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SLANT TOEPLITZ OPERATORS ON THE LEBESGUE SPACE OF THE TORUS

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ABSTRACT. This paper introduces the class of slant Toeplitz operators on the Lebesgue space of the torus. A characterization of these operators as the solutions of an operator equation is obtained. The paper describes various algebraic properties of these operators. The compactness, commutativity and essential commutativity of these operators are also discussed.

1. INTRODUCTION

The study of slant Toeplitz operators emerged with Ho [4], with the study of some elementary properties such as norms, eigen spaces and spectrum together with the discussion of some structural properties of the C*-algebra generated by these operators. If $\phi(\theta) = \sum_{n=-\infty}^{\infty} \phi_n e^{in\theta}$ is an L^∞ function on the unit circle $\mathbb{T}$, an operator S_ϕ on $L^2(\mathbb{T})$ is said to be a slant Toeplitz operator induced by the symbol ϕ if $\langle S_\phi e^{in\theta}, e^{im\theta}\rangle = \phi_{2m-n}$ for all integers n and m. The matrix of S_ϕ with respect to the basis $\{e_n(\theta) = e^{in\theta}\}_{n=-\infty}^{\infty}$ of $L^2(\mathbb{T})$ can be obtained by removing all the odd rows of the corresponding multiplication operator M_ϕ on $L^2(\mathbb{T})$ given by $\langle M_\phi e^{in\theta}, e^{im\theta}\rangle = \phi_{m-n}$.

Slant Toeplitz operators are closely associated with multiplication operators and composition operators. As a matter of fact, each slant Toeplitz operator S_ϕ can be expressed as $S_1 M_\phi$, where M_ϕ denotes the multiplication operator on $L^2(\mathbb{T})$, while the adjoint of a slant Toeplitz operator is a weighted composition operator. Slant Toeplitz operators are linked closely with wavelets, dynamical systems and Ruelle operators. We refer to [4, 5] and the references therein for a detailed study of these operators.

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A lot of progress has taken place in the study of Toeplitz operators on the Hardy space of the bidisk. The semi-commutator of Toeplitz operators on the bidisk is studied in [2], while commuting Hankel and Toeplitz operators on the Hardy space of the bidisk are described in [6]. More recently, a necessary and sufficient condition is obtained for two Toeplitz operators to be commuting on Hardy space of the bidisk [1].

With this paper, we extend in scope the study of slant Toeplitz operators to the usual Lebesgue space of the torus $\mathbb{T}^2$, where $\mathbb{T}^2$ is the Cartesian product of two copies $\mathbb{T}$ and is a subset of $\mathbb{C}^2$ (the symbol $\mathbb{C}$ refers to the complex plane). We denote the space of all complex valued measurable functions satisfying $\int_{\mathbb{T}^2} |f|^2 d\sigma < \infty$, where $d\sigma$ is the normalized Haar measure on $\mathbb{T}^2$, by $L^2(\mathbb{T}^2, d\sigma)$. If there is no confusion about the measure, we simply denote it by $L^2(\mathbb{T}^2)$. $L^2(\mathbb{T}^2)$ is a Hilbert space with norm induced by the inner product

$$\langle f, g\rangle = (\frac{1}{2\pi})^2 \int_0^{2\pi}\int_0^{2\pi} f(e^{i\theta_1}, e^{i\theta_2})\overline{g}(e^{i\theta_1}, e^{i\theta_2})d\theta_1 d\theta_2.$$

The set $\{e_{(m_1,m_2)}(z_1, z_2) = z_1^{m_1} z_2^{m_2} : m_1, m_2 \in \mathbb{Z}\}$ forms an orthonormal basis of $L^2(\mathbb{T}^2)$. The space of essentially bounded measurable functions on $\mathbb{T}^2$ is denoted by $L^\infty(\mathbb{T}^2)$.

We characterize slant Toeplitz operators on $L^2(\mathbb{T}^2)$ as the solutions X of the operator equation $M_{z_1^{i_1} z_2^{i_2}} X = X M_{z_1^{2i_1} z_2^{2i_2}}$, where i_1, i_2 are integers and $M_{z_1^{i_1} z_2^{i_2}}$ and $M_{z_1^{2i_1} z_2^{2i_2}}$ denote the multiplication operators on $L^2(\mathbb{T}^2)$ induced by the symbols $z_1^{i_1} z_2^{i_2}$ and $z_1^{2i_1} z_2^{2i_2}$ in $L^\infty(\mathbb{T}^2)$ respectively. We also study some structural and algebraic properties of these operators.

Throughout the paper, we use the symbols $\mathbb{Z}$ and $\mathbb{Z}_+$ to denote the set of all integers and the set of all non-negative integers respectively. The multiple Fourier series on the torus $\mathbb{T}^2$ can be viewed as the Fourier transformation on $L^1(\mathbb{T}^2)$ (see [2, 6] and references therein). Using multiple Fourier series, we have that

$$L^2(\mathbb{T}^2) = \{f | f(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2} z_1^{m_1} z_2^{m_2} : \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} |f_{m_1,m_2}|^2 < \infty\}.$$

The space $H^2(\mathbb{T}^2)$ is the collection of all those elements $f(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2} z_1^{m_1} z_2^{m_2}$ of $L^2(\mathbb{T}^2)$ for which $f_{m_1,m_2} = 0$, if either of the subscripts is negative. Thus, we obtain that for $f \in H^2(\mathbb{T}^2)$, $f(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}_+\times\mathbb{Z}_+} f_{m_1,m_2} z_1^{m_1} z_2^{m_2}$ and $\|f\|^2 = \sum_{(m_1,m_2)\in\mathbb{Z}_+\times\mathbb{Z}_+} |f_{m_1,m_2}|^2 < \infty$. The symbol $\mathfrak{B}(L^2(\mathbb{T}^2))$ denotes the set of all bounded linear operators on $L^2(\mathbb{T}^2)$, while $\mathcal{R}an(T)$ denotes the range of an operator T. Two operators A and B are said to commute essentially if $AB - BA$ is a compact operator.

2. Some basic properties and characterization of slant Toeplitz operators

For $\phi \in L^\infty(\mathbb{T})$, the slant Toeplitz operator [4] S_ϕ on the Hilbert space $L^2(\mathbb{T})$ is defined as $S_\phi = W M_\phi$, where W is an operator on $L^2(\mathbb{T})$ defined as $We_{2n} = e_n$

and $We_{2n+1} = 0$ for each integer n. The operator W eliminates the odd rows from the matrix of the multiplication operator M_ϕ. In fact, the operator W coincides with S_1, the slant Toeplitz operator on $L^2(\mathbb{T})$ induced by the symbol $\phi(z) = 1$.

Following a similar approach, we define a linear operator E on $L^2(\mathbb{T}^2)$ as

$$Ez_1^{m_1} z_2^{m_2} = \begin{cases} z_1^{\frac{m_1}{2}} z_2^{\frac{m_2}{2}} & \text{if both } m_1 \text{ and } m_2 \text{ are even integers} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the operator E is a bounded operator on $L^2(\mathbb{T}^2)$ with $\|E\| = 1$. The structure of E provides us the following, which can be obtained with simple computations.

Proposition 2.1. *For the operator E on $L^2(\mathbb{T}^2)$ as defined above, we have the following.*

(1) $EE^* = I$ *i.e. E is a co-isometry.*
(2) $E^*E = P_{ee}$, *where P_{ee} is the projection of $L^2(\mathbb{T}^2)$ onto the closed subspace generated by $\{z_1^{2m_1} z_2^{2m_2} : m_1$ and m_2 are integers$\}$.*
(3) $EM_{z_1z_2}E^* = 0$.
(4) *For $f, g \in L^2(\mathbb{T}^2)$ such that $fg \in L^2(\mathbb{T}^2)$, $E(fg) = (Ef)(Eg) + E[((I - P_{ee})f)((I-P_{ee})g)]$. In particular, $E(f(z_1^2, z_2^2)g) = f(Eg)$ and $E(fg(z_1^2, z_2^2)) = g(Ef)$.*

Proposition 2.2. *$\mathcal{R}an(P_2)$ is a reducing subspace of E, where P_2 denotes the orthogonal projection of $L^2(\mathbb{T}^2)$ onto $H^2(\mathbb{T}^2)$.*

Proof. It is easy to see that $P_2E(z_1^{m_1} z_2^{m_2}) = EP_2(z_1^{m_1} z_2^{m_2}) =$

$$\begin{cases} z_1^{\frac{m_1}{2}} z_2^{\frac{m_2}{2}} & \text{if both } m_1 \text{ and } m_2 \text{ are non-negative even integers} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $P_2E = EP_2$. This provides that $\mathcal{R}an(P_2)$ is a reducing subspace of E. $\square$

Next, we compute the adjoint of E and obtain the following.

Proposition 2.3. *The adjoint of E is an operator on $L^2(\mathbb{T}^2)$ given by $E^*z_1^{m_1} z_2^{m_2} = z_1^{2m_1} z_2^{2m_2}$ for each $(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$. In fact, E^* is a composition operator $C_H : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$, where $H : \mathbb{T}^2 \to \mathbb{T}^2$ is given by $H(z_1, z_2) = (z_1^2, z_2^2)$.*

Proof. We have for $(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$,

$$\begin{aligned} \langle E^*(z_1^{m_1} z_2^{m_2}), z_1^{i_1} z_2^{i_2} \rangle &= \langle z_1^{m_1} z_2^{m_2}, E(z_1^{i_1} z_2^{i_2}) \rangle \\ &= \begin{cases} 1 & \text{if } i_1 = 2m_1 \text{ and } i_2 = 2m_2 \\ 0 & \text{otherwise} \end{cases} \\ &= \langle z_1^{2m_1} z_2^{2m_2}, z_1^{i_1} z_2^{i_2} \rangle, \end{aligned}$$

for each $(i_1, i_2) \in \mathbb{Z} \times \mathbb{Z}$. Hence, $E^*z_1^{m_1} z_2^{m_2} = z_1^{2m_1} z_2^{2m_2}$ for each pair $(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$.

Let $f_{m_1,m_2}(z_1, z_2) = z_1^{m_1} z_2^{m_2} \in L^2(\mathbb{T}^2)$. Then, using the definition of a composition operator, we get that $C_H(z_1^{m_1} z_2^{m_2}) = C_H(f_{m_1,m_2}(z_1, z_2)) = (f_{m_1,m_2} oH)(z_1, z_2) = f_{m_1,m_2}\ (z_1^2, z_2^2) = z_1^{2m_1} z_2^{2m_2} = E^*(z_1^{m_1} z_2^{m_2})$, for each $(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$. Since the operators E and C_H agree on all basis elements, we conclude that $E = C_H$. This completes the proof. $\square$

Motivated by the approach of Ho [4], we now proceed to define slant Toeplitz operators on the space $L^2(\mathbb{T}^2)$.

Definition 2.4. Let $\phi \in L^\infty(\mathbb{T}^2)$. The slant Toeplitz operator A_ϕ induced by the symbol ϕ is an operator on $L^2(\mathbb{T}^2)$ defined as $A_\phi f = EM_\phi f$ for each $f \in L^2(\mathbb{T}^2)$, where M_ϕ denotes the multiplication operator on $L^2(\mathbb{T}^2)$ induced by ϕ.

It is trivial to see that the slant Toeplitz operator A_ϕ on $L^2(\mathbb{T}^2)$ induced by the symbol $\phi(z_1, z_2) = 1$ is nothing but the operator E defined above.

The operator A_ϕ is a bounded linear operator on $L^2(\mathbb{T}^2)$ with $\|A_\phi\| \le \|\phi\|_\infty$. Let $\phi(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1} z_2^{m_2}$, then for each $(n_1, n_2) \in \mathbb{Z}\times\mathbb{Z}$, we have that

$$\begin{aligned}
A_\phi(z_1^{n_1} z_2^{n_2}) &= E\Big(\sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1+n_1} z_2^{m_2+n_2}\Big) \\
&= \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1-n_1,m_2-n_2} E(z_1^{m_1} z_2^{m_2}) \\
&= \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{2m_1-n_1,2m_2-n_2} z_1^{m_1} z_2^{m_2}.
\end{aligned}$$

We also obtain

$$\begin{aligned}
\langle A_\phi^*(z_1^{n_1} z_2^{n_2}), z_1^{i_1} z_2^{i_2}\rangle &= \langle z_1^{n_1} z_2^{n_2}, \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{2m_1-i_1,2m_2-i_2} z_1^{m_1} z_2^{m_2}\rangle \\
&= \overline{\phi}_{2n_1-i_1,2n_2-i_2} \\
&= \langle \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \overline{\phi}_{2n_1-m_1,2n_2-m_2} z_1^{m_1} z_2^{m_2}, z_1^{i_1} z_2^{i_2}\rangle,
\end{aligned}$$

for every $(i_1, i_2) \in \mathbb{Z}\times\mathbb{Z}$, so that the adjoint A_ϕ^* of A_ϕ is given by

$$A_\phi^*(z_1^{n_1} z_2^{n_2}) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \overline{\phi}_{2n_1-m_1,2n_2-m_2} z_1^{m_1} z_2^{m_2}.$$

Proposition 2.5. *Let $\phi \in L^\infty(\mathbb{T}^2)$. Then, $A_\phi = 0$ if and only if $\phi = 0$.*

Proof. Nothing needs to be proved in "if" part. For "only if" part, let $A_\phi = 0$, where $\phi(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1} z_2^{m_2}$. Then, for each $(i_1, i_2) \in \mathbb{Z}\times\mathbb{Z}$, $A_\phi(z_1^{i_1} z_2^{i_2}) = 0$. The structure of A_ϕ provides that

$$\sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{2m_1-i_1,2m_2-i_2} z_1^{m_1} z_2^{m_2} = 0.$$

Consequently, we have $\phi_{2m_1-i_1,2m_2-i_2} = 0$ for each $(i_1, i_2), (m_1, m_2) \in \mathbb{Z}\times\mathbb{Z}$. Then, substituting $i_1 = m_1$ and $i_2 = m_2$, we obtain that $\phi_{m_1,m_2} = 0$ for each $(m_1, m_2) \in \mathbb{Z}\times\mathbb{Z}$. Thus we conclude that $\phi = 0$. This completes the proof. $\square$

It is now straightforward to state our next result.

Theorem 2.6. *$\phi \mapsto A_\phi$ is a linear one-one correspondence from $L^\infty(\mathbb{T}^2)$ to $\mathfrak{B}(L^2(\mathbb{T}^2))$.*

It is known that slant Toeplitz operators on $L^2(\mathbb{T})$ are characterized as the solutions X of the operator equation $M_z X = X M_{z^2}$ (see [4]). We aim to characterize slant Toeplitz operators on $L^2(\mathbb{T}^2)$ in a similar fashion. For this purpose, we shall need the following prerequisites.

Proposition 2.7. *For $\phi \in L^\infty(\mathbb{T}^2)$, we have the following for $(i_1, i_2) \in \mathbb{Z} \times \mathbb{Z}$.*

(1) $M_{z_1^{i_1} z_2^{i_2}} E = E M_{z_1^{2i_1} z_2^{2i_2}}$.

(2) $M_{z_1^{2i_1} z_2^{2i_2}} M_\phi = M_\phi M_{z_1^{2i_1} z_2^{2i_2}}$.

Proof. (1) follows since $M_{z_1^{i_1} z_2^{i_2}} E(z_1^{m_1} z_2^{m_2}) = E M_{z_1^{2i_1} z_2^{2i_2}} (z_1^{m_1} z_2^{m_2}) =$

$$\begin{cases} z_1^{\frac{m_1}{2}+i_1} z_2^{\frac{m_2}{2}+i_2} & \text{if both } m_1 \text{ and } m_2 \text{ are even integers} \\ 0 & \text{otherwise.} \end{cases}$$

To prove (2), consider $\phi \in L^\infty(\mathbb{T}^2)$ given as $\phi(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1} z_2^{m_2}$. Then, for each $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$, we find that $M_{z_1^{2i_1} z_2^{2i_2}} M_\phi (z_1^{n_1} z_2^{n_2}) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1+n_1+2i_1} z_2^{m_2+n_2+2i_2} = M_\phi M_{z_1^{2i_1} z_2^{2i_2}} (z_1^{n_1}\ z_2^{n_2})$. This completes the proof. □

We now extend the result of Ho [4] to the slant Toeplitz operators on $L^2(\mathbb{T}^2)$ in the following theorem.

Theorem 2.8. *$A \in \mathfrak{B}(L^2(\mathbb{T}^2))$ is a slant Toeplitz operator if and only if $M_{z_1^{i_1} z_2^{i_2}} A = A M_{z_1^{2i_1} z_2^{2i_2}}$ for all $(i_1, i_2) \in \mathbb{Z} \times \mathbb{Z}$.*

Proof. Let $A = A_\phi$ be a slant Toeplitz operator on $L^2(\mathbb{T}^2)$. Then, $M_{z_1^{i_1} z_2^{i_2}} A_\phi = M_{z_1^{i_1} z_2^{i_2}} E M_\phi = E M_{z_1^{2i_1} z_2^{2i_2}} M_\phi = E M_\phi M_{z_1^{2i_1} z_2^{2i_2}} = A_\phi M_{z_1^{2i_1} z_2^{2i_2}}$. Hence, A satisfies the operator equation $M_{z_1^{i_1} z_2^{i_1}} A = A M_{z_1^{2i_1} z_2^{2i_2}}$.

Conversely, let A be a bounded operator on $L^2(\mathbb{T}^2)$ satisfying $M_{z_1^{i_1} z_2^{i_2}} A = A M_{z_1^{2i_1} z_2^{2i_2}}$. For any element $f(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2}\ z_1^{m_1} z_2^{m_2} \in L^2(\mathbb{T}^2)$, we have

$$\begin{aligned} Af(z_1^2, z_2^2) &= A(\sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2} z_1^{2m_1} z_2^{2m_2}) \\ &= \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2} A(z_1^{2m_1} z_2^{2m_2}) \\ &= \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2} A M_{z_1^{2m_1} z_2^{2m_2}}(1) \\ &= \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2} M_{z_1^{m_1} z_2^{m_2}} A(1) = f(z_1, z_2) A(1). \end{aligned}$$

Working along similar lines, we obtain that $A(z_1 f(z_1^2, z_2^2)) = f(z_1, z_2) A(z_1)$, $A(z_2 f(z_1^2, z_2^2)) = f(z_1, z_2) A(z_2)$ and $A(z_1 z_2 f(z_1^2, z_2^2)) = f(z_1, z_2) A(z_1 z_2)$. We claim that each of the functions $\phi_{00} = A(1), \phi_{10} = A(z_1), \phi_{01} = A(z_2)$ and $\phi_{11} = A(z_1 z_2)$ belongs to $L^\infty(\mathbb{T}^2)$.

Observe that for each $i, j \in \{0, 1\}$,

$$\|\phi_{i,j} f\| = \|A(z_1^i z_2^j f(z_1, z_2))\| \leq \|A\| \|z_1^i z_2^j f(z_1, z_2)\| = \|A\| \|f\|.$$

This implies that each $\phi_{i,j}$ induces a bounded multiplication operator on $L^2(\mathbb{T}^2)$. Thus each $\phi_{i,j} \in L^\infty(\mathbb{T}^2)$ for $i, j \in \{0, 1\}$.

Lastly, using these ϕ_{ij}'s, we construct a ϕ in $L^\infty(\mathbb{T}^2)$ such that $A = A_\phi$. Let $\phi(z_1, z_2) = \phi_{00}(z_1^2, z_2^2) + \overline{z}_1 \phi_{10}(z_1^2, z_2^2) + \overline{z}_2 \phi_{01}(z_1^2, z_2^2) + \overline{z}_1 \overline{z}_2 \phi_{11}(z_1^2, z_2^2)$. Then, $\phi \in L^\infty(\mathbb{T}^2)$. Also, for any f in $L^2(\mathbb{T}^2)$,

$$f(z_1, z_2) = f_{00}(z_1^2, z_2^2) + z_1 f_{10}(z_1^2, z_2^2) + z_2 f_{01}(z_1^2, z_2^2) + z_1 z_2 f_{11}(z_1^2, z_2^2).$$

Clearly, f_{00}, f_{10}, f_{01} and $f_{11} \in L^2(\mathbb{T}^2)$. Consider now

$$\begin{aligned}
A_\phi f(z_1, z_2) &= E M_\phi f(z_1, z_2) \\
&= E(\phi_{00}(z_1^2, z_2^2) f_{00}(z_1^2, z_2^2) + \phi_{10}(z_1^2, z_2^2) f_{10}(z_1^2, z_2^2) + \phi_{01}(z_1^2, \\
&\quad z_2^2) f_{01}(z_1^2, z_2^2) + \phi_{11}(z_1^2, z_2^2) f_{11}(z_1^2, z_2^2) + \text{terms containing} \\
&\quad z_1^i z_2^j \text{ in which either } i \text{ or } j \text{ is odd}) \\
&= \phi_{00}(z_1, z_2) f_{00}(z_1, z_2) + \phi_{10}(z_1, z_2) f_{10}(z_1, z_2) + \phi_{01}(z_1, z_2) \\
&\quad f_{01}(z_1, z_2) + \phi_{11}(z_1, z_2) f_{11}(z_1, z_2) \\
&= f_{00}(z_1, z_2) A(1) + f_{10}(z_1, z_2) A(z_1) + f_{01}(z_1, z_2) A(z_2) + f_{11} \\
&\quad (z_1, z_2) A(z_1 z_2) \\
&= A(f_{00}(z_1^2, z_2^2) + z_1 f_{10}(z_1^2, z_2^2) + z_2 f_{01}(z_1^2, z_2^2) + z_1 z_2 f_{11}(z_1^2, \\
&\quad z_2^2)) \\
&= A f(z_1, z_2).
\end{aligned}$$

Thus $A = A_\phi$ and the proof is complete. □

It is straightforward to observe that $M_{\overline{z}_1^{i_1} \overline{z}_2^{i_2}} = M^*_{z_1^{i_1} z_2^{i_2}} = M^{-1}_{z_1^{i_1} z_2^{i_2}}$ (i.e. $M_{z_1^{i_1} z_2^{i_2}}$ is a unitary operator). Hence, slant Toeplitz operators on $L^2(\mathbb{T}^2)$ can also be characterized in the following manner.

Corollary 2.9. *A bounded operator A on $L^2(\mathbb{T}^2)$ is a slant Toeplitz operator if and only if $A = M_{\overline{z}_1^{i_1} \overline{z}_2^{i_2}} A M_{z_1^{2i_1} z_2^{2i_2}}$ for all $(i_1, i_2) \in \mathbb{Z} \times \mathbb{Z}$.*

Theorem 2.10. *The set of all slant Toeplitz operators on $L^2(\mathbb{T}^2)$ is weakly closed and hence strongly closed.*

Proof. Let $A_n \to A$ weakly, where each A_n is a slant Toeplitz operator on $L^2(\mathbb{T}^2)$. Then, for $f, g \in L^2(\mathbb{T}^2)$, $\langle A_n f, g \rangle \to \langle A f, g \rangle$. This provides that

$$\langle A_n z_1^{2i_1} z_2^{2i_2} f, z_1^{i_1} z_2^{i_2} g \rangle \to \langle A z_1^{2i_1} z_2^{2i_2} f, z_1^{i_1} z_2^{i_2} g \rangle = \langle M_{\overline{z}_1^{i_1} \overline{z}_2^{i_2}} A M_{z_1^{2i_1} z_2^{2i_2}} f, g \rangle.$$

Also, since each A_n is a slant Toeplitz operator, therefore making use of Corollary 2.9, $M_{\overline{z}_1^{i_1} \overline{z}_2^{i_2}} A_n M_{z_1^{2i_1} z_2^{2i_2}} = A_n$. Thus

$$\langle A_n z_1^{2i_1} z_2^{2i_2} f, z_1^{i_1} z_2^{i_2} g \rangle = \langle M_{\overline{z}_1^{i_1} \overline{z}_2^{i_2}} A_n M_{z_1^{2i_1} z_2^{2i_2}} f, g \rangle = \langle A_n f, g \rangle \to \langle A f, g \rangle.$$

Therefore by uniqueness of limits, we obtain that $\langle Af, g\rangle = \langle M_{\bar{z}_1^{i_1}\bar{z}_2^{i_1}} A\, M_{z_1^{2i_1} z_2^{2i_2}} f, g\rangle$ for all $f, g \in L^2(\mathbb{T}^2)$. Thus $A = M_{\bar{z}_1^{i_1}\bar{z}_2^{i_1}} A M_{z_1^{2i_1} z_2^{2i_2}}$. Therefore, A is a slant Toeplitz operator on $L^2(\mathbb{T}^2)$. This completes the proof. $\square$

Theorem 2.11. *For $\phi \in L^\infty(\mathbb{T}^2)$, A_ϕ^* is a slant Toeplitz operator on $L^2(\mathbb{T}^2)$ if and only if $\phi = 0$.*

Proof. If $\phi = 0$, nothing needs to be proved. Conversely, let A_ϕ^* be a slant Toeplitz operator, where $\phi(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1} z_2^{m_2}$. Then, using the characterization of slant Toeplitz operators, we have $M_{z_1z_2} A_\phi^* = A_\phi^* M_{z_1^2 z_2^2}$. Hence, $\langle M_{z_1z_2} A_\phi^* z_1^{n_1} z_2^{n_2}, z_1^{i_1} z_2^{i_2}\rangle = \langle A_\phi^* M_{z_1^2 z_2^2} z_1^{n_1} z_2^{n_2}, z_1^{i_1} z_2^{i_2}\rangle$ for each $(n_1, n_2), (i_1, i_2) \in \mathbb{Z}\times\mathbb{Z}$. The structure of A_ϕ^* provides that

$$\overline{\phi}_{2n_1-i_1+1,2n_2-i_2+1} = \overline{\phi}_{2n_1-i_1+4,2n_2-i_2+4}$$

for each $(n_1, n_2), (i_1, i_2) \in \mathbb{Z}\times\mathbb{Z}$. Once we put $n_1 = 0 = n_2$, we get that $\overline{\phi}_{-i_1+1,-i_2+1} = \overline{\phi}_{-i_1+4,-i_2+4}$ for each $(i_1, i_2) \in \mathbb{Z}\times\mathbb{Z}$. This yields that $\phi_{i_1,i_2} = \phi_{i_1+3m,i_2+3m}$ for each $m \geq 0$ and for each $(i_1, i_2) \in \mathbb{Z}\times\mathbb{Z}$. Since $\phi \in L^\infty(\mathbb{T}^2) \subseteq L^2(\mathbb{T}^2)$, therefore $\sum_{m\in\mathbb{Z}} |\phi_{i_1+3m,i_2+3m}|^2 < \infty$ and hence $\lim_{m\to\infty} |\phi_{i_1+3m,i_2+3m}| = 0$. This helps us to conclude that $\phi = 0$. $\square$

In order to discuss the compactness of slant Toeplitz operators on $L^2(\mathbb{T}^2)$, we first describe compact multiplication operators on $L^2(\mathbb{T}^2)$.

It is known [3] that the only compact multiplication operator on $L^2(\mathbb{T})$ is the zero operator. We obtain the same for multiplication operators on $L^2(\mathbb{T}^2)$ in the following result.

Lemma 2.12. *The multiplication operator M_ϕ on $L^2(\mathbb{T}^2)$, induced by $\phi \in L^\infty(\mathbb{T}^2)$, is compact if and only if the inducing symbol $\phi = 0$.*

Proof. Let $\phi(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1} z_2^{m_2} \in L^\infty(\mathbb{T}^2)$ and M_ϕ be compact. Since compact operators map weakly convergent sequences to strongly convergent ones, we obtain for each $(j_1, j_2) \in \mathbb{Z}\times\mathbb{Z}$ and $n_2 \in \mathbb{Z}$, $|\langle M_\phi z_1^{n_1} z_2^{n_2}, z_1^{n_1+j_1} z_2^{n_2+j_2}\rangle| \leq \|M_\phi z_1^{n_1} z_2^{n_2}\| \to 0$ as $n_1 \to \infty$.

Also, structure of M_ϕ provides that $|\langle M_\phi z_1^{n_1} z_2^{n_2}, z_1^{n_1+j_1} z_2^{n_2+j_2}\rangle| = |\langle \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1+n_1} z_2^{m_2+n_2}, z_1^{n_1+j_1} z_2^{n_2+j_2}\rangle| = |\phi_{j_1,j_2}|$. Therefore, we obtain that $\phi_{j_1,j_2} = 0$ for each $(j_1, j_2) \in \mathbb{Z}\times\mathbb{Z}$. Thus $\phi = 0$.

Converse is straightforward. $\square$

Now, we investigate the existence of compact slant Toeplitz operators on $L^2(\mathbb{T}^2)$ and obtain the following.

Theorem 2.13. *The only compact slant Toeplitz operator on $L^2(\mathbb{T}^2)$ is the zero operator.*

Proof. Let A_ϕ, where $\phi(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1} z_2^{m_2} \in L^\infty(\mathbb{T}^2)$, be compact. Then, each operator $A_\phi M_{z_1^p z_2^q} E^*$, for p and q equal to 0 and 1, is also compact. Also, for $(i_1, i_2) \in \mathbb{Z} \times \mathbb{Z}$ and $p, q \in \{0, 1\}$ we obtain that

$$\begin{aligned} A_\phi M_{z_1^p z_2^q} E^*(z_1^{i_1} z_2^{i_2}) &= A_\phi(z_1^{2i_1+p} z_2^{2i_2+q}) \\ &= E(\sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1+2i_1+p} z_2^{m_2+2i_2+q}) \\ &= \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{2m_1-p,2m_2-q} z_1^{m_1+i_1} z_2^{m_2+i_2} \\ &= M_{\xi_{p,q}}(z_1^{i_1} z_2^{i_2}), \end{aligned}$$

where $\xi_{p,q}(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{2m_1-p,2m_2-q} z_1^{m_1} z_2^{m_2} \in L^\infty(\mathbb{T}^2)$. Hence, the operators $M_{\xi_{p,q}}$, $p, q \in \{0, 1\}$ are compact. Making use of Lemma 2.12, we get that for any $(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$ and for each $p, q \in \{0, 1\}$, $\phi_{2m_1-p,2m_2-q} = 0$. This is turn yields that $\phi = 0$. Hence, $A_\phi = 0$ and the proof is complete. □

3. Algebraic properties

This section aims at the study of some algebraic properties of slant Toeplitz operators on $L^2(\mathbb{T}^2)$. We discuss sums and products of slant Toeplitz operators induced by different symbols. Commutativity and essential commutativity of these operators is also discussed.

Proposition 3.1. *The sum of two slant Toeplitz operators on $L^2(\mathbb{T}^2)$ is a slant Toeplitz operator.*

Proof. Let A_ϕ and A_ψ be two slant Toeplitz operators induced by symbols ϕ and ψ in $L^\infty(\mathbb{T}^2)$ respectively. Then, $M_{z_1^{i_1} z_2^{i_2}}(A_\phi + A_\psi) = M_{z_1^{i_1} z_2^{i_2}} A_\phi + M_{z_1^{i_1} z_2^{i_2}} A_\psi = A_\phi M_{z_1^{2i_1} z_2^{2i_2}} + A_\psi M_{z_1^{2i_1} z_2^{2i_2}} = (A_\phi + A_\psi) M_{z_1^{2i_1} z_2^{2i_2}}$. Hence, the sum $A_\phi + A_\psi$ is a slant Toeplitz operator.

In fact, $A_\phi + A_\psi = EM_\phi + EM_\psi = EM_{\phi+\psi} = A_{\phi+\psi}$, the slant Toeplitz operator on $L^2(\mathbb{T}^2)$ induced by the symbol $\phi + \psi$. □

In Proposition 2.7, we proved that $M_{z_1^{i_1} z_2^{i_1}} E = EM_{z_1^{2i_1} z_2^{2i_2}}$. We extend this result and find that for any general $\phi \in L^\infty(\mathbb{T}^2)$, $M_\phi E = EM_{\phi(z_1^2, z_2^2)}$.

Proposition 3.2. *Let $\phi \in L^\infty(\mathbb{T}^2)$. Then, $M_\phi E = EM_{\phi(z_1^2,z_2^2)}$ $(= A_{\phi(z_1^2,z_2^2)})$.*

Proof. Let $\phi(z_1, z_2) = \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{n_1,n_2} z_1^{n_1} z_2^{n_2} \in L^\infty(\mathbb{T}^2)$. For any pair $(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$, we obtain

$$M_\phi E(z_1^{2m_1} z_2^{2m_2}) = \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{n_1,n_2} z_1^{n_1+m_1} z_2^{n_2+m_2} = EM_{\phi(z_1^2,z_2^2)}(z_1^{2m_1} z_2^{2m_2}).$$

Also, for $(i, j) \in \{(1, 0), (0, 1), (1, 1)\}$,

$$M_\phi E(z_1^{2m_1+i} z_2^{2m_2+j}) = 0 = EM_{\phi(z_1^2,z_2^2)}(z_1^{2m_1+i} z_2^{2m_2+j}).$$

This completes the proof. □

Theorem 3.3. *Let $\phi, \psi \in L^\infty(\mathbb{T}^2)$. Then, $M_\phi A_\psi$ is a slant Toeplitz operator and $M_\phi A_\psi = A_{\phi(z_1^2,z_2^2)\psi}$.*

Proof. We have $M_{z_1^{i_1} z_2^{i_2}}(M_\phi A_\psi) = M_\phi M_{z_1^{i_1} z_2^{i_2}} A_\psi = (M_\phi A_\psi) M_{z_1^{2i_1} z_2^{2i_2}}$. Hence, $M_\phi A_\psi$ is a slant Toeplitz operator on $L^2(\mathbb{T}^2)$.

Also, utilizing Proposition 3.2, we obtain that $M_\phi A_\psi = M_\phi E M_\psi = E M_{\phi(z_1^2, z_2^2)} M_\psi$ $= E M_{\phi(z_1^2, z_2^2)\psi} = A_{\phi(z_1^2, z_2^2)\psi}$. ☐

Theorem 3.4. *For $\phi, \psi \in L^\infty(\mathbb{T}^2)$, $M_\phi A_\psi = A_\psi M_\phi$ if and only if $\phi(z_1^2, z_2^2)\psi$ $= \phi\psi$. If ψ is invertible, then $M_\phi A_\psi = A_\psi M_\phi$ if and only if ϕ is constant.*

Proof. By Theorem 3.3, we have $M_\phi A_\psi = A_{\phi(z_1^2, z_2^2)\psi}$. Also, $A_\psi M_\phi = E M_\psi M_\phi =$ $E M_{\phi\psi} = A_{\phi\psi}$.

Now, "only if" part is obvious, while "if" part follows using the injectivity of the mapping $\phi \mapsto A_\phi$.

Let $\phi(z_1, z_2) = \sum_{(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}} \phi_{m_1, m_2} z_1^{m_1} z_2^{m_2} \in L^\infty(\mathbb{T}^2)$ for $(z_1, z_2) \in \mathbb{T}^2$. Let ψ be invertible, then $M_\phi A_\psi = A_\psi M_\phi$, equivalently $\phi(z_1^2, z_2^2) = \phi$, or $\langle \phi(z_1^2, z_2^2) - \phi(z_1, z_2), z_1^{i_1} z_2^{i_2} \rangle = 0$ for each $(i_1, i_2) \in \mathbb{Z} \times \mathbb{Z}$. But $\langle \phi(z_1^2, z_2^2) - \phi(z_1, z_2), z_1^{i_1} z_2^{i_2} \rangle =$
$$\begin{cases} \phi_{\frac{i_1}{2}, \frac{i_2}{2}} - \phi_{i_1, i_2} & \text{if both } i_1, i_2 \text{ are even} \\ -\phi_{i_1, i_2} & \text{otherwise.} \end{cases}$$
Therefore, we obtain that $\phi_{i_1, i_2} = 0$ for each $(0,0) \neq (i_1, i_2) \in \mathbb{Z} \times \mathbb{Z}$ i.e. $\phi(z_1, z_2) =$ $\phi_{00}($ a constant$)$.

Hence, for an invertible ψ, $M_\phi A_\psi = A_\psi M_\phi$ if and only if ϕ is a constant function. ☐

It is evident from Proposition 3.2 that the operators E and M_ϕ do not commute in general. However, as a consequence of the above theorem, we find that for a constant ϕ, E and M_ϕ commute.

Corollary 3.5. *For $\phi \in L^\infty(\mathbb{T}^2)$, $E M_\phi = M_\phi E$ if and only if ϕ is constant.*

We now focus our attention towards the product $A_\phi A_\psi$ of two slant Toeplitz operators and obtain condition(s) for this product to be a slant Toeplitz operator. For, we first observe the following.

Proposition 3.6. *For $\phi \in L^\infty(\mathbb{T}^2)$, $E A_\phi$ is a slant Toeplitz operator if and only if $\phi = 0$.*

Proof. Let, if possible, $E A_\phi = A_\xi$ for some $\xi(z_1, z_2) = \sum_{(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}} \xi_{m_1, m_2}$ $z_1^{m_1} z_2^{m_2} \in L^\infty(\mathbb{T}^2)$. Let $\phi(z_1, z_2) = \sum_{(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}} \phi_{m_1, m_2} z_1^{m_1} z_2^{m_2} \in L^\infty(\mathbb{T}^2)$. Then, for each $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$, we have $E A_\phi(z_1^{n_1} z_2^{n_2}) = A_\xi(z_1^{n_1} z_2^{n_2})$. This provides that $\phi_{4m_1 - n_1, 4m_2 - n_2} = \xi_{2m_1 - n_1, 2m_2 - n_2}$ for each $m_1, m_2, n_1, n_2 \in \mathbb{Z}$. For $m_1 = m_2 = 0$, we get that $\phi_{n_1, n_2} = \xi_{n_1, n_2}$ for each $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$.

Hence, for each $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$, we obtain that $\phi_{2 - n_1, 2 - n_2} = \phi_{4 - n_1, 4 - n_2} =$ $\phi_{8 - n_1, 8 - n_2} = \cdots$. Since $\lim\limits_{m \to \infty} |\phi_{2^m - n_1, 2^m - n_2}| = 0$, so $\phi_{2 - n_1, 2 - n_2} = 0$ for each $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$. This provides that $\phi = 0$.

Converse is trivial. ☐

Theorem 3.7. *The operator $A_\phi A_\psi$ is a slant Toeplitz operator if and only if $\phi(z_1^2, z_2^2)\psi = 0$.*

Proof. By the definition of slant Toeplitz operator and using Proposition 3.2, we obtain that $A_\phi A_\psi = E(M_\phi E)M_\psi = E(EM_{\phi(z_1^2,z_2^2)})M_\psi = E(EM_{\phi(z_1^2,z_2^2)\psi}) = EA_{\phi(z_1^2,z_2^2)\psi}$. The result now follows using Proposition 3.6. □

Next, we discuss the compactness of the product $A_\phi A_\psi$ of two slant Toeplitz operators A_ϕ and A_ψ. We shall require the following.

Lemma 3.8. *Let $\phi \in L^\infty(\mathbb{T}^2)$. Then, we have the following.*

(1) *$A_1 A_\phi = 0$ if and only if $\phi = 0$.*
(2) *$A_1 A_\phi$ is compact if and only if $\phi = 0$.*

Proof. Let $\phi(z_1, z_2) = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{m_1,m_2} z_1^{m_1} z_2^{m_2} \in L^\infty(\mathbb{T}^2)$. For (1), let $A_1A_\phi = 0$. But, $A_1A_\phi = EM_1EM_\phi = E^2M_\phi$. Hence, we have

$$\begin{aligned}\langle A_1A_\phi z_1^{n_1}z_2^{n_2}, z_1^{i_1}z_2^{i_2}\rangle &= \langle E^2M_\phi z_1^{n_1}z_2^{n_2}, z_1^{i_1}z_2^{i_2}\rangle\\ &= \langle A_\phi z_1^{n_1}z_2^{n_2}, z_1^{2i_1}z_2^{2i_2}\rangle\\ &= \langle \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \phi_{2m_1-n_1,2m_2-n_2} z_1^{m_1}z_2^{m_2}, z_1^{2i_1}z_2^{2i_2}\rangle\\ &= \phi_{4i_1-n_1,4i_2-n_2}.\end{aligned}$$

Thus we obtain that $\phi_{4i_1-n_1,4i_2-n_2} = 0$ for each $(i_1,i_2),(n_1,n_2) \in \mathbb{Z}\times\mathbb{Z}$. Substituting $n_1 = 3i_1$ and $n_2 = 3i_2$, we obtain that $\phi_{i_1,i_2} = 0$ for each $(i_1,i_2) \in \mathbb{Z}\times\mathbb{Z}$. Thus $\phi = 0$.

Converse is obvious.

For (2), let A_1A_ϕ be compact. Then, we get that for each $p,q \in \{0,1,2,3\}$, the operator $E^2(A_1A_\phi M_{z_1^p z_2^q})^*$ is compact. Now for the case $p = q = 0$, we have

$$\begin{aligned}E^2(A_1A_\phi M_{z_1^0z_2^0})^* z_1^{m_1}z_2^{m_2} = E^2M_\phi^* z_1^{4m_1}z_2^{4m_2} &= E^2(\overline{\phi}(z_1z_2)z_1^{4m_1}z_2^{4m_2})\\ &= z_1^{m_1}z_2^{m_2}E^2(\overline{\phi}(z_1,z_2))\\ &= M_{E^2(\overline{\phi}(z_1,z_2))}(z_1^{m_1}z_2^{m_2}),\end{aligned}$$

for each $(m_1,m_2) \in \mathbb{Z}\times\mathbb{Z}$. Hence, $E^2(A_1A_\phi)^* = M_{E^2(\overline{\phi}(z_1,z_2))}$. Now, the compactness of $E^2(A_1A_\phi)^*$ provides that $M_{E^2(\overline{\phi}(z_1,z_2))}$ is a compact operator and hence $E^2(\overline{\phi}(z_1,z_2)) = 0$. This provides that

$$\begin{aligned}0 &= \langle E^2(\overline{\phi}(z_1,z_2)), z_1^{i_1}z_2^{i_2}\rangle\\ &= \langle \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} \overline{\phi}_{m_1,m_2} z_1^{-m_1}z_2^{-m_2}, z_1^{4i_1}z_2^{4i_2}\rangle = \overline{\phi}_{-4i_1,-4i_2},\end{aligned}$$

for each $(i_1,i_2) \in \mathbb{Z}\times\mathbb{Z}$.

Applying similar computations, we get that $E^2(A_1A_\phi M_{z_1^pz_2^q})^* = M_{E^2(\overline{z}_1^p\overline{z}_2^q\overline{\phi})}$ and the compactness of these operators helps to provide that $\overline{\phi}_{-4i_1-p,-4i_2-q} = 0$ for $(0,0) \neq (p,q) \in \{0,1,2,3\}\times\{0,1,2,3\}$.

Thus $\phi_{m_1,m_2} = 0$ for each $(m_1,m_2) \in \mathbb{Z}\times\mathbb{Z}$ and hence $\phi = 0$.

Nothing needs to be proved in the converse part. □

Proposition 3.9. *The product $A_\phi A_\psi$ of two slant Toeplitz operators is 0 if and only if $\phi(z_1^2, z_2^2)\psi = 0$.*

Proof. Since $A_\phi A_\psi = EA_{\phi(z_1^2,z_2^2)\psi}$, Lemma 3.8 together with the fact that $E = A_1$ helps to provide that $A_\phi A_\psi = 0$ if and only if $\phi(z_1^2, z_2^2)\psi = 0$. $\square$

We are now in a position to state the following result. It follows simply by using together Theorem 3.7, Lemma 3.8 and Proposition 3.9.

Theorem 3.10. *Let $\phi, \psi \in L^\infty(\mathbb{T}^2)$. Then, the following are equivalent:*

(1) *$A_\phi A_\psi$ is a compact operator.*
(2) *$A_\phi A_\psi$ is a slant Toeplitz operator.*
(3) *$A_\phi A_\psi = 0$.*
(4) *$\phi(z_1^2, z_2^2)\psi = 0$.*

Remark 3.11. The following are now easy to obtain:

(1) A non-zero slant Toeplitz operator on $L^2(\mathbb{T}^2)$ can not be idempotent.
(2) The product of two slant Toeplitz operators on $L^2(\mathbb{T}^2)$ can not be a non-zero slant Toeplitz operator on $L^2(\mathbb{T}^2)$.

Lastly, we obtain that the notions of commutativity and essential commutativity coincide for slant Toeplitz operators on $L^2(\mathbb{T}^2)$.

Theorem 3.12. *The following are equivalent:*

(1) *A_ϕ and A_ψ commute.*
(2) *A_ϕ and A_ψ essentially commute.*
(3) *$\phi(z_1^2, z_2^2)\psi - \psi(z_1^2, z_2^2)\phi = 0$.*

Proof. We shall first prove that (1) and (3) are equivalent. We obtain that $A_\phi A_\psi = A_\psi A_\phi$ if and only if $EA_{\phi(z_1^2,z_2^2)\psi} = EA_{\psi(z_1^2,z_2^2)\phi}$. This is equivalent to $E(A_{\phi(z_1^2,z_2^2)\psi-\psi(z_1^2,z_2^2)\phi}) = 0$.

Utilizing Lemma 3.8, we obtain that $A_\phi A_\psi = A_\psi A_\phi$ if and only if $\phi(z_1^2, z_2^2)\psi$ $-\psi(z_1^2, z_2^2)\phi = 0$.

The equivalence of (2) and (3) follows using Lemma 3.8 since $A_\phi A_\psi - A_\psi A_\phi = E(A_{\phi(z_1^2,z_2^2)\psi-\psi(z_1^2,z_2^2)\phi})$ and $E = A_1$. $\square$

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