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STRONGLY QUASILINEAR PARABOLIC SYSTEMS IN DIVERGENCE FORM WITH WEAK MONOTONICITY

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Abstract. The existence of solutions to the strongly quasilinear parabolic system

$$\frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,u,Du) + g(x,t,u,Du) = f,$$

is proved, where the source term f is assumed to belong to $L^{p'}(0,T; W^{-1,p'}(\Omega;\mathbb{R}^m))$. Further, we prove the existence of a weak solution by means of the Young measures under mild monotonicity assumptions on σ.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$. By ∂Q we denote the boundary of Q and $\mathbb{M}^{m\times n}$ the real vector space of $m \times n$ matrices equipped with the inner product $A : B = A_{i,j}B_{i,j}$ (with conventional summation). Consider first the quasilinear parabolic initial-boundary value system

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,u,Du) &= f \text{ in } Q, \\ u(x,t) &= 0 \text{ on } \partial Q, \\ u(x,0) &= u_0(x) \text{ in } \Omega, \end{aligned} \tag{1.1}$$

where $u : Q \to \mathbb{R}^m$. In (1.1) the right hand side f belongs to $L^{p'}(0,T; W^{-1,p'}(\Omega;\mathbb{R}^m))$ for some $p \in (1,\infty)$. In [13], Young introduced Young measure as a powerful tool

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to describe the weak limit of sequences. N. Hungerbühler [8] obtained the existence of a weak solution for (1.1) by using the concept of Young measures. The author assumed weak monotonicity assumptions on σ.

If $A(u) = -\operatorname{div}\sigma(x,t,u,Du)$, $u : Q \to \mathbb{R}$ and A is a classical operator of the Leray-Lions type with respect to the Sobolev space $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ for some $1 < p < \infty$, then the existence of solutions for (1.1) was proved in [3, 10, 11, 12]. The authors required the strict monotonicity or monotonicity in the variables $(u,F) \in \Omega \times \mathbb{R}^n$. Nevertheless, we will not use the previous type of monotonicity.

In this paper, we will be using the Young measures and Galerkin method to prove the existence result for the following strongly quasilinear parabolic system

$$\frac{\partial u}{\partial t} - \operatorname{div}\sigma(x,t,u,Du) + g(x,t,u,Du) = f \text{ in } Q, \tag{1.2}$$

$$u(x,t) = 0 \text{ on } \partial Q, \tag{1.3}$$

$$u(x,0) = u_0(x) \text{ in } \Omega. \tag{1.4}$$

The problem (1.2)-(1.4) can be seen as a more general form of (1.1), where $g : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$. Similar problems to (1.2)-(1.4) were studied, we refer the reader [1, 4, 6].

This article is organized as follows: in Section 2, we present our assumptions and main result. Section 3 is a brief review of Young measures. Section 4 deals with the Galerkin approximations and necessary *a priori* estimates. Section 5 concerns the identification of weak limits by means of Young measures, while Section 6 is devoted to the proof of the main result.

2. Assumptions and main results

Let Ω be a bounded open subset of $\mathbb{R}^n$ and set $Q = \Omega \times (0,T)$ for $T > 0$. Throughout this paper, we denote $Q_\tau = \Omega \times (0,\tau)$ for every $\tau \in [0,T]$. Consider the problem (1.2)–(1.4), where $\sigma : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ and $g : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ satisfy the following assumptions:

(H0) σ and g are Carathéodory functions (i.e., measurable w.r.t $(x,t) \in Q$ and continuous w.r.t other variables).

(H1) There exist $c_1 \geq 0$, $\beta > 0$, $d_1 \in L^{p'}(Q)$ and $d_2 \in L^1(Q)$ such that

$$|\sigma(x,t,u,A)| \leq d_1(x,t) + c_1\big(|u|^{p-1} + |A|^{p-1}\big),$$

$$\sigma(x,t,u,A) : A + g(x,t,u,A).A \geq -d_2(x,t) + \beta|A|^p.$$

(H2) σ satisfies one of the following conditions:

(a) For all $(x,t) \in Q$, $A \mapsto \sigma(x,t,u,A)$ is a C^1-function and is monotone, that is, for all $(x,t) \in Q$, $u \in \mathbb{R}^m$ and $A, B \in \mathbb{M}^{m\times n}$, we have

$$\big(\sigma(x,t,u,A) - \sigma(x,t,u,B)\big) : (A - B) \geq 0.$$

(b) There exists a function $W : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,t,u,A) = \frac{\partial W}{\partial A}(x,t,u,A)$ and $A \to W(x,t,u,A)$ is convex and C^1 for all $(x,t) \in Q$ and $u \in \mathbb{R}^m$.

(c) σ is strictly monotone, that is, σ is monotone and

$$\big(\sigma(x,t,u,A)-\sigma(x,t,u,B)\big):(A-B)=0 \Rightarrow A=B.$$

(d)

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,u,\lambda)-\sigma(x,t,u,\overline{\lambda})\big):(\lambda-\overline{\lambda})d\nu(\lambda)dxdt>0,$$

where $\overline{\lambda}=\langle \nu_{(x,t)}, id\rangle$ and $\nu=\{\nu_{(x,t)}\}_{(x,t)\in Q}$ is any family of Young measures generated by a bounded sequence in $L^p(Q)$ and not a Dirac measure for a.e. $(x,t)\in Q$.

(H3) g satisfies one of the following conditions:

(i) There exist $c_2 \geq 0$ and $d_2 \in L^{p'}(Q)$ such that

$$|g(x,t,u,A)| \leq d_2(x,t)+c_2\big(|u|^{p-1}+|A|^{p-1}\big).$$

(ii) The function g is independent of the fourth variable, or, for a.e. $(x,t)\in Q$ and all $u\in\mathbb{R}^m$, the mapping $A\to g(x,t,u,A)$ is linear.

Remark 2.1. Assumptions (H1) and (H3)(i) state standard growth and coercivity conditions. The assumption (H1)(b) allows to take a potential $W(x,t,u,A)$ which is only convex but not strictly convex in $A\in\mathbb{M}^{m\times n}$ and to consider (1.2) with $\sigma(x,t,u,A)=\frac{\partial W}{\partial A}(x,t,u,A)$. Note that if W is assumed to be strictly convex, then σ becomes strict monotone. Thus, the standard method may apply. Finally, (H2)(d) states the notion of strict p-quasimonotone in terms of gradient Young measures.

We shall prove the following existence theorem.

Theorem 2.2. *Suppose that the conditions* (H0)–(H1) *are satisfied. Let* $u_0 \in L^2(\Omega;\mathbb{R}^m)$ *and* $f\in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ *be given. Then*

(1) *if* σ *satisfies one of the conditions* (H2)(a) *or* (b)*, then for every* g *satisfying* (H3)(ii)*, the system* (1.2)–(1.4) *has a weak solution.*
(2) *if* σ *satisfies one of the conditions* (H2)(c) *or* (d)*, then for each* g *satisfying* (H3)(i)*, the system* (1.2)–(1.4) *has a weak solution.*

Remark 2.3. A simple model of our problem is as follows:

$$\frac{\partial u}{\partial t}-\operatorname{div}(|Du|^{p-2}Du)+|u|^{p-2}u=f \text{ in } Q,$$
$$u(x,t)=0 \text{ on } \partial Q,$$
$$u(x,0)=u_0(x) \text{ in } \Omega.$$

For the potential W, one can take $W:=\frac{1}{p}|A|^p$.

3. A review of Young measures

In the following, $\mathcal{C}_0(\mathbb{R}^m)$ denotes the closure of the space of continuous functions on $\mathbb{R}^m$ with compact support with respect to the $\|.\|_\infty$-norm. Its dual space can be identified with $\mathcal{M}(\mathbb{R}^m)$, the space of signed Radon measures with finite mass. The related duality pairing is given by

$$\langle \nu, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu(\lambda).$$

Note that, as $id(\lambda) = \lambda$ then $\langle \nu, id \rangle = \int_{\mathbb{R}^m} \lambda d\nu(\lambda)$.

Definition 3.1. Assume that the sequence $\{w_j\}_{j\geq 1}$ is bounded in $L^\infty(\Omega; \mathbb{R}^m)$. Then there exist a subsequence $\{w_k\}_k$ and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such that for almost each $\varphi \in \mathcal{C}(\mathbb{R}^m)$ we have

$$\varphi(w_k) \rightharpoonup^* \overline{\varphi} \text{ weakly in } L^\infty(\Omega),$$

where

$$\overline{\varphi}(x) = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu_x(\lambda).$$

We call $\nu = \{\nu_x\}_{x\in\Omega}$ the family of Young measures associated with the subsequences $\{w_k\}_k$.

The fundamental theorem on Young measures may be stated in the following lemma.

Lemma 3.2 ([5])**.** *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable (not necessarily bounded) and let $w_j : \Omega \to \mathbb{R}^m$, $j = 1, 2, \ldots$ be a sequence of Lebesgue measurable functions. Then there exist a subsequence w_k and a family $\{\nu_x\}_{x\in\Omega}$ of nonnegative Radon measures on $\mathbb{R}^m$, such that*

(i) *$\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} := \int_{\mathbb{R}^m} d\nu_x \leq 1$ for almost $x \in \Omega$.*
(ii) *$\varphi(w_k) \rightharpoonup^* \overline{\varphi}$ weakly in $L^\infty(\Omega)$ for all $\varphi \in \mathcal{C}_0(\mathbb{R}^m)$, where $\overline{\varphi}(x) = \langle \nu_x, \varphi \rangle$.*
(iii) *If*

$$\lim_{L\to\infty} \sup_k \left| \{x \in \Omega \cap B_R(0) : \ |w_k(x)| \geq L\} \right| = 0 \qquad (3.1)$$

for all $R > 0$, then $\|\nu_x\| = 1$ for a.e. $x \in \Omega$, and for all measurable $\Omega' \subset \Omega$, there holds $\varphi(w_k) \rightharpoonup \overline{\varphi} = \langle \nu_x, \varphi \rangle$ weakly in $L^1(\Omega')$ for a continuous function φ provided the sequence $\varphi(w_k)$ is weakly precompact in $L^1(\Omega')$.

The following lemmas are considered as the applications of the fundamental theorem on Young measures (Lemma 3.2), which will be needed in what follows.

Lemma 3.3 ([7])**.** *If $|\Omega| < \infty$ and ν_x is the Young measure generated by the (whole) sequence w_j, then there holds*

$$w_j \to w \text{ in measure } \Leftrightarrow \ \nu_x = \delta_{w(x)} \quad \text{for a.e. } x \in \Omega.$$

Lemma 3.4 ([7])**.** *Let $F : \Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and let $u_k : \Omega \to \mathbb{R}^m$ be a sequence of measurable functions such that $u_k \to u$ in measure and such that Du_k generates the Young measure ν_x, with $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for almost every $x \in \Omega$. Then*

$$\liminf_{k\to\infty} \int_{\Omega'} F(x, u_k(x), Du_k(x))dx \geq \int_{\Omega'} \int_{\mathbb{M}^{m\times n}} F(x, u, \lambda) d\nu_x(\lambda) dx,$$

provided that the negative part $F^-(x, u_k(x), Du_k(x))$ is equi-integrable.

Remark 3.5. (1) Lemma 3.4 is a Fatou-type inequality.

(2) Under condition (3.1), it was proved [2] that for any measurable $\Omega' \subset \Omega$,

$$\varphi(., u_k) \rightharpoonup \langle \nu_x, \varphi(x, .)\rangle \quad \text{in } L^1(\Omega'),$$

for every Carathéodory function $\varphi : \Omega' \times \mathbb{R}^m \to \mathbb{R}$ such that $\{\varphi(., u_k)\}$ is sequentially weakly relative compact in $L^1(\Omega')$. Further, the author showed that if u_k generates the Young measure ν_x, then for $\varphi \in L^1(\Omega; \mathcal{C}_0(\mathbb{R}^m))$ we have

$$\lim_{k\to\infty} \int_\Omega \varphi(x, u_k(x))dx = \int_\Omega \int_{\mathbb{R}^m} \varphi(x, \lambda)d\nu_x(\lambda)dx.$$

4. Galerkin approximation

We choose an $L^2(\Omega; \mathbb{R}^m)$-orthonormal base $\{w_i\}_{i\geq 1}$ such that

$$\{w_i\}_{i\geq 1} \subset \mathcal{C}_0^\infty(\Omega; \mathbb{R}^m), \quad \mathcal{C}_0^\infty(\Omega; \mathbb{R}^m) \subset \overline{\bigcup_{k\geq 1} V_k}^{\mathcal{C}^1(\overline{\Omega}; \mathbb{R}^m)},$$

where $V_k = \operatorname{span}\{w_1, \dots, w_k\}$. Define the following approach for approximating solutions of (1.2)–(1.4):

$$u_k(x, t) = \sum_{i=1}^k \alpha_{ki}(t) w_i(x), \tag{4.1}$$

where $\alpha_{ki} : [0, T] \to \mathbb{R}$ are measurable bounded functions. Assume that $u_k \in L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R}^m))$. Thus u_k satisfies the boundary condition (1.3) by construction. For the initial condition (1.4), one can choose the initial coefficients $\alpha_{ki}(0) := (u_0, w_i)_{L^2}$, with $(.,.)$ denotes the inner product of L^2, such that

$$u_k(., 0) = \sum_{i=1}^k \alpha_{ki}(0) w_i(.) \to u_0 \quad \text{in } L^2(\Omega)$$

as $k \to \infty$. To complete the construction of u_k, it remains to determine the coefficients $\alpha_{ki}(t)$. For this, let $k \in \mathbb{N}$ be fixed (for the moment), $0 < \tau < T$ and $I = [0, \tau]$. Furthermore, we choose $r > 0$ large enough, such that the set $B_r(0) := B(0, r) \subset \mathbb{R}^k$ contains the vectors $(\alpha_{1k}(0), \dots, \alpha_{kk}(0))$. Consider the function

$$\Theta : I \times \overline{B_r(0)} \to \mathbb{R}^k$$

$$(t, \alpha_1, \dots, \alpha_k) \mapsto \Big(\langle f(t), w_j \rangle - \int_\Omega \sigma(x, t, \sum_{i=1}^k \alpha_i w_i, \sum_{i=1}^k \alpha_i D w_i) : D w_j dx$$
$$- \int_\Omega g(x, t, \sum_{i=1}^k \alpha_i w_i, \sum_{i=1}^k \alpha_i D w_i).w_j dx \Big)_{j=1,\dots,k},$$

where $\langle ., . \rangle$ denotes the dual pairing of $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. The operator Θ is a Carathéodory function by the condition (H0). Next, we will estimate

Θ_j. By using the conditions (H1) and (H3)(i), one gets together with the Hölder inequality

$$\begin{aligned} &\Big|\int_\Omega \sigma(x,t,\sum_{i=1}^k \alpha_i w_i, \sum_{i=1}^k \alpha_i Dw_i) : Dw_j dx\Big| \\ &\quad\leq \Big(\int_\Omega \big|\sigma(x,t,\sum_{i=1}^k \alpha_i w_i, \sum_{i=1}^k \alpha_i Dw_i)\big|^{p'} dx\Big)^{\frac{1}{p'}} \Big(\int_\Omega |Dw_j|^p dx\Big)^{\frac{1}{p}} \\ &\quad\leq c\int_\Omega d_1(x,t)dx + c, \end{aligned} \tag{4.2}$$

and

$$\Big|\int_\Omega g(x,t,\sum_{i=1}^k \alpha_i w_i, \sum_{i=1}^k \alpha_i Dw_i).w_j dx\Big| \leq c\int_\Omega d_2(x,t)dx + c, \tag{4.3}$$

where c depends on k and r but not on t.

Note that (4.2) and (4.3) are obtained by the following arguments: firstly, we have $W_0^{s,2}(\Omega) \subset W_0^{1,p}(\Omega)$ for $s \geq 1+n(\frac{1}{2}-\frac{1}{p})$, secondly $Dw_j \in W^{s-1,2}(\Omega) \subset L^\infty(\Omega)$ for $w_j \in W^{s,2}(\Omega)$. For the first term in the definition of Θ, we have

$$\big|\langle f(t), w_j\rangle\big| \leq \|f(t)\|_{-1,p'}\|w_j\|_{1,p}.$$

As a consequence, the j^{th} term of Θ can be estimated as follows:

$$|\Theta_j(t,\alpha_1,\dots,\alpha_k)| \leq c(r,k)b(t) \tag{4.4}$$

uniformly on $I \times \overline{B_r(0)}$, where $c(r,k)$ is a constant, which depends on r and k, and where $b(t) \in L^1(I)$ does not depend on r and k. Thus, the Carathéodory existence result on ordinary differential equations (cf. Kamke [9]) applied to the system

$$\begin{cases} \alpha_j'(t) = \Theta_j(t,\alpha_1(t),\dots,\alpha_k(t)), \\ \alpha_j(0) = \alpha_{kj}(0), \end{cases} \tag{4.5}$$

(for $j \in \{1,\dots,k\}$) ensures the existence of a distributional, continuous solution α_j (depending on k) of (4.5) on a time interval $[0,\tau')$, where $\tau' > 0$, a priori, may depend on k. Furthermore, the corresponding integral equation

$$\alpha_j(t) = \alpha_j(0) + \int_0^t \Theta_j(t,\alpha_1(s),\dots,\alpha_k(s))ds$$

holds on $[0,\tau')$. Hence

$$u_k(x,t) = \sum_{i=1}^k \alpha_{ki}(t)w_i(x)$$

is the desired solution to the system of ordinary differential equations

$$\begin{aligned} \Big(\frac{\partial u_k}{\partial t}, w_j\Big)_{L^2} + \int_\Omega \sigma(x,t,u_k,Du_k) : Dw_j dx + \int_\Omega g(x,t,u_k,Du_k).w_j dx \\ = \langle f(t), w_j\rangle, \end{aligned} \tag{4.6}$$

with the initial condition $u_k(.,0) = \sum_{i=1}^{k} \alpha_{ki}(0) w_i(.) \to u_0$ in $L^2(\Omega)$ as $k \to \infty$. Now, we will extend the local solution defined on $[0, \tau')$ to a global one. For this, we multiply each side of (4.6) by $\alpha_{kj}(t)$ and we sum. This gives for an arbitrary time $\tau \in [0, T)$

$$\int_{Q_\tau} \frac{\partial u_k}{\partial t} u_k dxdt + \int_{Q_\tau} \big(\sigma(x,t,u_k,Du_k) : Du_k + g(x,t,u_k,Du_k).u_k\big) dxdt = \int_0^\tau \langle f(t), u_k \rangle dt,$$

which is denoted as $I_1 + I_2 = I_3$. By integrating and (H1), we have

$$I_1 = \frac{1}{2} \|u_k(.,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2} \|u_k(.,0)\|^2_{L^2(\Omega)},$$

and

$$I_2 \geq -\int_{Q_\tau} d_2(x,t) dxdt + \beta \int_{Q_\tau} |Du_k|^p dxdt.$$

By Hölder's inequality

$$|I_3| \leq \|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \|u_k\|_{L^p(0,T;W_0^{1,p}(\Omega))}.$$

From the estimations on I_ϵ, $\epsilon = 1,2,3$, we deduce

$$\big|\big(\alpha_{ki}(\tau)\big)_{i=1,\dots,k}\big|^2_{\mathbb{R}^k} = \|u_k(.,\tau)\|^2_{L^2(\Omega)} \leq \bar{c},$$

where $\bar{c}$ is a constant independent of τ (and of k).

Consider now

$$M := \big\{ t \in [0,T) : \text{there exists a weak solution of (4.5) on } [0,t) \big\}.$$

We have M is nonempty, because it contains a local solution. Moreover, thanks to [8], we then have M is an open set which is also closed. Thus $M = [0, T)$.

From the estimations on I_ϵ, $\epsilon = 1,2,3$, we conclude that the sequence $(u_k)_k$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$. Therefore, by extracting a suitable subsequence (still denoted by $(u_k)_k$), we may assume

$$u_k \rightharpoonup u \text{ in } L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)), \tag{4.7}$$

$$u_k \rightharpoonup^* u \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m)). \tag{4.8}$$

The function $u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$ is a candidate to be a weak solution for the problem (1.2)–(1.4). Using the growth condition in (H1) and (H3), together with (4.7), we can extract a suitable subsequence of $\big\{-\operatorname{div}\sigma(x,t,u_k,Du_k)\big\}$ and $\big\{g(x,t,u_k,Du_k)\big\}$ such that

$$-\operatorname{div}\sigma(x,t,u_k,Du_k) \rightharpoonup \chi \text{ in } L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)) \tag{4.9}$$

and

$$g(x,t,u_k,Du_k) \rightharpoonup \xi \text{ in } L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)), \tag{4.10}$$

where $\chi, \xi \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$.

Since $(u_k)_k$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, there exists a subsequence, which is again denoted by $(u_k)_k$, such that

$$u_k(.,T) \rightharpoonup z \text{ in } L^2(\Omega;\mathbb{R}^m) \text{ as } k \to \infty.$$

We will prove that $z = u(.,T)$ and $u(.,0) = u_0(.)$. For simplicity, we denote $u(.,T)$ as $u(T)$ and $u(.,0)$ as $u(0)$. For every $\phi \in \mathcal{C}^\infty([0,T])$ and $v \in V_j$, $j \leq k$, we have

$$\int_Q \frac{\partial u_k}{\partial t} v\phi dxdt + \int_Q \sigma(x,t,u_k,Du_k) : Dv\phi dxdt + \int_Q g(x,t,u_k,Du_k).v\phi dxdt = \int_Q f.v\phi dxdt.$$

After integrating, one gets

$$\int_\Omega u_k(T)\phi(T)vdx - \int_\Omega u_k(0)\phi(0)vdx = \int_Q f.v\phi dxdt - \int_Q \sigma(x,t,u_k,Du_k) : Dv\phi dxdt - \int_Q g(x,t,u_k,Du_k).v\phi dxdt + \int_Q u_k v\phi' dxdt.$$

We pass to the limit as $k \to \infty$ in the previous equality

$$\int_\Omega z\phi(T)vdx - \int_\Omega u_0\phi(0)vdx = \int_Q f.v\phi dxdt - \int_Q \chi.v\phi dxdt - \int_Q \xi.v\phi dxdt + \int_Q uv\phi' dxdt.$$

Let $\phi(0) = \phi(T) = 0$. Then

$$-\int_Q \chi.v\phi dxdt - \int_Q \xi.v\phi dxdt + \int_Q f.v\phi dxdt = -\int_Q uv\phi' dxdt = \int_Q \phi vu' dxdt.$$

Consequently, we obtain

$$\begin{aligned}\int_\Omega z\phi(T)vdx - \int_\Omega u_0\phi(0)vdx &= \int_Q \phi vu' dxdt + \int_Q uv\phi' dxdt \\ &= \int_\Omega u\phi vdx\Big|_0^T \\ &= \int_\Omega u(T)\phi(T)vdx - \int_\Omega u(0)\phi(0)vdx.\end{aligned}$$

If we take $\phi(T) = 0$ and $\phi(0) = 1$, then we have $u(0) = u_0$; if $\phi(T) = 1$ and $\phi(0) = 0$, then $u(T) = z$.

The principal difficulty will be to identify χ with $-\operatorname{div}\sigma(x,t,u,Du)$ and ξ with $g(x,t,u,Du)$.

5. Identification of weak limits by means of Young measures

The Young measure is a device that comes to overcome the difficulty that may arises when weak convergence does not behave as one desire with respect to nonlinear functionals and operators. The following lemma describes limit points of gradient sequences of approximating solutions.

Lemma 5.1. *If $(Du_k)_k$ is bounded in $L^p(0,T;L^p(\Omega))$, then (Du_k) can generates the Young measure $\nu_{(x,t)}$ which satisfy $\|\nu_{(x,t)}\| = 1$, and there is a subsequence of (Du_k) weakly convergent to $\int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda)$ in $L^1(0,T;L^1(\Omega;\mathbb{R}^m))$.*

Proof. To prove the first part of Lemma 5.1, it is sufficient to show that (Du_k) satisfies equation the (3.1). Since (Du_k) is bounded, it follows that there exists $c \geq 0$ such that

$$c \geq \int_Q |Du_k|^p dxdt \geq \int_{\{(x,t):\,|Du_k(x,t)|\geq L\}} |Du_k|^p dxdt \geq L^p \big|\{(x,t) :\, |Du_k(x,t)| \geq L\}\big|.$$

Thus

$$\sup_{k\in\mathbb{N}} \big|\{(x,t) :\, |Du_k(x,t)| \geq L\}\big| \leq \frac{c}{L^p} \to 0, \text{ as } L \to \infty.$$

According to Lemma 3.2(iii), $\|\nu_{(x,t)}\| = 1$.

For the remaining part, the reflexivity of $L^p(0,T;L^p(\Omega))$ implies the existence of a subsequence (still denoted by (Du_k)) weakly convergent in $L^p(0,T;L^p(\Omega))$, thus weakly convergent in $L^1(0,T;L^1(\Omega))$. By Lemma 3.2(iii) and by taking φ as the identity mapping id, it result that

$$Du_k \rightharpoonup \langle \nu_{(x,t)}, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) \text{ weakly in } L^1(0,T;L^1(\Omega)).$$

□

Lemma 5.2. *For almost every $(x,t) \in Q$, $\nu_{(x,t)}$ satisfies the following identification*

$$\langle \nu_{(x,t)}, id\rangle = Du(x,t).$$

Proof. Since $u_k \rightharpoonup u$ in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ and $u_k \to u$ in $L^p(0,T;L^p(\Omega))$, we have

$$Du_k \rightharpoonup Du \text{ in } L^p(0,T;L^p(\Omega)).$$

Moreover, $Du_k \rightharpoonup Du$ in $L^1(0,T;L^1(\Omega))$ (up to a subsequence). By virtue of Lemma 5.1, we can infer that

$$Du(x,t) = \langle \nu_{(x,t)}, id\rangle \text{ for a.e. } (x,t) \in Q.$$

□

The following lemma, namely div-curl inequality, is the key ingredient to pass to the limit in the approximating equations and to prove that the weak limit u of the Galerkin approximations u_k is indeed a solution of (1.2)–(1.4).

Lemma 5.3. *The Young measure $\nu_{(x,t)}$ generated by the gradients Du_k of the Galerkin approximations u_k has the following property:*

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : (\lambda - Du) d\nu_{(x,t)}(\lambda) dxdt \leq 0.$$

Proof. Let us consider the sequence

$$\begin{aligned} J_k &:= \big(\sigma(x,t,u_k,Du_k)-\sigma(x,t,u,Du)\big):(Du_k-Du)\\ &= \sigma(x,t,u_k,Du_k):(Du_k-Du)-\sigma(x,t,u,Du):(Du_k-Du)\\ &=: J_{k,1}+J_{k,2}. \end{aligned}$$

We have by the growth condition (H1) that

$$\int_Q |\sigma(x,t,u,Du)|^{p'}dxdt \le c\int_Q \big(|d_1(x,t)|^{p'}+|u|^p+|Du|^p\big)dxdt,$$

and since $u\in L^p(0,T;W_0^{1,p}(\Omega))$ we obtain $\sigma\in L^{p'}(0,T;W^{-1,p'}(\Omega))$. By virtue of Lemma 5.1, it follows that

$$J_{k,2}\rightharpoonup \sigma(x,t,u,Du):\Big(\int_{\mathbb{M}^{m\times n}}\lambda d\nu_{(x,t)}(\lambda)-Du\Big),$$

which gives by Lemma 5.2 that $J_{k,2}\to 0$ as $k\to\infty$.
Since (u_k) is bounded, then $u_k\rightharpoonup u$ in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ and in measure on Q. It follows from the equi-integrability of $\sigma(x,t,u_k,Du_k)$ and Lemma 3.4, that

$$\begin{aligned} J:=\liminf_{k\to\infty}\int_Q J_k dxdt &= \liminf_{k\to\infty}\int_Q J_{k,1}dxdt\\ &\ge \int_Q\int_{\mathbb{M}^{m\times n}}\sigma(x,t,u,\lambda):(\lambda-Du)d\nu_{(x,t)}(\lambda)dxdt. \end{aligned}\tag{5.1}$$

To get the result, it is sufficient to prove that $J\le 0$. On the one hand, we have

$$\begin{aligned} \liminf_{k\to\infty} -\int_Q \sigma(x,t,u_k,Du_k):Dudxdt &= -\int_0^T\langle\chi,u\rangle dt\\ &= \frac12\|u(.,T)\|_{L^2}^2-\frac12\|u(.,0)\|_{L^2}^2-\int_0^T\langle f,u\rangle dt+\int_Q \xi.udxdt, \end{aligned}\tag{5.2}$$

where we have used the following energy equality related to χ and ξ:

$$\frac12\|u(.,s)\|_{L^2}^2+\int_0^s\langle\chi,u\rangle dt+\int_0^s\langle\xi,u\rangle dt=\int_0^s\langle f,u\rangle dt+\frac12\|u(.,0)\|_{L^2}^2$$

for all $s\in[0,T]$. On the other hand, by the Galerkin equations

$$\begin{aligned} &\int_Q \sigma(x,t,u_k,Du_k):Du_kdxdt\\ &\qquad = \int_0^T\langle f,u_k\rangle dt-\int_Q\frac{\partial u_k}{\partial t}u_kdt-\int_Q g(x,t,u_k,Du_k).u_kdxdt. \end{aligned}$$

We pass to the limit inf in the last equation and using the fact that $u_k(.,0)\to u_0(x)=u(x,0)$ and $u_k(.,T)\rightharpoonup u(.,T)$ in $L^2(\Omega;\mathbb{R}^m)$, we get

$$\begin{aligned} &\liminf_{k\to\infty}\int_Q\sigma(x,t,u_k,Du_k):Du_kdxdt\\ &\qquad \le \int_0^T\langle f,u\rangle dt-\frac12\|u(.,T)\|_{L^2}^2+\frac12\|u(.,0)\|_{L^2}^2-\int_Q\xi.udxdt. \end{aligned}\tag{5.3}$$

Due to (5.2) and (5.3)

$$J = \liminf_{k\to\infty} \int_Q \sigma(x,t,u_k,Du_k) : (Du_k - Du)dxdt \leq 0.$$

According to Lemma 5.2

$$\int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,Du) : (\lambda - Du)d\nu_{(x,t)}(\lambda)dxdt = 0.$$

This together with (5.1) imply the needed result. ☐

Lemma 5.4. *Suppose that $\nu_{(x,t)}$ satisfies the inequality of Lemma 5.3. Then for a.e. $(x,t) \in Q$ we have*

$$\big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : (\lambda - Du) = 0 \quad \textit{on} \quad \operatorname{supp} \nu_{(x,t)}.$$

Proof. We have by Lemma 5.3, that

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : (\lambda - Du)d\nu_{(x,t)}(\lambda)dxdt \leq 0.$$

The above integral is nonnegative, and this is according to the monotonicity assumption of σ. Hence, it must vanish almost everywhere with respect to the product measure $d\nu_{(x,t)}(\lambda) \otimes dx \otimes dt$. Consequently, for almost all $(x,t) \in Q$

$$\big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : (\lambda - Du) = 0 \quad \text{on} \quad \operatorname{supp} \nu_{(x,t)}.$$

☐

6. Proof of the main result

In this section, we give the proof of Theorem 2.2 based on the two cases listed in. We start with the case (2) where we have supposed that σ satisfies the condition (c) or (d).

Note that, in these cases, we will prove that we may extract a subsequence with the property

$$Du_k \to Du \quad \text{in measure on} \quad Q. \tag{6.1}$$

Case (c): By strict monotonicity, it follows from Lemma 5.4 that $\operatorname{supp} \nu_{(x,t)} = \{Du(x,t)\}$, thus $\nu_{(x,t)} = \delta_{Du(x,t)}$ for a.e. $(x,t) \in Q$.

Case (d): Suppose that $\nu_{(x,t)}$ is not a Dirac measure on a set $(x,t) \in Q' \subset Q$ of positive Lebesgue measure $|Q'| > 0$. Since $\|\nu_{(x,t)}\| = 1$ and $Du(x,t) = \langle \nu_{(x,t)}, id\rangle = \overline{\lambda}$, it follows from the strict p-quasimonotone that

$$\begin{aligned} 0 &< \int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,\overline{\lambda})\big) : (\lambda - \overline{\lambda})d\nu_{(x,t)}(\lambda)dxdt \\ &= \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : (\lambda - \overline{\lambda})d\nu_{(x,t)}(\lambda)dxdt. \end{aligned}$$

Hence

$$\int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda)dxdt > \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : Du\, d\nu_{(x,t)}(\lambda)dxdt.$$

From Lemma 5.3 and the above inequality, we get

$$\int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : Du\, d\nu_{(x,t)}(\lambda)dxdt \geq \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda)dxdt$$
$$> \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : Du\, d\nu_{(x,t)}(\lambda)dxdt$$

which is a contradiction. Hence $\nu_{(x,t)}$ is a Dirac measure. Assume that $\nu_{(x,t)} = \delta_{h(x,t)}$. Then

$$h(x,t) = \int_{\mathbb{M}^{m\times n}} \lambda d\delta_{h(x,t)}(\lambda) = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) = Du(x,t).$$

Thus $\nu_{(x,t)} = \delta_{Du(x,t)}$.

To complete the proof of this part, we argue as follows: we have $\nu_{(x,t)} = \delta_{Du(x,t)}$ for a.e. $(x,t) \in Q$. Then by Lemma 3.3 $Du_k \to Du$ in measure on Q as $k \to \infty$, and thus $\sigma(x,t,u_k,Du_k) \to \sigma(x,t,u,Du)$ and $g(x,t,u_k,Du_k) \to g(x,t,u,Du)$ almost everywhere on Q (up to extraction of a further subsequence). Since by (H1) and (H3)(i) the sequences $\sigma(x,t,u_k,Du_k)$ and $g(x,t,u_k,Du_k)$ are bounded. It follows that $\sigma(x,t,u_k,Du_k) \to \sigma(x,t,u,Du)$ and $g(x,t,u_k,Du_k) \to g(x,t,u,Du)$ in $L^\beta(Q)$, for all $\beta \in [1,p')$ by the Vitali convergence theorem. It then follows that

$$-\operatorname{div}\sigma(x,t,u_k,Du_k) \rightharpoonup \chi = -\operatorname{div}\sigma(x,t,u,Du) \tag{6.2}$$

and

$$g(x,t,u_k,Du_k) \rightharpoonup \xi = g(x,t,u,Du). \tag{6.3}$$

These properties are sufficient to pass to the limit in the Galerkin equations and to conclude the proof of the part (2) of Theorem 2.2.

For the remaining part (i.e., the first part) of Theorem 2.2, we note that the property (6.1) does not hold (in general), but we will obtain $\sigma(x,t,u_k,Du_k) \rightharpoonup \sigma(x,t,u,Du)$ and $g(x,t,u_k,Du_k) \rightharpoonup g(x,t,u,Du)$ in $L^{p'}(Q)$. To do this, we need the convergence in measure of the sequence u_k. Since $(u_k)_k$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$, we have then $u_k \rightharpoonup u$ in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ and in measure on Q as $k \to \infty$.

Case (a): We prove that for a.e. $(x,t) \in Q$ and every $\mu \in \mathbb{M}^{m\times n}$ the following equation holds on $\operatorname{supp}\nu_{(x,t)}$

$$\sigma(x,t,u,\lambda) : \mu = \sigma(x,t,u,Du) : \mu + \big(\nabla\sigma(x,t,u,Du)\big) : (\lambda - Du), \tag{6.4}$$

where ∇ denotes the derivative with respect to the third variable of σ. Due to the monotonicity of σ, we have for all $\tau \in \mathbb{R}$

$$\begin{aligned} 0 &\leq \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du+\tau\mu)\big) : (\lambda - Du - \tau\lambda) \\ &= \sigma(x,t,u,\lambda) : (\lambda - Du) - \sigma(x,t,u,\lambda) : \tau\mu - \sigma(x,t,u,Du+\tau\mu) : (\lambda - Du - \tau\mu) \\ &= \sigma(x,t,u,Du) : (\lambda - Du) - \sigma(x,t,u,\lambda) : \tau\mu - \sigma(x,t,u,Du+\tau\mu) : (\lambda - Du - \tau\mu), \end{aligned}$$

by Lemma 5.4. Hence

$$-\sigma(x,t,u,\lambda) : \tau\mu \geq -\sigma(x,t,u,Du) : (\lambda - Du) + \sigma(x,t,u,Du+\tau\mu) : (\lambda - Du - \tau\mu).$$

Using the fact that

$$\sigma(x,t,u,Du+\tau\mu)=\sigma(x,t,u,Du)+\nabla\sigma(x,t,u,Du)\tau\mu+o(\tau)$$

to deduce

$$-\sigma(x,t,u,\lambda):\tau\mu\geq\tau\Big(\big(\nabla\sigma(x,t,u,Du)\mu\big):(\lambda-Du)-\sigma(x,t,u,Du):\mu\Big)+o(\tau).$$

Since the sign of τ is arbitrary in $\mathbb{R}$, the above inequality implies (6.4). On the other hand, the equiintegrability of $\sigma(x,t,u_k,Du_k)$ implies that its weak L^1-limit $\overline{\sigma}$ is given by

$$\begin{aligned}\overline{\sigma}&=\int_{\operatorname{supp}\nu_{(x,t)}}\sigma(x,t,u,\lambda)d\nu_{(x,t)}(\lambda)\\&\overset{(6.4)}{=}\int_{\operatorname{supp}\nu_{(x,t)}}\big(\sigma(x,t,u,Du)+\nabla\sigma(x,t,u,Du):(Du-\lambda)\big)d\nu_{(x,t)}(\lambda)\\&=\sigma(x,t,u,Du),\end{aligned}$$

where we have used $\|\nu_{(x,t)}\|=1$ and $\int_{\operatorname{supp}\nu_{(x,t)}}(Du-\lambda)d\nu_{(x,t)}(\lambda)=0$.
Evidently,

$$\sigma(x,t,u_k,Du_k)\rightharpoonup\sigma(x,t,u,Du)\quad\text{in}\quad L^{p'}(Q).$$

Case (b): We start by proving that for almost all $(x,t)\in Q$, $\operatorname{supp}\nu_{(x,t)}\subset K_{(x,t)}$, where

$$K_{(x,t)}:=\big\{\lambda\in\mathbb{M}^{m\times n}:W(x,t,u,\lambda)=W(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du)\big\}.$$

If $\lambda\in\operatorname{supp}\nu_{(x,t)}$, then by Lemma 5.4

$$(1-\tau):\big(\sigma(x,t,u,\lambda)-\sigma(x,t,u,Du)\big):(\lambda-Du)=0\quad\forall\tau\in[0,1].\tag{6.5}$$

The monotonicity of σ together with (6.5) imply

$$\begin{aligned}0&\leq(1-\tau):\big(\sigma(x,t,u,Du+\tau(\lambda-Du))-\sigma(x,t,u,\lambda)\big):(Du-\lambda)\\&=(1-\tau):\big(\sigma(x,t,u,Du+\tau(\lambda-Du))-\sigma(x,t,u,Du)\big):(Du-\lambda).\end{aligned}\tag{6.6}$$

Again by the monotonicity of σ and $\tau\in[0,1]$, it follows that the right hand side of (6.6) is nonpositive, because

$$\big(\sigma(x,t,u,Du+\tau(\lambda-Du))-\sigma(x,t,u,Du)\big):\tau(\lambda-Du)\geq 0,$$

which implies for all $\tau\in[0,1]$

$$\big(\sigma(x,t,u,Du+\tau(\lambda-Du))-\sigma(x,t,u,Du)\big):(1-\tau)(\lambda-Du)\geq 0.$$

Thus, for all $\tau\in[0,1]$

$$\big(\sigma(x,t,u,Du+\tau(\lambda-Du))-\sigma(x,t,u,Du)\big):(\lambda-Du)=0,$$

whenever $\lambda\in\operatorname{supp}\nu_{(x,t)}$. From the hypothesis of the potential W we get

$$\begin{aligned}W(x,t,u,\lambda)&=W(x,t,u,Du)+\int_0^1\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big):(\lambda-Du)d\tau\\&=W(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du).\end{aligned}$$

We conclude that $\lambda \in K_{(x,t)}$, i.e., $\operatorname{supp}\nu_{(x,t)} \subset K_{(x,t)}$. Due to the convexity of W, we have for all $\lambda \in \mathbb{M}^{m\times n}$

$$W(x,t,u,\lambda) \geq W(x,t,u,Du) + \sigma(x,t,u,Du) : (\lambda - Du).$$

For all $\lambda \in K_{(x,t)}$, put

$$F(\lambda) = W(x,t,u,\lambda) \quad \text{and} \quad G(\lambda) = W(x,t,u,Du) + \sigma(x,t,u,Du) : (\lambda - Du).$$

Since $\lambda \mapsto F(\lambda)$ is continuous and differentiable, it follows for $\mu \in \mathbb{M}^{m\times n}$ and $\tau \in \mathbb{R}$

$$\begin{aligned} \frac{F(\lambda+\tau\mu) - F(\lambda)}{\tau} &\geq \frac{G(\lambda+\tau\mu) - G(\lambda)}{\tau} \quad \text{if } \tau > 0, \\ \frac{F(\lambda+\tau\mu) - F(\lambda)}{\tau} &\leq \frac{G(\lambda+\tau\mu) - G(\lambda)}{\tau} \quad \text{if } \tau < 0. \end{aligned}$$

Consequently, $DF = DG$, i.e.,

$$\sigma(x,t,u,\lambda) = \sigma(x,t,u,Du) \quad \forall\, \lambda \in K_{(x,t)} \supset \operatorname{supp}\nu_{(x,t)}.$$

Hence

$$\begin{aligned} \overline{\sigma} = \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda)d\nu_{(x,t)}(\lambda) &= \int_{\operatorname{supp}\nu_{(x,t)}} \sigma(x,t,u,\lambda)d\nu_{(x,t)}(\lambda) \\ &= \int_{\operatorname{supp}\nu_{(x,t)}} \sigma(x,t,u,Du)d\nu_{(x,t)}(\lambda) \\ &= \sigma(x,t,u,Du). \end{aligned} \tag{6.7}$$

This shows that $\sigma(x,t,u_k,Du_k) \rightharpoonup \sigma(x,t,u,Du)$ in $L^1(Q)$, and we will show the strong convergence. Consider the Carathéodory function

$$h(x,t,s,\lambda) = \big|\sigma(x,t,s,\lambda) - \overline{\sigma}(x,t)\big|, \quad s \in \mathbb{R}^m,\ \lambda \in \mathbb{M}^{m\times n}.$$

We have $\sigma(x,t,u_k,Du_k)$ is weakly convergent in $L^{p'}(Q)$, hence equi-integrable. This implies the equi-integrability of $h_k(x,t) := h(x,t,u_k,Du_k)$ and

$$h_k \rightharpoonup \overline{h} \quad \text{in} \quad L^1(Q),$$

where

$$\begin{aligned} \overline{h}(x,t) &= \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} h(x,t,s,\lambda)d\delta_{u(x,t)}(s) \otimes d\nu_{(x,t)}(\lambda) \\ &= \int_{\mathbb{M}^{m\times n}} \big|\sigma(x,t,u,\lambda) - \overline{\sigma}(x,t)\big|d\nu_{(x,t)}(\lambda) \\ &= \int_{\operatorname{supp}\nu_{(x,t)}} \big|\sigma(x,t,u,\lambda) - \overline{\sigma}(x,t)\big|d\nu_{(x,t)}(\lambda) = 0, \end{aligned}$$

by (6.7). Since $h_k \geq 0$, it follows that

$$h_k \to 0 \quad \text{in} \quad L^1(Q).$$

Using the fact that h_k is bounded in $L^{p'}(Q)$ together with the Vitali convergence theorem, we conclude that $\sigma(x,t,u_k,Du_k) \rightharpoonup \sigma(x,t,u,Du)$ in $L^{p'}(Q)$.

From cases (a) and (b), we have

$$\sigma(x,t,u_k,Du_k) \rightharpoonup \sigma(x,t,u,Du) \quad \text{in} \quad L^{p'}(Q).$$

It remains then to prove that $g(x,t,u_k,Du_k) \rightharpoonup g(x,tu,Du)$ in $L^{p'}(Q)$. If g does not depend on the third variable, then by the convergence in measure of u_k to u and the continuity of g, we get the needed result. On the other hand, if g is linear in $A \in \mathbb{M}^{m\times n}$, then

$$\begin{aligned} g(x,t,u_k,Du_k) \rightharpoonup & \int_{\mathbb{M}^{m\times n}} g(x,t,u,\lambda) d\nu_{(x,t)}(\lambda) \\ &= g(x,t,u,.) \circ \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) \\ &= g(x,t,u,.) \circ Du = g(x,t,u,Du), \end{aligned}$$

where we have used $Du(x,t) = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda)$.

In conclusion, we can now pass to the limit in the Galerkin equations. Note that the energy equality

$$\frac{1}{2}\|u(.,T)\|^2_{L^2(\Omega)} + \int_0^T \langle \chi, u\rangle dt + \int_0^T \langle \xi, u\rangle dt = \int_0^T \langle f, u\rangle dt + \frac{1}{2}\|u(.,0)\|^2_{L^2(\Omega)}$$

holds true with χ replaced by $-\operatorname{div}\sigma(x,t,u,Du)$ and ξ by $g(x,t,u,Du)$.

References

1. L. Aharouch, E. Azroul and M. Rhoudaf, *Existence result for variational degenerated parabolic problems via pseudo-monotonicity*, in: Proceedings of the 2005 Oujda International Conference on Nonlinear Analysis, pp. 9–20, Electron. J. Differ. Equ. Conf. 14, Southwest Texas State Univ. San Marcos, TX, 2006.
2. J.M. Ball, *A version of the fundamental theorem for Young measures*, in: PDEs and continuum models of phase transitions (Nice, 1988), pp. 207–215, Lecture Notes in Phys. 344, Springer, Berlin, 1989.
3. H. Brézis and F.E. Browder, *Strongly nonlinear parabolic initial-boundary value problems*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), no. 1, 38–40.
4. J. Chabrowski and K.W. Zhang, *Quasi-monotonicity and perturbated systems with critical growth*, Indiana Univ. Math. J. **41** (1992), no. 2, 483–504.
5. G. Dolzmann, N. Hungerühler and S. Muller, *Nonlinear elliptic systems with measure-valued right hand side*, Math. Z. **226** (1997) 545–574.
6. P. Dreyfuss and N. Hungerbühler, *Navier-Stokes systems with quasimonotone viscosity tensor*, Int. J. Differ. Equ. Appl. **9** (2004), no. 1, 59–79.
7. N. Hungerbühler, *Young measures and nonlinear PDEs,* Birmingham, 1999.
8. N. Hungerbühler, *Quasi-linear parabolic systems in divergence form with weak monotonicity*, Duke Math. J. **107** (2001), no. 3, 497–519.
9. E. Kamke, *Das Lebesgue-Stieltjes-Integral*, Teubner, Leipzig, 1960.
10. R. Landes, *On Galerkin's method in the existence theory of quasilinear elliptic equations,* J. Funct. Anal. **39** (1980) 123–148.
11. R. Landes, *On weak solutions of quasilinear parabolic equations*, Nonlinear Anal. **9** (1985) 887–904.
12. R. Landes and V. Mustonen, *On parabolic initial-boundary value problems with critical growth for the gradient*, Ann. Henri Poincaré, **11** (1994) 135–158.
13. L.C. Young, *Generalized curves and the existence of an attained absolute minimum in the calculus of variations*, Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, **30** (1937) 212–234.

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