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APPROXIMATING SOLUTIONS OF THIRD-ORDER NONLINEAR HYBRID DIFFERENTIAL EQUATIONS VIA DHAGE ITERATION PRINCIPLE

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Abstract. We prove the existence and approximation of solutions of the initial value problems of nonlinear third-order hybrid differential equations. The main tool employed here is the Dhage iteration principle in a partially ordered normed linear space. An example is also given to illustrate the main results.

1. Introduction

Let $J = [t_0, t_0 + a]$ be a closed and bounded interval of the real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $t_0 \geq 0$ and $a > 0$. We consider the existence of solutions for the following initial value problem (in short IVP) of the nonlinear third-order hybrid differential equation

$$\begin{cases} \left(\dfrac{x(t)}{p(t) + \int_{t_0}^{t} g(s, x(s))ds} \right)''' = f(t, x(t)), \ t \in J, \\ \left. \left(\dfrac{x(t)}{p(t) + \int_{t_0}^{t} g(s, x(s))ds} \right)^{(k)} \right|_{t=t_0} = \alpha_k, \quad k = 0, 1, 2, \end{cases} \tag{1.1}$$

where $g, f : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $p : J \to \mathbb{R}$ is a continuous function such that

$$p(t) + \int_{t_0}^{t} g(s, x(s))ds > 0 \quad \text{for all } t \in J,$$

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and

$$\alpha_0 + \alpha_1(t - t_0) + \frac{1}{2}\alpha_2(t - t_0)^2 + \frac{1}{2}\int_{t_0}^{t}(t - s)^2 f(x, x(s))ds \geq 0 \quad \text{for all } t \in J.$$

By a solution of the IVP (1.1), we mean a function $x \in C^3(J, \mathbb{R})$ that satisfies (1.1), where $C^3(J, \mathbb{R})$ is the space of thrice continuously differentiable real-valued functions defined on J.

Differential equations arise from a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows, and so on; see [1, 4–9]. The special cases of the IVP (1.1) are well known in the literature and are discussed at length for existence as well as other aspects of the solutions (see [7, 9]). The purpose of this paper is to use the Dhage iteration principle to show the existence and approximation of solutions of (1.1) under weaker partially continuity and partially compactness type conditions.

The article is organized as follows. In Section 2, we give some preliminaries and key fixed point theorem that will be used in later sections. In Section 3, we prove some sufficient conditions of the existence and approximation of solutions of (1.1) by using the Dhage iteration principle. For details on the Dhage iteration principle, we refer the reader to [3]. Finally, an example is given to illustrate our main results.

2. Preliminaries

Let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$. It is known that E is regular if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in [8] and the references therein.

Definition 2.1. A mapping $\mathcal{A} : E \to E$ is called isotone or monotone nondecreasing if it preserves the order relation $\preceq$, that is, $x \preceq y$ implies $\mathcal{A}x \preceq \mathcal{A}y$ for all $x, y \in E$. Similarly, $\mathcal{A}$ is called monotone nonincreasing if $x \preceq y$ implies $\mathcal{A}x \succeq \mathcal{A}y$ for all $x, y \in E$. Finally, $\mathcal{A}$ is called monotonic or simply monotone if it is either monotone nondecreasing or monotone nonincreasing on E.

Definition 2.2. An operator $\mathcal{A}$ on a normed linear space E into itself is called compact if $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{A}$ is called totally bounded if for any bounded subset S of E, $\mathcal{A}(S)$ is a relatively compact subset of E. If $\mathcal{A}$ is continuous and totally bounded, then it is called completely continuous on E.

Definition 2.3 (Dhage [3])**.** A mapping $\mathcal{A} : E \to E$ is called partially continuous at a point $a \in E$ if for $\epsilon > 0$, there exists $\delta > 0$ such that $\|\mathcal{A}x - \mathcal{A}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{A}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{A}$ is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 2.4 (Dhage [2,3]). An operator $\mathcal{A}$ on a partially normed linear space E into itself is called partially bounded if $\mathcal{A}(C)$ is bounded for every chain C in E. $\mathcal{A}$ is called uniformly partially bounded if all chains $\mathcal{A}(C)$ in E are bounded by a unique constant. $\mathcal{A}$ is called partially compact if $\mathcal{A}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{A}$ is called partially totally bounded if for any totally ordered and bounded subset C of E, $\mathcal{A}(C)$ is a relatively compact subset of E. If $\mathcal{A}$ is partially continuous and partially totally bounded, then it is called partially completely continuous on E.

Definition 2.5 (Dhage [2]). The order relation $\preceq$ and the metric d on a non-empty set E are said to be compatible if $\{x_n\}$ is monotone, that is, monotone nondecreasing or monotone nondecreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function has this property.

Theorem 2.6 (Dhage [3]). *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in E. Let $\mathcal{A} : E \to E$ be a partially continuous, nondecreasing, and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0$ or $x_0 \succeq \mathcal{A}x_0$, then the operator equation $\mathcal{A}x = x$ has a solution x^* in E and the sequence $\{\mathcal{A}^n x_0\}$ of successive iterations converges monotonically to x^*.*

3. Main results

The equivalent integral formulation of the IVP (1.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)|, \tag{3.1}$$

$$x \leq y \iff x(t) \leq y(t), \tag{3.2}$$

for all $t \in J$. Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered with respect to the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ has some nice properties with respect to the above order relation in it.

Lemma 3.1 (see [5]). *Let $(C(J, \mathbb{R}), \leq, \|\cdot\|)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2), respectively. Then $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset of $C(J, \mathbb{R})$.*

We need the following definition in what follows.

Definition 3.2. A function $u \in C^3(J, R)$ is said to be a lower solution of the IVP (1.1) if it satisfies

$$\begin{cases} \left(\dfrac{u(t)}{p(t) + \int_{t_0}^{t} g(s, u(s))ds} \right)''' \leq f(t, u(t)), \quad t \in J, \\ \left. \left(\dfrac{u(t)}{p(t) + \int_{t_0}^{t} g(s, u(s))ds} \right)^{(k)} \right|_{t=t_0} \leq \alpha_k, \quad k = 0, 1, 2. \end{cases} \tag{3.3}$$

Similarly, an upper solution $v \in C^3(J, R)$ for the IVP (1.1) is defined on J, by reversing the above inequalities.

We consider the following set of assumptions:

(B1) There exist constants $K_g, K_f > 0$ such that

$$|g(t, x)| \leq K_g \text{ and } |f(t, x)| \leq K_f \text{ for all } t \in J \text{ and } x \in \mathbb{R}.$$

(B2) There exists constant $K_p > 0$ such that

$$|p(t_2) - p(t_1)| \leq K_p |t_2 - t_1| \text{ for all } t_1, t_2 \in J.$$

(B3) $g(t, x)$ and $f(t, x)$ are monotone nondecreasing functions in x for all $t \in J$.

(B4) The IVP (1.1) has a lower solution $u \in C^3(J, \mathbb{R})$.

Lemma 3.3. *The IVP*

$$\begin{cases} \left(\dfrac{x(t)}{q(t)} \right)''' = h(t), \quad t \in J, \\ \left. \left(\dfrac{x(t)}{q(t)} \right)^{(k)} \right|_{t=t_0} = \alpha_k, \quad k = 0, 1, 2, \end{cases} \tag{3.4}$$

is equivalent to the integral equation

$$x(t) = q(t) \left(\alpha_0 + \alpha_1 (t - t_0) + \frac{1}{2} \alpha_2 (t - t_0)^2 + \frac{1}{2} \int_{t_0}^{t} (t - s)^2 h(s) ds \right), \quad t \in J. \tag{3.5}$$

Theorem 3.4. *Assume that hypotheses* $(B1) - (B4)$ *hold. Then the IVP* (1.1) *has a solution* x^* *defined on* J *and the sequence* $\{x_n\}$ *of successive approximations defined by*

$$\begin{aligned} x_{n+1}(t) = &\left(p(t) + \int_{t_0}^{t} g(s, x_n(s)) ds \right) \\ &\times \left(\alpha_0 + \alpha_1 (t - t_0) + \frac{1}{2} \alpha_2 (t - t_0)^2 + \frac{1}{2} \int_{t_0}^{t} (t - s)^2 f(s, x_n(s)) ds \right), \end{aligned} \tag{3.6}$$

for all $t \in J$, *where* $x_0 = u$ *converges monotonically to* x^*.

Proof. Set $E = C(J, \mathbb{R})$. Then by Lemma 3.1, every compact chain in E is compatible with respect to the norm $\| \cdot \|$ and order relation $\leq$. Define the operator $\mathcal{A}$ on E by

$$\begin{aligned}(\mathcal{A}x)(t) = &\left(p(t) + \int_{t_0}^{t} g(s, x(s))ds\right) \\ &\times \left(\alpha_0 + \alpha_1 (t - t_0) + \frac{1}{2}\alpha_2 (t - t_0)^2 + \frac{1}{2}\int_{t_0}^{t} (t-s)^2 f(s, x(s))\, ds\right), \quad t \in J.\end{aligned} \tag{3.7}$$

From the continuity of the integral, it follows that $\mathcal{A}$ defines the map $\mathcal{A} : E \to E$. Now, by Lemma 3.3, the IVP (1.1) is equivalent to the operator equation

$$(\mathcal{A}x)(t) = x(t), \quad t \in J. \tag{3.8}$$

We shall show that the operators $\mathcal{A}$ satisfies all the conditions of Theorem 2.6. This is achieved in the series of following steps.

Step I: $\mathcal{A}$ is nondecreasing operator on E. Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (B3), we obtain

$$\begin{aligned}(\mathcal{A}x)(t) &= \left(p(t) + \int_{t_0}^{t} g(s, x(s))ds\right) \\ &\qquad \times \left(\alpha_0 + \alpha_1 (t - t_0) + \frac{1}{2}\alpha_2 (t - t_0)^2 + \frac{1}{2}\int_{t_0}^{t} (t-s)^2 f(s, x(s))\, ds\right) \\ &\leq \left(p(t) + \int_{t_0}^{t} g(s, y(s))ds\right) \\ &\qquad \times \left(\alpha_0 + \alpha_1 (t - t_0) + \frac{1}{2}\alpha_2 (t - t_0)^2 + \frac{1}{2}\int_{t_0}^{t} (t-s)^2 f(s, y(s))\, ds\right) \\ &= (\mathcal{A}y)(t)\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{A}$ is nondecreasing operator on E into E.

Step II: $\mathcal{A}$ is a partially continuous operator on E. Let $\{x_n\}$ be a sequence in a chain C in E such that $x_n \to x$, when $n \to \infty$. Then, by the dominated convergence theorem, we have

$$\begin{aligned}&\lim_{n\to\infty} (\mathcal{A}x_n)(t) \\ &= \lim_{n\to\infty} \left[\left(p(t) + \int_{t_0}^{t} g(s, x_n(s))ds\right)\right. \\ &\quad \left.\times \left(\alpha_0 + \alpha_1 (t - t_0) + \frac{1}{2}\alpha_2 (t - t_0)^2 + \frac{1}{2}\int_{t_0}^{t} (t-s)^2 f(s, x_n(s))\, ds\right)\right] \\ &= \left(p(t) + \int_{t_0}^{t} \left[\lim_{n\to\infty} g(s, x_n(s))\right] ds\right) \\ &\quad \times \left(\alpha_0 + \alpha_1 (t - t_0) + \frac{1}{2}\alpha_2 (t - t_0)^2 + \frac{1}{2}\int_{t_0}^{t} (t-s)^2 \left[\lim_{n\to\infty} f(s, x_n(s))\right] ds\right) \\ &= \left(p(t) + \int_{t_0}^{t} g(s, x(s))ds\right) \\ &\quad \times \left(\alpha_0 + \alpha_1 (t - t_0) + \frac{1}{2}\alpha_2 (t - t_0)^2 + \frac{1}{2}\int_{t_0}^{t} (t-s)^2 f(s, x(s))\, ds\right) \\ &= (\mathcal{A}x)(t)\end{aligned}$$

for all $t \in J$. This shows that $\{\mathcal{A}x_n\}$ converges pointwise to $\mathcal{A}x$ on J.

Next, we show that $\{\mathcal{A}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then, for all $n \in \mathbb{N}$

$$\begin{aligned}
&\left|(\mathcal{A}x_n)(t_2) - (\mathcal{A}x_n)(t_1)\right| \\
&\leq \left(|p(t_1)| + \int_{t_0}^{t_1} |g(s, x_n(s))|\, ds\right) \\
&\quad \times \Bigg(|\alpha_1| |t_2 - t_1| + a |\alpha_2| |t_2 - t_1| \\
&\quad + \frac{1}{2} \left| \int_{t_0}^{t_2} (t_2 - s)^2 f(s, x_n(s))\, ds - \int_{t_0}^{t_1} (t_1 - s)^2 f(s, x_n(s))\, ds \right| \Bigg) \\
&\quad + \left(|p(t_2) - p(t_1)| + \left| \int_{t_0}^{t_2} g(s, x_n(s)) ds - \int_{t_0}^{t_1} g(s, x_n(s)) ds \right| \right) \\
&\quad \times \left(|\alpha_0| + |\alpha_1| (t_2 - t_0) + \frac{1}{2} |\alpha_2| (t_2 - t_0)^2 + \frac{1}{2} \int_{t_0}^{t_2} (t_2 - s)^2 |f(s, x_n(s))|\, ds \right) \\
&\leq (|p(t_1)| + K_g a) \Bigg((|\alpha_1| + a |\alpha_2|) |t_2 - t_1| \\
&\quad + \frac{a}{2} \left| \int_{t_0}^{t_1} (t_2 - t_1) f(s, x_n(s))\, ds \right| + \frac{1}{2} \left| \int_{t_1}^{t_2} (t_2 - s)^2 f(s, x_n(s))\, ds \right| \Bigg) \\
&\quad + (K_p |t_2 - t_1| + K_g |t_2 - t_1|) \left(|\alpha_0| + |\alpha_1| a + \frac{1}{2} |\alpha_2| a^2 + K_f \frac{a^3}{6} \right) \\
&\leq (|p(t_1)| + K_g a) \Bigg((|\alpha_1| + a |\alpha_2|) |t_2 - t_1| \\
&\quad + \frac{a K_f}{2} \int_{t_0}^{t_0 + a} |t_2 - t_1|\, ds + \frac{1}{2} K_f \int_{t_1}^{t_2} (t_2 - s)^2\, ds \Bigg) \\
&\quad + (K_p + K_g) \left(|\alpha_0| + |\alpha_1| a + \frac{1}{2} |\alpha_2| a^2 + K_f \frac{a^3}{6} \right) |t_2 - t_1| \\
&\leq \Bigg[(|p(t_1)| + K_g a) \left(|\alpha_1| + a |\alpha_2| + K_f \left(\frac{a^2}{2} + \frac{(t_0 + a)^2}{2} \right) \right) \\
&\quad + (K_p + K_g) \left(|\alpha_0| + |\alpha_1| a + \frac{1}{2} |\alpha_2| a^2 + K_f \frac{a^3}{6} \right) \Bigg] |t_2 - t_1| \\
&\to 0,
\end{aligned}$$

uniformly, as $t_2 - t_1 \to 0$. This shows that the convergence $\mathcal{A}x_n \to \mathcal{A}x$ is uniformly and hence $\mathcal{A}$ is partially continuous on E.

Step III: $\mathcal{A}$ is a partially compact operator on E. Let C be an arbitrary chain in E. We show that $\mathcal{A}(C)$ is a uniformly bounded and equicontinuous set in E.

First we show that $\mathcal{A}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then

$$\begin{aligned}
&|(\mathcal{A}x)(t)| \\
&\leq \left(|p(t)| + \left|\int_{t_0}^{t} g(s,x(s))ds\right|\right) \\
&\quad \times \left(|\alpha_0| + |\alpha_1|(t-t_0) + \frac{1}{2}|\alpha_2|(t-t_0)^2 + \frac{1}{2}\left|\int_{t_0}^{t}(t-s)^2 f(s,x(s))\,ds\right|\right) \\
&\leq \left(K_p(t-t_0) + |p(t_0)| + \int_{t_0}^{t_0+a}|g(s,x(s))|\,ds\right) \\
&\quad \times \left(|\alpha_0| + |\alpha_1|a + \frac{1}{2}|\alpha_2|a^2 + \frac{1}{2}\int_{t_0}^{t_0+a}(t-s)^2|f(s,x(s))|\,ds\right) \\
&\leq (K_p a + |p(t_0)| + K_g a)\left(|\alpha_0| + |\alpha_1|a + \frac{1}{2}|\alpha_2|a^2 + K_f\frac{a^3}{6}\right) \\
&= r,
\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{A}x\| \leq r$ for all $x \in C$. Hence $\mathcal{A}$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{A}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for all $x \in C$

$$\begin{aligned}
&|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)| \\
&\leq \left(|p(t_1)| + \int_{t_0}^{t_1}|g(s,x(s))|\,ds\right) \\
&\quad \times \Bigg(|\alpha_1||t_2-t_1| + a|\alpha_2||t_2-t_1| \\
&\quad + \frac{1}{2}\left|\int_{t_0}^{t_2}(t_2-s)^2 f(s,x(s))\,ds - \int_{t_0}^{t_1}(t_1-s)^2 f(s,x(s))\,ds\right|\Bigg) \\
&\quad + \left(|p(t_2)-p(t_1)| + \left|\int_{t_0}^{t_2} g(s,x(s))ds - \int_{t_0}^{t_1} g(s,x(s))ds\right|\right) \\
&\quad \times \left(|\alpha_0| + |\alpha_1|(t_2-t_0) + \frac{1}{2}|\alpha_2|(t_2-t_0)^2 + \frac{1}{2}\int_{t_0}^{t_2}(t_2-s)^2|f(s,x(s))|\,ds\right) \\
&\leq \Bigg[(|p(t_1)| + K_g a)\left(|\alpha_1| + a|\alpha_2| + K_f\left(\frac{a^2}{2} + \frac{(t_0+a)^2}{2}\right)\right) \\
&\quad + (K_p + K_g)\left(|\alpha_0| + |\alpha_1|a + \frac{1}{2}|\alpha_2|a^2 + K_f\frac{a^3}{6}\right)\Bigg]|t_2-t_1| \\
&\to 0
\end{aligned}$$

uniformly, as $t_2 - t_1 \to 0$. Hence $\mathcal{A}(C)$ is a compact subset of E, and consequently, $\mathcal{A}$ is a partially compact operator on E into itself.

Step IV: u satisfies the operator inequality $u \leq \mathcal{A}u$. By hypothesis $(B4)$, the IVP (1.1) has a lower solution u on J. Then we have

$$\left(\frac{u(t)}{p(t)+\int_{t_0}^{t} g(s,u(s))ds}\right)''' \leq f(t,u(t)), \quad t \in J, \tag{3.9}$$

satisfying

$$\left.\left(\frac{u(t)}{p(t)+\int_{t_0}^{t} g(s,u(s))ds}\right)^{(k)}\right|_{t=t_0} \leq \alpha_k, \quad k=0,1,2.$$

Integrating (3.9) thrice which together with the definition of the operator $\mathcal{A}$ implies that $u(t) \leq (\mathcal{A}u)(t)$ for all $t \in J$. Consequently, u is a lower solution to the operator equation $x = \mathcal{A}x$.

Thus $\mathcal{A}$ satisfies all the conditions of Theorem 2.6 with $x_0 = u$ and we apply it to conclude that the operator equation $\mathcal{A}x = x$ has a solution. Consequently, the integral equation and the IVP (1.1) have a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.6) converges monotonically to x^*. This completes the proof. □

Remark 3.5. The conclusion of Theorem 3.4 also remains true if we replace the hypothesis $(B4)$ with the following one:

(B4') The IVP (1.1) has an upper solution $v \in C^3(J,R)$.

Example 3.6. Given a closed and bounded interval $J=[0,1]$ in $\mathbb{R}$, consider the IVP,

$$\begin{cases} \left(\dfrac{x(t)}{\pi+\sin t+\int_0^t \arctan x(s)\,ds}\right)''' = \tanh x(t), \quad t \in J, \\ \left.\left(\dfrac{x(t)}{\pi+\sin t+\int_0^t \arctan x(s)\,ds}\right)^{(k)}\right|_{t=0} = 0, \quad k=0,1, \\ \left.\left(\dfrac{x(t)}{\pi+\sin t+\int_0^t \arctan x(s)\,ds}\right)^{(2)}\right|_{t=0} = 1, \end{cases} \tag{3.10}$$

where $g(t,x) = \arctan x$, $f(t,x) = \tanh x$, and $p(t) = \pi + \sin t$. Clearly, the functions g and f are continuous on $J \times \mathbb{R}$, p is continuous on J, and

$$\pi + \sin t + \int_0^t \arctan x(s)\,ds > 0 \quad \text{for all } t \in J.$$

The functions g and f satisfy the hypothesis $(B1)$ with $K_g = \pi/2$ and $K_f = 1$. The function p satisfies the hypothesis $(B2)$ with $K_p = 1$. Moreover, the functions g and f are nondecreasing in x for each $t \in J$ and so the hypothesis $(B3)$ is satisfied. Finally the IVP (3.10) has a lower solution

$$u(t) = \left(\frac{3\pi}{2} + \sin t\right)\left(\frac{t^2}{2} - \frac{t^3}{6}\right),$$

defined on J. Thus all hypotheses of Theorem 3.4 are satisfied. Hence we apply Theorem 3.4 and conclude that the IVP (3.10) has a solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = \left(\pi + \sin t + \int_0^t \arctan x_n(s)\, ds\right)\left(\frac{1}{2}t^2 + \frac{1}{2}\int_0^t (t-s)^2 \tanh x_n(s) ds\right), \tag{3.11}$$

for all $t \in J$, where $x_0 = u$, converges monotonically to x^*.

Remark 3.7. In view of Remark 3.5, the existence of the solutions x^* of the IVP (3.10) may be obtained under the upper solution

$$v(t) = \left(\frac{\pi}{2} + \sin t\right)\left(\frac{t^2}{2} + \frac{t^3}{6}\right), \quad t \in J.$$

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