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ON STARLIKENESS, CONVEXITY, AND CLOSE-TO-CONVEXITY OF HYPER-BESSEL FUNCTION

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Abstract. In the present investigation, our main aim is to derive some conditions on starlikeness, convexity, and close-to-convexity of normalized hyper-Bessel functions. Also we give some similar results for classical Bessel functions by using the relationships between hyper-Bessel and Bessel functions. As a result of the obtained conditions, some examples are also given.

1. Introduction and preliminaries

Bessel and related functions are frequently used in engineering and applied sciences. For this reason, they have a long history in mathematical studies. Most of mathematicians have investigated some properties of Bessel and related functions in different directions. Some of the most important properties of these functions are geometric properties like univalence, starlikeness, convexity, and close-to convexity. In 1960, first studies on the univalence of Bessel function have been done by Brown in [20], while Kreyszig and Todd determined the radius of univalence of Bessel functions in [25]. In 1984, De Branges has solved famous Bieberbach conjecture by using hyper geometric functions. After this solution, geometric properties of some special functions have become very attractive since hyper geometric series and Bessel type special functions are closely related. As a result, most of mathematicians have begun to study on geometric properties of special functions like Bessel, Struve, Lommel, Mittag–Leffler, Wright, and their some extensions. Some of the obtained geometric properties of above mentioned functions can be found in [1, 3–10, 12–14, 16–19]. In fact, the authors have used

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some properties of zeros of the mentioned functions to investigate their geometric properties. For the properties of zeros of some special functions, one can refer to papers [11, 15, 21, 23, 24, 27, 28] and the references therein.

The Bessel function is defined by the following infinite series:

$$J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \quad z \in \mathbb{C}, \tag{1.1}$$

where $\Gamma(z)$ denotes the familiar gamma function. In the literature, there are many investigations on Bessel and related functions. For example, some geometric properties of Lommel functions have been studied by using basic concepts of geometric function theory in [29]. In addition, some geometric properties of hyper-Bessel function have been investigated in papers [1, 2, 5]. Motivated by some earlier works, in this study, our main aim is to obtain some new geometric properties of hyper-Bessel functions.

Now, we would like to remind the definition of hyper-Bessel function. The hyper-Bessel function is defined by (see [22])

$$J_{\gamma_d}(z) = \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{d+1}\right)^{n(d+1)+\gamma_1+\cdots+\gamma_d}}{n!\Gamma(\gamma_1+n+1)\ldots\Gamma(\gamma_d+n+1)}. \tag{1.2}$$

Now, we are going to remind some basic definitions in geometric function theory and give a lemma, which will be used in order to prove our main results.

Let $\mathbb{D}_r$ be the open disk $\{z \in \mathbb{C} : |z| < r\}$ with radius $r > 0$ and $\mathbb{D}_1 = \mathbb{D}$. Let $\mathcal{A}$ denote the class of analytic functions $f : \mathbb{D}_r \to \mathbb{C}$,by

$$f(z) = z + \sum_{n\geq 2} a_n z^n,$$

which satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. By $\mathcal{S}$ we mean the class of functions belonging to $\mathcal{A}$, which are univalent in $\mathbb{D}_r$. Also, for $0 \leq \alpha < 1$, by $\mathcal{S}^\star(\alpha)$, $\mathcal{C}(\alpha)$, and $\mathcal{K}(\alpha)$ we denote the subclasses of $\mathcal{A}$ consisting of functions which are starlike, convex, and close-to convex of order α, respectively. The analytic characterizations of these subclasses are

$$\mathcal{S}^\star(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \text{ for } z \in \mathbb{D} \right\},$$

$$\mathcal{C}(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \text{ for } z \in \mathbb{D} \right\},$$

and

$$\mathcal{K}(\alpha) = \left\{ f : f \in \mathcal{A}, \Re\left(\frac{f'(z)}{g'(z)}\right) > \alpha \text{ for } z \in \mathbb{D} \text{ and } g \in \mathcal{C}(\alpha) \right\},$$

respectively.

The following result, which is given by Owa et al. [26, p. 67, Corollary 2], will be required in order to prove the close-to-convexity of the function $f_{\gamma_d}(z)$.

Lemma 1.1. *If the function* $f \in \mathcal{A}$ *satisfies the inequality*

$$|zf''(z)| < \frac{1-\alpha}{4} \qquad (z \in \mathbb{D}, 0 \leq \alpha < 1),$$

then

$$\Re\left(f'(z)\right) > \frac{1+\alpha}{2} \qquad (z \in \mathbb{D}, 0 \leq \alpha < 1).$$

Since the hyper-Bessel function $z \mapsto J_{\gamma_d}$, which is given by (1.2), does not belong to the class $\mathcal{A}$, we first perform a natural normalization. The normalized hyper-Bessel function $\mathcal{J}_{\gamma_d}(z)$ is defined by

$$J_{\gamma_d}(z) = \frac{\left(\frac{z}{d+1}\right)^{\gamma_1+\cdots+\gamma_d}}{\Gamma(\gamma_1+1)\ldots\Gamma(\gamma_d+1)}\mathcal{J}_{\gamma_d}(z). \tag{1.3}$$

By combining equalities (1.2) and (1.3), we get the following infinite series representation:

$$\mathcal{J}_{\gamma_d}(z) = \sum_{n\geq 0} \frac{(-1)^n\left(\frac{z}{d+1}\right)^{n(d+1)}}{n!(\gamma_1+1)_n\ldots(\gamma_d+1)_n},$$

where $(\beta)_n$ is the known Pochhammer symbol and it is defined by $(\beta)_0 = 1$ and

$$(\beta)_n = \beta(\beta+1)\ldots(\beta+n-1) = \frac{\Gamma(\beta+n)}{\Gamma(\beta)}$$

for $n \geq 1$. As a result, we have that the function

$$f_{\gamma_d}(z) = z\mathcal{J}_{\gamma_d}(z) = z + \sum_{n\geq 1} \frac{(-D)^n}{n!(\gamma_1+1)_n\ldots(\gamma_d+1)_n} z^{n(d+1)+1}$$

is in the class $\mathcal{A}$, where $D = \frac{1}{(d+1)^{d+1}}$.

We would like to remind here that the following well-known inequalities

$$(\beta)^n \leq (\beta)_n \tag{1.4}$$

and

$$2^{n-1} \leq n! \tag{1.5}$$

are true for $n \in \{1, 2, \ldots\}$. Also, we are going to use the following well-known triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2| \qquad (z_1, z_2 \in \mathbb{C}) \tag{1.6}$$

and reverse triangle inequality

$$|z_1 - z_2| \geq ||z_1| - |z_2|| \qquad (z_1, z_2 \in \mathbb{C}) \tag{1.7}$$

in order to prove our assertions. In addition, the following geometric series sums

$$\sum_{n\geq 1} r^{n-1} = \frac{1}{1-r} \qquad (|r| < 1), \tag{1.8}$$

$$\sum_{n\geq 1} nr^{n-1} = \frac{1}{(1-r)^2} \qquad (|r| < 1), \tag{1.9}$$

and

$$\sum_{n\geq 1} n^2 r^{n-1} = \frac{1+r}{(1-r)^3} \qquad (|r|<1) \tag{1.10}$$

will be used to prove our results.

2. Main results

In this section, we present our main results.

Theorem 2.1. *Let* $\alpha \in [0,1)$*,* $\kappa_1 = \prod_{i=1}^{d}(\gamma_i+1)$*, and* $\kappa_2 = \prod_{i=1}^{d}(\gamma_i+2) > 0$*. If*

$$\frac{4\kappa_2}{(d+1)^d(2\kappa_2 - D)\left[\kappa_1(2\kappa_2-D) - 2D\kappa_2\right]} < 1-\alpha,$$

then for all $z \in \mathbb{D}$ *the hyper-Bessel function* $f_{\gamma_d}(z)$ *is starlike of order* α*.*

Proof. In order to prove the starlikeness of order α of the function $z \mapsto f_{\gamma_d}$, it is enough to show that the inequality $\left|\frac{zf'_{\gamma_d}(z)}{f_{\gamma_d}(z)} - 1\right| < 1-\alpha$ holds true for $\alpha \in [0,1)$ and $z \in \mathbb{D}$. By using the infinite series representation of the function $z \mapsto f_{\gamma_d}$, the identity $(\beta)_n = \beta(\beta+1)_{n-1}$ and the inequalities, which are given by (1.4), (1.5), and (1.6), we can write that

$$\begin{aligned}
\left|f'_{\gamma_d}(z) - \frac{f_{\gamma_d}(z)}{z}\right| &= \left|\sum_{n\geq 1} \frac{n(d+1)(-D)^n}{n!\prod_{i=1}^{d}(\gamma_i+1)_n} z^{n(d+1)}\right| \\
&= \frac{d+1}{\kappa_1}\left|\sum_{n\geq 1} \frac{n(-D)^n}{n!\prod_{i=1}^{d}(\gamma_i+2)_{n-1}} z^{n(d+1)}\right| \\
&\leq \frac{(d+1)D}{\kappa_1}\left|\sum_{n\geq 1} \frac{n(-D)^{n-1}}{2^{n-1}\prod_{i=1}^{d}(\gamma_i+2)^{n-1}}\right| \\
&= \frac{(d+1)D}{\kappa_1}\sum_{n\geq 1} n\left(\frac{D}{2\kappa_2}\right)^{n-1}.
\end{aligned}$$

Now, using the known geometric series sum, which is given by (1.9), for $\left|\frac{D}{2\kappa_2}\right| < 1$, we get

$$\left|f'_{\gamma_d}(z) - \frac{f_{\gamma_d}(z)}{z}\right| \leq \frac{4{\kappa_2}^2}{(d+1)^d\kappa_1(2\kappa_2-D)^2}. \tag{2.1}$$

In addition, using the reverse triangle inequality which is given by (1.7) implies that

$$\begin{aligned}
\left|\frac{f_{\gamma_d}(z)}{z}\right| &= \left|1 + \sum_{n\geq 1} \frac{(-D)^n}{n!\prod_{i=1}^{d}(\gamma_i+1)_n} z^{n(d+1)}\right| \\
&\geq 1 - \frac{D}{\kappa_1}\sum_{n\geq 1}\left(\frac{D}{2\kappa_2}\right)^{n-1}.
\end{aligned}$$

For $\left|\frac{D}{2\kappa_2}\right| < 1$, we obtain that

$$\left|\frac{f_{\gamma_d}(z)}{z}\right| \geq \frac{\kappa_1(2\kappa_2 - D) - 2D\kappa_2}{\kappa_1(2\kappa_2 - D)}. \tag{2.2}$$

By considering inequalities (2.1) with (2.2), we have that

$$\left|\frac{zf'_{\gamma_d}(z)}{f_{\gamma_d}(z)} - 1\right| \leq \frac{4\kappa_2}{(d+1)^d\,(2\kappa_2 - D)\left[\kappa_1\,(2\kappa_2 - D) - 2D\kappa_2\right]}.$$

Thus, the function $z \mapsto f_{\gamma_d}$ is starlike of order α under the assumption. □

Theorem 2.2. *Let $\alpha \in [0,1)$, $\kappa_1 = \prod_{i=1}^{d}(\gamma_i + 1)$, and $\kappa_2 = \prod_{i=1}^{d}(\gamma_i + 2) > 0$. If*

$$\frac{4(d+1)D{\kappa_2}^2[2\kappa_2(d+2) + dD]}{(2\kappa_2 - D)\left\{\kappa_1(2\kappa_2 - D)^2 - 2\kappa_2 D[2\kappa_2\,(d+2) - D]\right\}} < 1 - \alpha,$$

then for all $z \in \mathbb{D}$, the hyper-Bessel function $f_{\gamma_d}(z)$ is convex of order α.

Proof. For the convexity of order α of the function $f_{\gamma_d}(z)$, it is enough to show that the inequality $\left|\frac{zf''_{\gamma_d}(z)}{f'_{\gamma_d}(z)}\right| < 1 - \alpha$ holds true for $\alpha \in [0,1)$ and $z \in \mathbb{D}$. From the infinite series representation of the function $f_{\gamma_d}(z)$ and the inequalities, which are given by (1.4), (1.5), and (1.6), we can write that

$$\begin{aligned}
\left|zf''_{\gamma_d}(z)\right| &= \left|\sum_{n\geq 1}\frac{n(d+1)[n(d+1)+1](-D)^n}{n!\prod_{i=1}^{d}(\gamma_i+1)_n}z^{n(d+1)}\right| \\
&= \left|\sum_{n\geq 1}\frac{n^2(d+1)^2(-D)^n}{n!\prod_{i=1}^{d}(\gamma_i+1)_n}z^{n(d+1)} + \sum_{n\geq 1}\frac{n(d+1)(-D)^n}{n!\prod_{i=1}^{d}(\gamma_i+1)_n}z^{n(d+1)}\right| \\
&\leq \frac{(d+1)^2 D}{\kappa_1}\sum_{n\geq 1}n^2\frac{D^{n-1}}{n!\prod_{i=1}^{d}(\gamma_i+2)_{n-1}} \\
&\quad + \frac{(d+1)D}{\kappa_1}\sum_{n\geq 1}n\frac{D^{n-1}}{n!\prod_{i=1}^{d}(\gamma_i+2)_{n-1}} \\
&\leq \frac{(d+1)^2 D}{\kappa_1}\sum_{n\geq 1}n^2\left(\frac{D}{2\kappa_2}\right)^{n-1} + \frac{(d+1)D}{\kappa_1}\sum_{n\geq 1}n\left(\frac{D}{2\kappa_2}\right)^{n-1}.
\end{aligned}$$

Now, using the known geometric series sums which are given by (1.9) and (1.10) for $\left|\frac{D}{2\kappa_2}\right| < 1$, we have

$$\left|zf''_{\gamma_d}(z)\right| \leq \frac{4(d+1)D\kappa_2^2[\kappa_2(2d+4) + dD]}{\kappa_1(2\kappa_2 - D)^3}. \tag{2.3}$$

From the inequalities, which are given by (1.4), (1.5), and (1.7), it can be easily seen that

$$\begin{aligned}\left|f'_{\gamma_d}(z)\right| &= \left|1+\sum_{n\geq 1}\frac{[n(d+1)+1](-D)^n}{n!\prod_{i=1}^{d}(\gamma_i+1)_n}z^{n(d+1)}\right| \\ &\geq 1-\sum_{n\geq 1}\frac{[n(d+1)+1]D^n}{n!\prod_{i=1}^{d}(\gamma_i+1)_n} \\ &= 1-\left[\frac{(d+1)D}{\kappa_1}\sum_{n\geq 1}n\frac{D^{n-1}}{n!\prod_{i=1}^{d}(\gamma_i+2)_{n-1}}+\frac{D}{\kappa_1}\sum_{n\geq 1}\frac{D^{n-1}}{n!\prod_{i=1}^{d}(\gamma_i+2)_{n-1}}\right] \\ &\geq 1-\left[\frac{(d+1)D}{\kappa_1}\sum_{n\geq 1}n\left(\frac{D}{2\kappa_2}\right)^{n-1}+\frac{D}{\kappa_1}\sum_{n\geq 1}\left(\frac{D}{2\kappa_2}\right)^{n-1}\right].\end{aligned}$$

Now, by making use of the geometric series sums which are given (1.8) and (1.9), we get

$$\left|f'_{\gamma_d}(z)\right| \geq \frac{\kappa_1(2\kappa_2-D)^2-2\kappa_2 D(\kappa_2(2d+4)-D)}{\kappa_1(2\kappa_2-D)^2} \tag{2.4}$$

for $\left|\frac{D}{2\kappa_2}\right|<1$. Finally, if we consider inequality (2.3) with (2.4), then we have

$$\left|\frac{zf''_{\gamma_d}(z)}{f'_{\gamma_d}(z)}\right| \leq \frac{4(d+1)D{\kappa_2}^2[2\kappa_2(d+2)+dD]}{(2\kappa_2-D)\left\{\kappa_1(2\kappa_2-D)^2-2\kappa_2 D[2\kappa_2\,(d+2)-D]\right\}}.$$

As a consequence, the proof is completed. □

Theorem 2.3. *Let $\alpha\in[0,1)$, $\kappa_1=\prod_{i=1}^{d}(\gamma_i+1)$, and $\kappa_2=\prod_{i=1}^{d}(\gamma_i+2)>0$. If*

$$\frac{16(d+1)D{\kappa_2}^2[2\kappa_2(d+2)+dD]}{\kappa_1(2\kappa_2-D)^3}<1-\alpha,$$

then for all $z\in\mathbb{D}$ the hyper-Bessel function $f_{\gamma_d}(z)$ is close-to-convex of order $\frac{1+\alpha}{2}$ and so $\Re\left(f'_{\gamma_d}(z)\right)>\frac{1+\alpha}{2}$.

Proof. It is known from inequality (2.3) that

$$\left|zf''_{\gamma_d}(z)\right| \leq \frac{4(d+1)D\kappa_2^2[\kappa_2(2d+4)+dD]}{\kappa_1(2\kappa_2-D)^3}$$

for all $z\in\mathbb{D}$. By using Lemma(1.1), it is clear that

$$\left|zf''_{\gamma_d}(z)\right| < \frac{1-\alpha}{4}$$

for

$$0\leq\alpha<1-\frac{16(d+1)D{\kappa_2}^2[2\kappa_2(d+2)+dD]}{\kappa_1(2\kappa_2-D)^3}.$$

This implies that the hyper-Bessel function $f_{\gamma_d}(z)$ is close-to-convex of order $\frac{1+\alpha}{2}$ and so $\Re\left(f'_{\gamma_d}(z)\right)>\frac{1+\alpha}{2}$. □

It is important to mention here that there is a close relationship between hyper-Bessel functions and classical Bessel functions. More precisely, by putting $d = 1$ and $\gamma_1 = \nu$ in expressions (1.2), we have the classical Bessel function which is given by (1.1). By considering this close relationship in our main theorems, we have the following results.

Corollary 2.4. *Let* $\alpha \in [0,1)$ *and* $z \in \mathbb{D}$. *If* $\frac{8(\nu+2)}{(8\nu+15)(8\nu^2+21\nu+11)} < 1-\alpha$, *then the function* $f_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu} J_\nu(z)$ *is starlike of order* α.

Corollary 2.5. *Let* $\alpha \in [0,1)$ *and* $z \in \mathbb{D}$. *If* $\frac{32(\nu+2)^2(24\nu+49)}{(8\nu+15)(64\nu^3+256\nu^2+275\nu+37)} < 1-\alpha$, *then the function* $f_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu} J_\nu(z)$ *is convex of order* α.

Corollary 2.6. *Let* $\alpha \in [0,1)$ *and* $z \in \mathbb{D}$. *If* $\frac{128(\nu+2)^2(24\nu+49)}{(\nu+1)(8\nu+15)^3} < 1-\alpha$, *then the function* $f_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu} J_\nu(z)$ *is close-to-convex of order* $\frac{1+\alpha}{2}$.

3. Applications

It is well-known from [9, p. 13–14] that the basic trigonometric functions can be represented by the classical Bessel function J_ν for appropriate values of the parameter ν. Clearly, for $\nu = -\frac{1}{2}$, $\nu = \frac{1}{2}$ and $\nu = \frac{3}{2}$, respectively, we have the following equalities:

$$J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi z}} \cos z,\; J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi z}} \sin z \text{ and } J_{\frac{3}{2}} = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z \right).$$

By considering the above special cases, some examples can be given.

Example 3.1. Let $\alpha \in [0,1)$ and $z \in \mathbb{D}$. The following assertions hold true:

i. For $\alpha < \alpha_1 \cong 0.564$, the function $f_{-\frac{1}{2}}(z) = z\cos z$ is starlike of order α.

ii. For $\alpha < \alpha_2 \cong 0.955$, the function $f_{\frac{1}{2}}(z) = \sin z$ is starlike of order α.

iii. For $\alpha < \alpha_3 \cong 0.983$, the function $f_{\frac{3}{2}}(z) = \frac{3}{z^2}(\sin z - z\cos z)$ is starlike of order α.

iv. For $\alpha < \alpha_4 \cong 0.005$, the function $f_{\frac{3}{2}}(z) = \frac{3}{z^2}(\sin z - z\cos z)$ is convex of order α.

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