)=f(x)$, uniformly in each compact subset of I.*

Proof. In view of Lemma 2.2, we get

$$\lim_{n\to\infty} D_{n,\alpha,\beta}^{(1/n)}(e_i;x)=x^i, \quad i=0,1,2,$$

uniformly in each compact subset of I. Applying the Bohman–Korovkin theorem, it follows that $\lim_{n\to\infty} D_{n,\alpha,\beta}^{(1/n)}(f;x) = f(x)$, uniformly in each compact subset of I. □

3.1. **Voronovskaja type theorem.** In this section, we prove the Voronvoskaya type asymptotic theorem for the operators $D_{n,\alpha,\beta}^{(1/n)}$.

Theorem 3.2. *Let f be a bounded and integrable function on I, and let the second derivative of f exist at a fixed point $x\in I$. Then*

$$\lim_{n\to\infty} n\left(D_{n,\alpha,\beta}^{(1/n)}(f;x)-f(x)\right)=\left((\alpha+1)-(\beta+2)x\right)f'(x)+\frac{3}{2}x(1-x)f''(x).$$

Proof. Let $x\in I$ be fixed. Using Taylor's expansion formula of the function f implies that

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2, \tag{3.1}$$

where $r(t,x)$ is a bounded function and $\lim_{t\to x} r(t,x)=0$.

Applying $D_{n,\alpha,\beta}^{(1/n)}$ on both sides of (3.1), we get

$$\begin{aligned} n\left(D_{n,\alpha,\beta}^{(1/n)}(f;x)-f(x)\right) &= nf'(x)D_{n,\alpha,\beta}^{(1/n)}\left((t-x);x\right)+\frac{1}{2}nf''(x)D_{n,\alpha,\beta}^{(1/n)}\left((t-x)^2;x\right)\\ &\quad +nD_{n,\alpha,\beta}^{(1/n)}\left((t-x)^2r(t,x);x\right).\end{aligned}$$

In view of Remark 2.4, we have

$$\lim_{n\to\infty} nD_{n,\alpha,\beta}^{(1/n)}\left((t-x);x\right)=(\alpha+1)-(\beta+2)x \tag{3.2}$$

and

$$\lim_{n\to\infty} nD_{n,\alpha,\beta}^{(1/n)}\left((t-x)^2;x\right)=3x(1-x). \tag{3.3}$$

Now, we shall show that

$$\lim_{n\to\infty} nD_{n,\alpha,\beta}^{(1/n)}\left(r(t,x)(t-x)^2;x\right)=0.$$

Using the Cauchy–Schwarz inequality, we have

$$D^{(1/n)}_{n,\alpha,\beta}\Big(r(t,x)(t-x)^2;x\Big) \;\le\; \Big(D^{(1/n)}_{n,\alpha,\beta}(r^2(t,x);x)\Big)^{1/2}\Big(D^{(1/n)}_{n,\alpha,\beta}((t-x)^4;x)\Big)^{1/2}. \quad (3.4)$$

We observe that $r^2(x,x)=0$ and $r^2(.,x)\in C_B(I)$. Then, it follows that

$$\lim_{n\to\infty} D^{(1/n)}_{n,\alpha,\beta}(r^2(t,x);x)=r^2(x,x)=0. \quad (3.5)$$

Now, from (3.4) and (3.5), we obtain

$$\lim_{n\to\infty} nD^{(1/n)}_{n,\alpha,\beta}\left(r(t,x)(t-x)^2;x\right)=0. \quad (3.6)$$

From (3.2), (3.3) and (3.6), we get the required result. □

The next theorem uses the asymptotic formulas fulfilled by $D^{(1/n)}_{n,\alpha,\beta}$ and $D^{(1/n)}_{n}$ to state a sort of weak result that shows that for certain family of illustrative functions the new sequence approximates better than the previous operators.

Theorem 3.3. *Let $f\in C^2(I)$. Assume that there exists $n_0\in N$, such that*

$$f(x)\le D^{(1/n)}_{n,\alpha,\beta}(f;x)\le D^{(1/n)}_{n}(f;x) \quad (3.7)$$

for all $n\ge n_0$ and $x\in(0,1)$. Then

$$\frac{3}{2}x(1-x)f''(x)\ge(\alpha-\beta x)f'(x)\ge 0 \quad x\in(0,1). \quad (3.8)$$

In particular, $f'(x)\ge 0$ and $f''(x)\ge 0$.

Conversely, if (3.8) holds with strict inequalities at a given point $x\in(0,1)$, then there exists $n_0\in N$ such that for $n\ge n_0$

$$f(x)<D^{(1/n)}_{n,\alpha,\beta}(f;x)<D^{(1/n)}_{n}(f;x).$$

Proof. From (3.7), we have

$$0\le n(D^{(1/n)}_{n,\alpha,\beta}(f;x)-f(x))\le n(D^{(1/n)}_{n}(f;x)-f(x))$$

for all $n\ge n_0$ and $x\in(0,1)$.

Then, using Theorem 3.2 and [14] implies that

$$0\le(\alpha-\beta x)f'(x)\le\frac{3}{2}x(1-x)f''(x),$$

from which (3.8) follows directly.

Conversely, if (3.8) holds with strict inequalities for a given $x\in(0,1)$, then directly

$$0<(\alpha-\beta x)f'(x)<\frac{3}{2}x(1-x)f''(x),$$

and using again Theorem 3.2 and [14] completes the proof. □

3.2. **Local approximation.** We begin by recalling the following K-functional:

$$K_2(f,\delta) \;=\; \inf\{\| f-g \| +\delta \| g'' \| : g \in W^2\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B(I) : g', g'' \in C_B(I)\}$. By, [7, p.177, Theorem 2.4], there exists an absolute constant $M > 0$ such that

$$K_2(f,\delta) \leq M\omega_2(f,\sqrt{\delta}), \tag{3.9}$$

where

$$\omega_2(f,\sqrt{\delta}) \;=\; \sup_{0<h\leq\sqrt{\delta}} \sup_{x,x+2h\in I} \mid f(x+2h)-2f(x+h)+f(x) \mid$$

is the second order modulus of smoothness of $f \in C_B(I)$. We denote the first order modulus of continuity of $f \in C_B(I)$ by

$$\omega(f,\delta) \;=\; \sup_{0<h\leq\delta} \sup_{x,x+h\in I} \mid f(x+h)-f(x) \mid .$$

Theorem 3.4. *Let $f \in C_B(I)$. Then, for every $x \in I$, we have*

$$\mid D_{n,\alpha,\beta}^{(1/n)}(f;x) - f(x) \mid \;\leq\; M\omega_2\left(f, \chi_{n,\alpha,\beta}^{(1/n)}(x)\right) + \omega\left(f, \xi_{n,\alpha,\beta}^{(1/n)}\right),$$

where M is a positive constant and

$$\chi_{n,\alpha,\beta}^{(1/n)}(x) \;=\; \left(\zeta_{n,\alpha,\beta}^{(1/n)}(x) + \left(\xi_{n,\alpha,\beta}^{(1/n)}\right)^2\right)^{1/2}.$$

Proof. For $x \in I$, we consider the auxiliary operators $\overline{D}_{n,\alpha,\beta}^{(1/n)}$ defined by

$$\overline{D}_{n,\alpha,\beta}^{(1/n)}(f;x) \;=\; D_{n,\alpha,\beta}^{(1/n)}(f;x) - f\left(\frac{n^2x+n(\alpha+1)+2\alpha}{(n+\beta)(n+2)}\right) + f(x). \tag{3.10}$$

From Lemma 2.2, we observe that the operators $\overline{D}_{n,\alpha,\beta}^{(1/n)}$ are linear and reproduce the linear functions.

Hence

$$\overline{D}_{n,\alpha,\beta}^{(1/n)}((t-x);x) \;=\; 0. \tag{3.11}$$

Let $g \in W^2$ and $x,t \in I$. By Taylor's expansion, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv.$$

Applying $\overline{D}_{n,\alpha,\beta}^{(1/n)}$ on both sides of the above equation and using (3.11), we get

$$\overline{D}_{n,\alpha,\beta}^{(1/n)}(g;x) - g(x) \;=\; \overline{D}_{n,\alpha,\beta}^{(1/n)}\left(\int_x^t (t-v)g''(v)dv, x\right).$$

Thus, by (3.10), we get

$$\begin{aligned}|\overline{D}_{n,\alpha,\beta}^{(1/n)}(g;x)-g(x)| &\leq D_{n,\alpha,\beta}^{(1/n)}\left(\left|\int_x^t (t-v)g''(v)dv\right|,x\right)\\ &\quad+\left|\int_x^{\frac{n^2x+n(\alpha+1)+2\alpha}{(n+\beta)(n+2)}}\left(\frac{n^2x+n(\alpha+1)+2\alpha}{(n+\beta)(n+2)}-v\right)g''(v)dv\right|\\ &\leq \left(\zeta_{n,\alpha,\beta}^{(1/n)}(x)+\left(\xi_{n,\alpha,\beta}^{(1/n)}(x)\right)^2\right)\parallel g''\parallel\\ &\leq \left(\chi_{n,\alpha,\beta}^{(1/n)}(x)\right)^2\parallel g''\parallel. \end{aligned}\tag{3.12}$$

On the other hand, by (3.10) and Lemma 2.3, we have

$$|\overline{D}_{n,\alpha,\beta}^{(1/n)}(f;x)| \leq \parallel f\parallel. \tag{3.13}$$

Using (3.12) and (3.13) in (3.10), we obtain

$$\begin{aligned}|D_{n,\alpha,\beta}^{(1/n)}(f;x)-f(x)| &\leq |\overline{D}_{n,\alpha,\beta}^{(1/n)}(f-g;x)|+|(f-g)(x)|+|\overline{D}_{n,\alpha,\beta}^{(1/n)}(g;x)-g(x)|\\ &\quad+\left|f\left(\frac{n^2x+n(\alpha+1)+2\alpha}{(n+\beta)(n+2)}\right)-f(x)\right|\\ &\leq 2\parallel f-g\parallel+\left(\chi_{n,\alpha,\beta}^{(1/n)}(x)\right)^2\parallel g''\parallel\\ &\quad+\left|f\left(\frac{n^2x+n(\alpha+1)+2\alpha}{(n+\beta)(n+2)}\right)-f(x)\right|.\end{aligned}$$

Taking infimum over all $g\in W^2$, we get

$$|\,D_{n,\alpha,\beta}^{(1/n)}(f;x)-f(x)\,| \leq K_2\left(f,(\chi_{n,\alpha,\beta}^{(1/n)}(x))^2\right)+\omega\left(f,\xi_{n,\alpha,\beta}^{(1/n)}(x)\right).$$

In view of (3.9), we get

$$|\,D_{n,\alpha,\beta}^{(1/n)}(f;x)-f(x)\,| \leq M\omega_2\left(f,\chi_{n,\alpha,\beta}^{(1/n)}(x)\right)+\omega\left(f,\xi_{n,\alpha,\beta}^{(1/n)}(x)\right),$$

which completes the proof. □

Let $a_1,a_2>0$ be fixed. We define the following Lipschitz-type space (see [33]):

$$\mathrm{Lip}_M^{(a_1,a_2)}(\eta)=\left\{f\in C(I):|f(t)-f(x)|\leq M\frac{|t-x|^\eta}{(t+a_1x^2+a_2x)^{\eta/2}};\ x,t\in(0,1]\right\},$$

where M is a positive constant and $0<\eta\leq 1$.

Theorem 3.5. *Let $f\in \mathrm{Lip}_M^{(a_1,a_2)}(\eta)$. Then, for all $x\in(0,1]$, we have*

$$|D_{n,\alpha,\beta}^{(1/n)}(f;x)-f(x)|\leq M\left(\frac{\zeta_{n,\alpha,\beta}^{(1/n)}(x)}{a_1x^2+a_2x}\right)^{\eta/2}.$$

Proof. First, we prove the result for the case $\eta = 1$. Then, for $f \in \mathrm{Lip}_M^{(a_1,a_2)}(1)$, and $x \in (0,1]$, we have

$$\begin{aligned} |D_{n,\alpha,\beta}^{(1/n)}(f;x) - f(x)| &\leq D_{n,\alpha,\beta}^{(1/n)}(|f(t)-f(x)|;x) \\ &\leq M D_{n,\alpha,\beta}^{(1/n)}\left(\frac{|t-x|}{(t+a_1x^2+a_2x)^{1/2}};x\right) \\ &\leq \frac{M}{(a_1x^2+a_2x)^{1/2}} D_{n,\alpha,\beta}^{(1/n)}(|t-x|;x). \end{aligned}$$

Applying the Cauchy–Schwarz inequality, we get

$$\begin{aligned} |D_{n,\alpha,\beta}^{(1/n)}(f;x) - f(x)| &\leq \frac{M}{(a_1x^2+a_2x)^{1/2}}\left(D_{n,\alpha,\beta}^{(1/n)}((t-x)^2;x)\right)^{1/2} \\ &\leq M\left(\frac{\zeta_{n,\alpha,\beta}^{(1/n)}(x)}{a_1x^2+a_2x}\right)^{1/2}. \end{aligned}$$

Thus the result holds for $\eta = 1$.

Now, we prove that the result is true for the case $0 < \eta < 1$. For $f \in \mathrm{Lip}_M^{(a_1,a_2)}(\eta)$ and $x \in (0,1]$, we get

$$|D_{n,\alpha,\beta}^{(1/n)}(f;x) - f(x)| \leq \frac{M}{(a_1x^2+a_2x)^{\eta/2}} D_{n,\alpha,\beta}^{(1/n)}(|t-x|^\eta;x).$$

Taking $p = \frac{1}{\eta}$ and $q = \frac{p}{p-1}$, applying the Hölders inequality, we have

$$|D_{n,\alpha,\beta}^{(1/n)}(f;x) - f(x)| \leq \frac{M}{(a_1x^2+a_2x)^{\eta/2}}\left(D_{n,\alpha,\beta}^{(1/n)}(|t-x|;x)\right)^\eta.$$

Finally by the Cauchy–Schwarz inequality, we get

$$|D_{n,\alpha,\beta}^{(1/n)}(f;x) - f(x)| \leq M\left(\frac{\zeta_{n,\alpha,\beta}^{(1/n)}(x)}{a_1x^2+a_2x}\right)^{\eta/2}.$$

Thus, the proof is completed. □

3.3. **Pointwise estimates.** In the present section, we obtain some pointwise estimates of the rate of convergence of the operators $D_{n,\alpha,\beta}^{(1/n)}$. First, we give the relationship between the local smoothness of f and local approximation.

We know that a function $f \in C(I)$ is in $\mathrm{Lip}_{M_f}(\eta)$ on E, where $\eta \in (0,1]$ and $E \subset I$ if it satisfies the condition

$$|f(t) - f(x)| \leq M_f|t-x|^\eta, \quad t \in E \text{ and } x \in I,$$

where M_f is a constant depending only on η and f.

Theorem 3.6. *Let $f \in C(I) \cap \mathrm{Lip}_{M_f}(\eta)$, $\eta \in (0,1]$ and let E be any bounded subset of the interval I. Then, for each $x \in I$, we have*

$$|D_{n,\alpha,\beta}^{(1/n)}(f;x) - f(x)| \leq M_f\left(\left(\zeta_{n,\alpha,\beta}^{(1/n)}(x)\right)^{\eta/2} + 2(d(x,E))^\eta\right),$$

where M_f is a constant depending on η and f and $d(x,E)$ is the distance between x and E defined as

$$d(x,E) = \inf\{|t-x| : t \in E\}.$$

Proof. Let $\overline{E}$ be the closure of E in I. Then, there exists at least one point $x_0 \in \overline{E}$ such that

$$d(x,E) = |x - x_0|.$$

From the triangle inequality, we have

$$|f(t) - f(x)| \leq |f(t) - f(x_0)| + |f(x) - f(x_0)|.$$

Using the definition of $\mathrm{Lip}_{M_f}(\eta)$, we get

$$\begin{aligned} |D^{(1/n)}_{n,\alpha,\beta}(f;x) - f(x)| &\leq D^{(1/n)}_{n,\alpha,\beta}(|f(t) - f(x_0)|;x) + D^{(1/n)}_{n,\alpha,\beta}(|f(x) - f(x_0)|;x) \\ &\leq M_f\left(D^{(1/n)}_{n,\alpha,\beta}(|t-x_0|^\eta;x) + |x-x_0|^\eta\right) \\ &\leq M_f\left(D^{(1/n)}_{n,\alpha,\beta}(|t-x|^\eta;x) + 2|x-x_0|^\eta\right). \end{aligned}$$

Now, applying the Hölder's inequality with $p = \dfrac{2}{\eta}$ and $\dfrac{1}{q} = 1 - \dfrac{1}{p}$, we obtain

$$|D^{(1/n)}_{n,\alpha,\beta}(f;x) - f(x)| \leq M_f\left(\{D^{(1/n)}_{n,\alpha,\beta}(|t-x|^2;x)\}^{\eta/2} + 2(d(x,E))^\eta\right),$$

from which the desired result is obtained immediately. □

Next, we obtain the local direct estimate of the operators defined in (1.4), using the Lipschitz-type maximal function of order η introduced by B. Lenze [25] as

$$\widetilde{\omega}_\eta(f,x) = \sup_{t\neq x,\ t\in I} \frac{|f(t)-f(x)|}{|t-x|^\eta} \quad x \in I \text{ and } \eta \in (0,1]. \tag{3.14}$$

Theorem 3.7. *Let $f \in C_B(I)$ and $0 < \eta \leq 1$. Then, for all $x \in I$, we have*

$$|D^{(1/n)}_{n,\alpha,\beta}(f;x) - f(x)| \leq \widetilde{\omega}_\eta(f,x)\left(\zeta^{(1/n)}_{n,\alpha,\beta}(x)\right)^{\eta/2}.$$

Proof. In view of (3.14), we have

$$|f(t) - f(x)| \leq \widetilde{\omega}_\eta(f,x)|t-x|^\eta$$

and

$$|D^{(1/n)}_{n,\alpha,\beta}(f;x) - f(x)| \leq \widetilde{\omega}_\eta(f,x) D^{(1/n)}_{n,\alpha,\beta}(|t-x|^\eta;x).$$

Applying the Hölder's inequality with $p = \dfrac{2}{\eta}$ and $\dfrac{1}{q} = 1 - \dfrac{1}{p}$, we get

$$\begin{aligned} |D^{(1/n)}_{n,\alpha,\beta}(f;x) - f(x)| &\leq \widetilde{\omega}_\eta(f,x) D^{(1/n)}_{n,\alpha,\beta}((t-x)^2;x)^{\eta/2} \\ &\leq \widetilde{\omega}_\eta(f,x)\left(\zeta^{(1/n)}_{n,\alpha,\beta}(x)\right)^{\eta/2}. \end{aligned}$$

Thus, the proof is completed. □

4. King type modification

In this section, we discuss better convergence rates by King type operators. To make the convergence faster, King [17] proposed an approach to modify the classical Bernstein polynomial, so that the sequence preserve test functions e_0 and e_2, where $e_i(t) = t^i, i = 0, 1, 2$. After this approach many researcher contributed in this direction.

As the operator $D^{(1/n)}_{n,\alpha,\beta}(f; x)$ defined in (1.4) preserves only the constant functions, so further modification of these operators is proposed to be made, so that the modified operators preserve the constant as well as linear functions.

For this purpose the modification of (1.4) is defined as

$$\hat{D}^{(1/n)}_{n,\alpha,\beta}(f; x) = (n+1)\sum_{k=0}^{n} p^{(1/n)}_{n,k}(r_n(x)) \int_0^1 p_{n,k}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt \qquad (4.1)$$

where $r_n(x) = \dfrac{(n+\beta)(n+2)x - n(\alpha+1) - 2\alpha}{n^2}$ and $x \in I_n = [\frac{\alpha}{n+\beta}, 1]$.

Lemma 4.1. *For every $x \in I_n$, we have*

(1) $\hat{D}^{(1/n)}_{n,\alpha,\beta}(1; x) = 1$,
(2) $\hat{D}^{(1/n)}_{n,\alpha,\beta}(t; x) = x$,
(3) $\hat{D}^{(1/n)}_{n,\alpha,\beta}(t^2; x) = \dfrac{(n-1)(n+2)x^2}{(n+1)(n+3)} + \dfrac{(3n^2+6n\alpha+n+10\alpha)x}{(n+\beta)(n+1)(n+3)}$ $+\frac{5n^2\alpha^2-22n^2\alpha-11n\alpha^2-2n^3}{(n+\beta)^2(n+1)(n+2)(n+3)}$.

Consequently, for each $x \in I_n$, we have the following equalities:

$$\hat{D}^{(1/n)}_{n,\alpha,\beta}((t-x); x) = 0,$$

and

$$\begin{aligned}\hat{D}^{(1/n)}_{n,\alpha,\beta}((t-x)^2, x) &= \frac{-(3n+5)x^2}{(n+1)(n+3)} + \frac{(3n^2+6n\alpha+n+10\alpha)x}{(n+\beta)(n+1)(n+3)} \\ &\quad + \frac{5n^2\alpha^2-22n^2\alpha-11n\alpha^2-2n^3}{(n+\beta)^2(n+1)(n+2)(n+3)} \\ &= \lambda^{(1/n)}_{n,\alpha,\beta}(x). \end{aligned} \qquad (4.2)$$

Theorem 4.2. *For $f \in C_B(I_n)$, we have*

$$|\hat{D}^{(1/n)}_{n,\alpha,\beta}(f; x) - f(x)| \leq M'\omega_2\left(f, \sqrt{\lambda^{(1/n)}_{n,\alpha,\beta}(x)}\right),$$

where $\lambda^{(1/n)}_{n,\alpha,\beta}(x)$ is given by (4.2) and M' is a positive constant.

Proof. Let $g \in W^2$ and $x, t \in I_n$. Using Taylor's expansion, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv.$$

Applying $\hat{D}^{(1/n)}_{n,\alpha,\beta}$, we get

$$\hat{D}^{(1/n)}_{n,\alpha,\beta}(g;x) - g(x) = \hat{D}^{(1/n)}_{n,\alpha,\beta}\left(\int_x^t (t-v)g''(v)dv; x\right).$$

Obviously, we have $\left|\int_x^t (t-v)g''(v)dv\right| \le (t-x)^2\|g''\|$. Therefore,

$$\mid \hat{D}^{(1/n)}_{n,\alpha,\beta}(g;x) - g(x) \mid\le \hat{D}^{(1/n)}_{n,\alpha,\beta}((t-x)^2;x) \parallel g'' \parallel = \lambda^{(1/n)}_{n,\alpha,\beta}(x) \parallel g'' \parallel .$$

Since $\mid \hat{D}^{(1/n)}_{n,\alpha,\beta}(f;x) \mid\le \|f\|$, we get

$$\begin{aligned} &\mid \hat{D}^{(1/n)}_{n,\alpha,\beta}(f;x) - f(x) \mid \\ &\qquad \le\mid \hat{D}^{(1/n)}_{n,\alpha,\beta}(f-g;x) \mid + \mid (f-g)(x) \mid + \mid \hat{D}^{(1/n)}_{n,\alpha,\beta}(g;x) - g(x) \mid \\ &\qquad \le 2\|f-g\| + \lambda^{(1/n)}_{n,\alpha,\beta}(x)\|g''\|. \end{aligned}$$

Finally, taking the infimum over all $g \in W^2$ and using (3.9), we obtain

$$\mid \hat{D}^{(1/n)}_{n,\alpha,\beta}(f;x) - f(x) \mid\le M'\omega_2\left(f, \sqrt{\lambda^{(1/n)}_{n,\alpha,\beta}(x)}\right),$$

which proves the theorem. □

Theorem 4.3. *Let $f \in C_B(I_n)$. If f' and f'' exist at a fixed point $x \in I_n$, then we have*

$$\lim_{n\to\infty} n\left(\hat{D}^{(1/n)}_{n,\alpha,\beta}(f;x) - f(x)\right) = \frac{3}{2}x(1-x)f''(x).$$

The proof follows along the lines of Theorem 3.2.

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[1] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, Tamil Nadu, India

[2] L. 1627 Awadh Puri Colony Beniganj, Phase - III, Opposite - Industrial Training Institute (I.T.I.), Ayodhya Main Road, Faizabad 224 001, Uttar Pradesh, India
E-mail address: lakshminarayanmishra04@gmail.com; l_n_mishra@yahoo.co.in

[3] Department of Computer Science, Dev Sanskriti Vishwavidyalaya, Haridwar-249411, Uttarakhand, India
E-mail address: alokkpma@gmail.com