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COMMUTING CONJUGACY CLASS GRAPH OF FINITE CA-GROUPS

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ABSTRACT. Let G be a finite nonabelian group. The commuting conjugacy class graph $\Gamma(G)$ is a simple graph with the noncentral conjugacy classes of G as its vertex set and two distinct vertices X and Y in $\Gamma(G)$ are adjacent if and only if there are $x \in X$ and $y \in Y$ with this property that $xy = yx$. The aim of this paper is to obtain the structure of the commuting conjugacy class graph of finite CA-groups. It is proved that this graph is a union of some complete graphs. The commuting conjugacy class graph of certain groups are also computed.

1. INTRODUCTION AND PRELIMINARIES

Suppose that $\Delta = (V, E)$ is a simple graph and that π is a partition of the vertex set V. Define the *quotient graph* $\frac{\Delta}{\pi}$ to be a simple graph with vertex set $\frac{V}{\pi}$ and two blocks $\frac{a}{\pi}$ and $\frac{b}{\pi}$ are adjacent if and only if there are $x \in \frac{a}{\pi}$ and $y \in \frac{b}{\pi}$ such that $xy \in E$. The *commuting graph* of a group G with center $Z(G)$ is the graph with vertex set $G \setminus Z(G)$ and in which two vertices are adjacent if and only if they commute. This graph was studied by Brauer and Fowler [3], and we denote it by $\Delta(G)$. The *commuting conjugacy class graph* of a nonabelian group G, $\Gamma(G)$, is the quotient graph $\frac{\Delta(G)}{\pi}$, where π is the set of all noncentral conjugacy classes of G; see [8]. This graph was first studied by Herzog, Longobardi, and Maj [5], but these authors considered the nonidentity conjugacy classes of the group as a vertex set.

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A CA-group is a group in which every noncentral element has an abelian centralizer; see [10]. We refer the interested readers to consult [7] for more properties of this important class of finite groups. The aim of this paper is to calculate the commuting conjugacy class graph of CA-groups. In addition, the commuting conjugacy class graph of dihedral and the following groups will be obtained:

$$\begin{aligned}
T_{4n} &= \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle, \\
U_{(n,m)} &= \langle a, b \mid a^{2n} = b^m = 1, a^{-1}ba = b^{-1}\rangle, \\
V_{8n} &= \langle a, b \mid a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b\rangle, \\
SD_{8n} &= \langle a, b \mid a^{4n} = b^2 = 1, bab = a^{2n-1}\rangle, \\
G(p, m, n) &= \langle a, b \mid a^{p^m} = b^{p^n} = [a, b]^p = 1, [a, [a, b]] = [b, [a, b]] = 1\rangle.
\end{aligned}$$

In the definition of $U_{(n,m)}$, if we put $m = 3$, then the resulting group is denoted by U_{6n}. This group was first introduced by James and Liebeck in their famous book [6] in which the character table of this group was calculated. The group $G(p, m, n)$ was introduced by Abbaspour and Behravesh [1] in which the conjugacy classes and character table of this group were given.

Throughout this paper, our notation are standard and mainly taken from [9]. Our calculations are done with the aid of GAP; see [11].

2. Commuting conjugacy class graph of certain groups

In this section, the graph structures of the groups presented in the last section are calculated. Since the dihedral groups have simple structure, we start our calculations by this group.

Proposition 2.1. *The commuting conjugacy class graph of dihedral groups can be computed by the following formula:*

$$\Gamma(D_{2n}) = \begin{cases} K_{\frac{n-1}{2}} \cup K_1, & n \text{ is odd,} \\ K_{\frac{n}{2}-1} \cup 2K_1, & n \text{ and } \frac{n}{2} \text{ are even,} \\ K_{\frac{n}{2}-1} \cup K_2, & n \text{ is even and } \frac{n}{2} \text{ is odd.} \end{cases}$$

Proof. Consider the presentation $D_{2n} = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ for dihedral groups. The proposition is proved as follows:

(1) *n is odd.* In this case, the dihedral group is centerless and the noncentral conjugacy classes are

$$a^{D_{2n}}, (a^2)^{D_{2n}}, (a^3)^{D_{2n}}, \ldots, (a^{\frac{n-1}{2}})^{D_{2n}}, b^{D_{2n}}.$$

By simple calculations, one can easily see that the conjugacy classes $a^{D_{2n}}$, $(a^2)^{D_{2n}}$, $\ldots$, $(a^{\frac{n}{2}-1})^{D_{2n}}$ are all adjacent together and so

$$\Gamma(D_{2n}) = K_{\frac{n-1}{2}} \cup K_1.$$

(2) *n is even* . In this case, $Z(D_{2n}) = \{1, a^{\frac{n}{2}}\}$, $(a^{\frac{n}{2}}b)b = b(a^{\frac{n}{2}}b)$, and the noncentral conjugacy classes of D_{2n} are

$$a^{D_{2n}}, (a^2)^{D_{2n}}, (a^3)^{D_{2n}}, \ldots, (a^{\frac{n}{2}-1})^{D_{2n}}, b^{D_{2n}}, (ab)^{D_{2n}}.$$

By the presentation of D_{2n}, we can see that all conjugacy classes $a^{D_{2n}}$, $(a^2)^{D_{2n}}$, $\ldots$, $(a^{\frac{n}{2}-1})^{D_{2n}}$ commute to each other. Furthermore, if $\frac{n}{2}$ is even, then $a^{\frac{n}{2}}b \in b^{D_{2n}}$, and if $\frac{n}{2}$ is odd, then $a^{\frac{n}{2}}b \in (ab)^{D_{2n}}$. Therefore,

$$\Gamma(D_{2n}) = \begin{cases} K_{\frac{n}{2}-1} \cup 2K_1, & \frac{n}{2} \text{ is even,} \\ K_{\frac{n}{2}-1} \cup K_2, & \frac{n}{2} \text{ is odd.} \end{cases}$$

This completes our argument. $\square$

Our second group that we consider into account is the dicyclic group T_{4n}. If n is a power of 2, then this group is the well-known generalized quaternion group.

Proposition 2.2. *The commuting conjugacy class graph of the dicyclic group T_{4n} is as follows:*

$$\Gamma(T_{4n}) = \begin{cases} K_{n-1} \cup 2K_1, & n \text{ is even,} \\ K_{n-1} \cup K_2, & n \text{ is odd.} \end{cases}$$

Proof. We first notice that $Z(T_{4n}) = \{1, a^n\}$ and $T_{4n} = 1^{T_{4n}} \cup (a^n)^{T_{4n}} \cup a^{T_{4n}} \cup (a^2)^{T_{4n}} \cdots \cup (a^{n-1})^{T_{4n}} \cup b^{T_{4n}} \cup (ab)^{T_{4n}}$. Thus, the noncentral conjugacy classes of T_{4n} are $a^{T_{4n}}$, $(a^2)^{T_{4n}}$, $(a^3)^{T_{4n}}$, $\ldots$, $(a^{n-1})^{T_{4n}}$, $b^{T_{4n}}$, and $(ab)^{T_{4n}}$. By the presentation of dicyclic groups, $a^{T_{4n}}$, $(a^2)^{T_{4n}}$, $\ldots$, $(a^{n-1})^{T_{4n}}$ have elements that commute to each other. Furthermore, if n is even, then $(a^n b) \in b^{T_{4n}}$, and if n is odd, then $(a^n b) \in (ab)^{T_{4n}}$. Therefore,

$$\Gamma(T_{4n}) = \begin{cases} K_{n-1} \cup 2K_1 & n \text{ is even,} \\ K_{n-1} \cup K_2 & n \text{ is odd,} \end{cases}$$

proving the result. $\square$

Proposition 2.3. *The commuting conjugacy class graph of the group $U_{(n,m)}$ is computed as follows:*

$$\Gamma(U_{(n,m)}) = \begin{cases} 2K_n \cup K_{n(\frac{m}{2}-1)}, & m \text{ is even,} \\ K_n \cup K_{n(\frac{m-1}{2})}, & m \text{ is odd.} \end{cases}$$

Proof. By the definition, $ba^2 = baa = ab^{-1}a = aab = a^2b$ and so $\langle a^2 \rangle \leq Z(U_{(n,m)})$. If i is odd, then $ba^i = a^ib^{-1}$ and for each j, $b^ja = ab^{-j}$. Furthermore, $b^2a = bba = bab^{-1} = ab^{-1}b^{-1} = ab^{-2}$ and $b^3a = bb^2a = bab^{-2} = ab^{-1}b^{-2} = ab^{-3}$. Therefore,

$$b^ja^i = \begin{cases} a^ib^j, & 2 \mid i, \\ a^ib^{-j}, & 2 \nmid i, \end{cases}$$

and we can see that

$$a^{-i}b^ta^i = \begin{cases} b^t, & 2 \mid i, \\ b^{-t}, & 2 \nmid i, \end{cases} \tag{2.1}$$
$$b^{-j}ab^j = ab^{2j}.$$

Since $b^m = 1$, $b^{\frac{m}{2}} = b^{-\frac{m}{2}}$, when m is even. Thus, $b^{\frac{m}{2}}a = ab^{\frac{m}{2}}$ and hence $b^{\frac{m}{2}} \in Z(U_{(n,m)})$. Therefore,

$$Z(U_{(n,m)}) = \begin{cases} \langle a^2 \rangle \times \langle b^{\frac{m}{2}} \rangle, & 2 \mid m, \\ \langle a^2 \rangle, & 2 \nmid m. \end{cases}$$

We now consider the following two cases:

(1) *m is even.* Then $Z(U_{(n,m)}) = \langle a^2 \rangle \times \langle b^{\frac{m}{2}} \rangle$, and the elements of $U_{(n,m)}$ are as follows:

$$\begin{array}{ccccccc}
\boxed{1} & a & \boxed{a^2} & a^3 & \cdots & \boxed{a^{2n-2}} & a^{2n-1} \\
b & ab & a^2b & a^3b & \cdots & a^{2n-2}b & a^{2n-1}b \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\boxed{b^{\frac{m}{2}}} & ab^{\frac{m}{2}} & \boxed{a^2b^{\frac{m}{2}}} & a^3b^{\frac{m}{2}} & \cdots & \boxed{a^{2n-2}b^{\frac{m}{2}}} & a^{2n-1}b^{\frac{m}{2}} \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
b^{m-1} & ab^{m-1} & a^2b^{m-1} & a^3b^{m-1} & \cdots & a^{2n-2}b^{m-1} & a^{2n-1}b^{m-1}
\end{array}$$

By Equation (2.1), the noncentral conjugacy classes of $U_{(n,m)}$ are as follows:

$$\begin{aligned}
(a^{2k}b^t)^{U_{(n,m)}} &= \{a^{2k}b^t, a^{2k}b^{-t}\}, \quad 0 \le k \le n-1,\ 1 \le t \le \frac{m}{2} - 1, \\
(a^{2k+1})^{U_{(n,m)}} &= \{a^{2k+1}b^{2j} \mid 0 \le j \le \frac{m}{2} - 1\}, \quad 0 \le k \le n-1, \\
(a^{2k+1}b)^{U_{(n,m)}} &= \{a^{2k+1}b^{2j+1} \mid 0 \le j \le \frac{m}{2} - 1\}, \quad 0 \le k \le n-1.
\end{aligned}$$

All conjugacy classes of the form $(a^{2k+1})^{U_{(n,m)}}$ are adjacent in the commuting conjugacy class graph. Since $a^2 \in Z(U_{(n,m)})$, we have $(a^{2k_1}b^{t_1})(a^{2k_2}b^{t_2}) = (a^{2k_2}b^{t_2})(a^{2k_1}b^{t_1})$ and so all classes in the form $(a^{2k}b^t)^{U_{(n,m)}}$ are also adjacent in $\Gamma(U_{(n,m)})$. On the other hand, $(ab)^2 = abab = aab^{-1}b = a^2$ and hence $(ab)^{2k+1} = (ab)^{2k}(ab) = a^{2k+1}b$, where $0 \le k \le n-1$. This shows that all conjugacy classes in the form $(a^{2k+1}b)^{U_{(n,m)}}$ are adjacent. Therefore, $\Gamma(U_{(n,m)}) = 2K_n \cup K_{n(\frac{m}{2}-1)}$.

(2) *m is odd.* In this case, $Z(U_{(n,m)}) = \langle a^2 \rangle$ and so all elements of $U_{(n,m)}$ can be written in the following array:

$$\begin{array}{ccccccc}
\boxed{1} & a & \boxed{a^2} & a^3 & \cdots & \boxed{a^{2n-2}} & a^{2n-1} \\
b & ab & a^2b & a^3b & \cdots & a^{2n-2}b & a^{2n-1}b \\
b^2 & ab^2 & a^2b^2 & a^3b^2 & \cdots & a^{2n-2}b^2 & a^{2n-1}b^2 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
b^{m-2} & ab^{m-2} & a^2b^{m-2} & a^3b^{m-2} & \cdots & a^{2n-2}b^{m-2} & a^{2n-1}b^{m-2} \\
b^{m-1} & ab^{m-1} & a^2b^{m-1} & a^3b^{m-1} & \cdots & a^{2n-2}b^{m-1} & a^{2n-1}b^{m-1}
\end{array}$$

By Equation (2.1), all of noncentral conjugacy classes of $U_{(n,m)}$ are as follows:

$$\begin{aligned}
(a^{2k}b^t)^G &= \{a^{2k}b^t, a^{2k}b^{-t}\}, \quad 0 \le k \le n-1,\ 1 \le t \le \frac{m-1}{2}, \\
(a^{2k+1})^G &= \{a^{2k+1}b^j \mid 0 \le j \le m-1\}, \quad 0 \le k \le n-1.
\end{aligned}$$

All of conjugacy classes in the form $(a^{2k+1})^{U_{(n,m)}}$ are adjacent in the commuting conjugacy class graph. Since $a^2 \in Z(U_{(n,m)})$, an argument similar to case (1) shows that the conjugacy classes in the form $(a^{2k}b^t)^{U_{(n,m)}}$ are also adjacent to each other. Therefore, $\Gamma(U_{(n,m)}) = K_n \cup K_{n(\frac{m-1}{2})}$. Hence the result holds. □

The group $U_{(n,m)}$ is a generalization of the known group $U_{6n} = U_{(n,3)}$ introduced by James and Liebeck in their famous book [6]. Hence by the previous theorem, $\Gamma(U_{6n}) = 2K_n$.

Proposition 2.4. *The commuting conjugacy class graph of the group V_{8n} can be computed by the following formula:*

$$\Gamma(V_{8n}) = \begin{cases} K_{2n-2} \cup 2K_2, & 2 \mid n, \\ K_{2n-1} \cup 2K_1, & 2 \nmid n. \end{cases}$$

Proof. By the presentation of V_{8n}, we have $ab = b^{-1}a^{-1}$ and $ab^{-1} = ba^{-1}$. Hence $b^2a = ab^2$, which proves that $b^2 \in Z(V_{8n})$. On the other hand,

$$a^ib = \begin{cases} ba^{-i}, & 2 \mid i, \\ b^{-1}a^{-i}, & 2 \nmid i. \end{cases} \tag{2.2}$$

Therefore, when n is even, we have $a^n \in Z(V_{8n})$. This shows that,

$$Z(V_{8n}) = \begin{cases} \{1, b^2, a^n, a^nb^2\}, & 2 \mid n, \\ \{1, b^2\}, & 2 \nmid n. \end{cases}$$

We now consider two cases that n is odd or even.

(1) n *is even.* In this case, the members of V_{8n} are as follows:

$$\begin{array}{ccccccccc} \boxed{1} & a & \cdots & a^{n-1} & \boxed{a^n} & a^{n+1} & \cdots & a^{2n-2} & a^{2n-1} \\ b & ab & \cdots & a^{n-1}b & a^nb & a^{n+1}b & \cdots & a^{2n-2}b & a^{2n-1}b \\ \boxed{b^2} & ab^2 & \cdots & a^{n-1}b^2 & \boxed{a^nb^2} & a^{n+1}b^3 & \cdots & a^{2n-2}b^2 & a^{2n-1}b^2 \\ b^3 & ab^3 & \cdots & a^{n-1}b^3 & a^nb^3 & a^{n+1}b^3 & \cdots & a^{2n-2}b^3 & a^{2n-1}b^3 \end{array}$$

By Equation (2.2), we have

$$\begin{aligned} b^{-1}a^ib &= \begin{cases} a^{-i}, & 2 \mid i, \\ a^{-i}b^2, & 2 \nmid i, \end{cases} \\ b^{-1}(a^ib^2)b &= \begin{cases} a^{-i}b^2, & 2 \mid i, \\ a^{-i}, & 2 \nmid i. \end{cases} \end{aligned} \tag{2.3}$$

Since $Z(V_{8n}) = \{1, b^2, a^n, a^nb^2\}$, the conjugacy classes of members contained in the first and third rows are as follows:

$$\begin{gathered} \{1\}, \{a^n\}, \{b^2\}, \\ \{a^nb^2\}, \{a^{2k}, a^{-2k}\}, \quad 1 \le k \le \tfrac{n}{2} - 1, \\ \{a^{2k-1}, a^{-2k+1}b^2\}, \quad 1 \le k \le n, \\ \{a^{2k}b^2, a^{-2k}b^2\}, \quad 1 \le k \le \tfrac{n}{2} - 1. \end{gathered}$$

On the other hand, by Equation (2.2), it can be easily seen that

$$ba^i = \begin{cases} a^{-i}b, & 2 \mid i, \\ a^{-i}b^{-1}, & 2 \nmid i, \end{cases} \qquad b^{-1}a^i = \begin{cases} a^{-i}b^{-1}, & 2 \mid i, \\ a^{-i}b, & 2 \nmid i. \end{cases}$$

Therefore,

$$a^{-i}ba^{i}=\begin{cases}a^{-2i}b, & 2\mid i,\\ a^{-2i}b^{-1}, & 2\nmid i,\end{cases}\qquad (a^{i}b)^{-1}b(a^{i}b)=\begin{cases}a^{2i}b, & 2\mid i,\\ a^{2i}b^{-1}, & 2\nmid i,\end{cases}$$

$$a^{-i}(ab)a^{i}=\begin{cases}a^{-2i+1}b, & 2\mid i,\\ a^{-2i+1}b^{-1}, & 2\nmid i,\end{cases}\qquad (a^{i}b)^{-1}(ab)(a^{i}b)=\begin{cases}a^{2i-1}b^{-1}, & 2\mid i,\\ a^{2i-1}b, & 2\nmid i,\end{cases}$$

$$a^{-i}(a^2b)a^{i}=\begin{cases}a^{-2i+2}b, & 2\mid i,\\ a^{-2i+2}b^{-1}, & 2\nmid i,\end{cases}\qquad (a^{i}b)^{-1}(a^2b)(a^{i}b)=\begin{cases}a^{2i-2}b, & 2\mid i,\\ a^{2i-2}b^{-1}, & 2\nmid i,\end{cases}$$

$$a^{-i}(a^3b)a^{i}=\begin{cases}a^{-2i+3}b, & 2\mid i,\\ a^{-2i+3}b^{-1}, & 2\nmid i,\end{cases}\qquad (a^{i}b)^{-1}(a^3b)(a^{i}b)=\begin{cases}a^{2i-3}b^{-1}, & 2\mid i,\\ a^{2i-3}b, & 2\nmid i.\end{cases}$$

Therefore, the conjugacy classes of members of the second and fourth rows can be written as follows:

$$\begin{aligned}
b^{V_{8n}} &= \left\{a^{4k}b, a^{4k+2}b^{-1} \mid 1\leq k\leq \frac{n}{2}\right\},\\
(ab)^{V_{8n}} &= \left\{a^{4k+1}b, a^{4k+3}b^{-1} \mid 1\leq k\leq \frac{n}{2}\right\},\\
(b^3)^{V_{8n}} &= \left\{a^{4k+2}b, a^{4k}b^{-1} \mid 1\leq k\leq \frac{n}{2}\right\},\\
(ab^3)^{V_{8n}} &= \left\{a^{4k+3}b, a^{4k+1}b^{-1} \mid 1\leq k\leq \frac{n}{2}\right\}.
\end{aligned}$$

Since $b^2\in Z(V_{8n})$, the conjugacy classes $b^{V_{8n}}$ and $(ab)^{V_{8n}}$ are adjacent with the conjugacy classes $(b^3)^{V_{8n}}$ and $(ab^3)^{V_{8n}}$, respectively. If $a^nb\in b^{V_{8n}}$ (or $a^nb^3\in b^{V_{8n}}$), then $a^nb^3\in (b^3)^{V_{8n}}$ (or $a^nb\in (b^3)^{V_{8n}}$). Finally, the conjugacy classes of members in the first and third rows are adjacent to each other and there are $2n-2$ conjugacy classes of this type. Therefore, $\Gamma(V_{8n})=K_{2n-2}\cup 2K_2$.

(2) *n is odd.* The elements of V_{8n} can be written in the following array:

$$\begin{array}{ccccccccc}
\boxed{1} & a & \cdots & a^{n-1} & a^{n} & a^{n+1} & \cdots & a^{2n-2} & a^{2n-1}\\
b & ab & \cdots & a^{n-1}b & a^{n}b & a^{n+1}b & \cdots & a^{2n-2}b & a^{2n-1}b\\
\boxed{b^2} & ab^2 & \cdots & a^{n-1}b^2 & a^{n}b^2 & a^{n+1}b^3 & \cdots & a^{2n-2}b^2 & a^{2n-1}b^2\\
b^3 & ab^3 & \cdots & a^{n-1}b^3 & a^{n}b^3 & a^{n+1}b^3 & \cdots & a^{2n-2}b^3 & a^{2n-1}b^3
\end{array}.$$

By Equation (2.3), we have $Z(V_{8n})=\{1,b^2\}$. Thus the conjugacy classes of members of the first and third rows are

$$\{1\},\{b^2\},\{a^{2k},a^{-2k}\},\quad 1\leq k\leq \tfrac{n-1}{2},$$
$$\{a^{2k-1},a^{-2k+1}b^2\},\quad 1\leq k\leq n,\ \{a^{2k}b^2,a^{-2k}b^2\},\quad 1\leq k\leq \tfrac{n-1}{2},$$

respectively. So, in this case, there are only two conjugacy classes as $b^{V_{8n}}=\{a^{2k}b,a^{2k}b^{-1}\mid 1\leq k\leq n\}$ and $(ab)^{V_{8n}}=\{a^{2k+1}b,a^{2k+1}b^{-1}\mid 1\leq k\leq n\}$. Now it is easy to see that only conjugacy classes of members in the first and third rows are adjacent to each other and there are $2n-1$ such conjugacy classes. Therefore, $\Gamma(V_{8n})=K_{2n-1}\cup 2K_1$. This proves our result. $\square$

It is merit to mention here that the group V_{8n} was first introduced by James and Liebeck [6], for the case that n is odd. The conjugacy classes and character table of this group for the cases that n is odd are taken from the mentioned book. When n is even, the conjugacy classes and character table of this group were computed by Darafsheh and Poursalavati [4].

Proposition 2.5. *The commuting conjugacy class graph of the semi-dihedral group SD_{8n} is computed as follows:*

$$\Gamma(SD_{8n}) = \begin{cases} K_{2n-1} \cup 2K_1, & n \text{ is even}, \\ K_{2n-2} \cup K_4, & n \text{ is odd}. \end{cases}$$

Proof. Since $bab = a^{2n-1}$ and $b^{-1} = b$, we have $ab = ba^{2n-1}$ and so

$$a^i b = \begin{cases} ba^{-i}, & 2 \mid i, \\ ba^{2n-i}, & 2 \nmid i. \end{cases} \tag{2.4}$$

If $i = 2n$, then $a^{2n}b = ba^{-2n} = ba^{2n}$ and so $a^{2n} \in Z(SD_{8n})$. If $i = n$, then

$$a^n b = \begin{cases} ba^{-n}, & 2 \mid n, \\ ba^{2n-n} = ba^n, & 2 \nmid n. \end{cases}$$

On the other hand, if n is odd, then $a^n \in Z(SD_{8n})$ and hence

$$Z(SD_{8n}) = \begin{cases} \{1, a^{2n}\}, & 2 \mid n, \\ \{1, a^n, a^{2n}, a^{3n}\}, & 2 \nmid n. \end{cases}$$

Our main proof considers two cases, n is odd or even, as follows:

(1) *n is even.* In this case, the elements of SD_{8n} can be partitioned as follows:

$$\begin{array}{c} \boxed{1}, a, \ldots, a^n, \ldots, \boxed{a^{2n}}, \ldots, a^{3n}, \ldots, a^{4n-1} \\ b, ab, \ldots, a^n b, \ldots, a^{2n}b, \ldots, a^{3n}b, \ldots, a^{4n-1}b \end{array}$$

By Equation (2.4), we have

$$b^{-1}a^i b = \begin{cases} a^{-i}, & 2 \mid i, \\ a^{2n-i}, & 2 \nmid i. \end{cases} \tag{2.5}$$

Since $Z(SD_{8n}) = \{1, a^{2n}\}$, the conjugacy classes of elements of the first row are as follows:

$$\begin{array}{c} \{1\}, \{a^{2n}\}, \{a^{2k}, a^{-2k}\}, \quad 1 \leq k \leq n-1, \\ \{a^{2k+1}, a^{2n-(2k+1)}\}, \quad -\frac{n}{2} \leq k \leq \frac{n}{2} - 1. \end{array}$$

On the other hand, by Equation (2.4), we have

$$a^{-i}ba^i = \begin{cases} a^{-2i}b, & i = 2k, \\ a^{2n-2i}b, & i = 2k+1, \end{cases} \qquad (a^i b)^{-1} b (a^i b) = \begin{cases} a^{2i}b, & i = 2k, \\ a^{-2n+2i}b, & i = 2k+1, \end{cases}$$
$$a^{-i}(ab)a^i = \begin{cases} a^{-2i+1}b, & i = 2k, \\ a^{2n-2i+1}b, & i = 2k+1, \end{cases} \quad (a^i b)^{-1}(ab)(a^i b) = \begin{cases} a^{2n+2i-1}b, & i = 2k, \\ a^{2i-1}b, & i = 2k+1, \end{cases} \tag{2.6}$$

and so the conjugacy classes in the second row are $b^{SD_{8n}} = \{a^{2k}b \mid 1 \leq k \leq 2n\}$ and $(ab)^{SD_{8n}} = \{a^{2k-1}b \mid 1 \leq k \leq 2n\}$. On the other hand, the elements of the first row is divided into $2n-1$ conjugacy classes, which are adjacent to each other in the commuting conjugacy class graph. Therefore, $\Gamma(SD_{8n}) = K_{2n-1} \cup 2K_1$.

(2) *n is odd.* In this case, the elements of SD_{8n} are as follows:

$$\begin{array}{c}\boxed{1}\,,\ a\ ,\ \ldots,\boxed{a^n},\ldots,\boxed{a^{2n}},\ldots,\boxed{a^{3n}},\ldots,a^{4n-1}\\ b, ab, \ldots, a^nb, \ldots, a^{2n}b, \ldots, a^{3n}b, \ldots, a^{4n-1}b\end{array}$$

By Equation (2.5) and the fact that $Z(SD_{8n}) = \{1, a^n, a^{2n}, a^{3n}\}$, the conjugacy classes of the elements in the first row are

$$\begin{array}{c}\{1\}, \{a^n\}, \{a^{2n}\}, \{a^{3n}\}, \{a^{2k}, a^{-2k}\}, \quad 1 \leq k \leq n-1,\\ \{a^{2k+1}, a^{2n-(2k+1)}\}, \quad -\frac{n-1}{2} \leq k \leq \frac{n-1}{2} - 1.\end{array}$$

On the other hand, by Equation (2.6), we have $b^{SD_{8n}} = \{a^{4k}b \mid 1 \leq k \leq n\}$ and $(ab)^{SD_{8n}} = \{a^{4k+1}b \mid 1 \leq k \leq n\}$, and by Equation (2.4), we have

$$a^{-i}(a^2b)a^i = \begin{cases} a^{-2i+2}b, & 2 \mid i,\\ a^{2n-2i+2}b, & 2 \nmid i,\end{cases} \qquad (a^ib)^{-1}(a^2b)(a^ib) = \begin{cases} a^{2i-2}b, & 2 \mid i,\\ a^{-2n+2i-2}b, & 2 \nmid i,\end{cases}$$

$$a^{-i}(a^3b)a^i = \begin{cases} a^{-2i+3}b, & 2 \mid i,\\ a^{2n-2i+3}b, & 2 \nmid i,\end{cases} \qquad (a^ib)^{-1}(a^3b)(a^ib) = \begin{cases} a^{2n+2i-3}b, & 2 \mid i,\\ a^{2i-3}b, & 2 \nmid i.\end{cases}$$

Therefore $(a^2b)^{SD_{8n}} = \{a^{4k+2}b \mid 1 \leq k \leq n\}$ and $(a^3b)^{SD_{8n}} = \{a^{4k+3}b \mid 1 \leq k \leq n\}$. Note that $a^{2n}b \in (a^2b)^{SD_{8n}}$ and so

(1) if $a^nb \in (ab)^{SD_{8n}}$, then $a^{3n}b \in (a^3b)^{SD_{8n}}$;
(2) if $a^{3n}b \in (ab)^{SD_{8n}}$ then $a^nb \in (a^3b)^{SD_{8n}}$.

Since $Z(SD_{8n}) = \{1, a^n, a^{2n}, a^{3n}\}$, the elements b, a^nb, $a^{2n}b$, and $a^{3n}b$ are commuting to each other. On the other hand, there are $2n-2$ conjugacy classes of elements in the first row, which are pairwise adjacent p. Therefore, $\Gamma(SD_{8n}) = K_{2n-2} \cup K_4$. This completes our argument. □

Proposition 2.6. *The commuting conjugacy class graph of $G(p,m,n)$ is isomorphic to $K_{p^{m-1}(p^n-p^{n-1})} \cup K_{p^{n-1}(p^m-p^{m-1})} \cup (p^n - p^{n-1})K_{p^{m-n}(p^n-p^{n-1})}$.*

Proof. By the definition of the group $G(p,m,n)$, we have $c \in Z(G(p,m,n))$, $ba = abc^{-1}$, and $b^{-1}a = ab$. So, for every i and j,

$$b^ja^i = a^ib^jc^{-ij}. \tag{2.7}$$

In the last equation, if we put $i = p$ and $j = 1$, then $a^p \in Z(G(p,m,n))$. Similarly, in Equation (2.7), we put $i = 1$ and $j = p$ to show that $b^p \in Z(G(p,m,n))$. Therefore, $Z(G(p,m,n)) = \langle a^p\rangle \times \langle b^p\rangle \times \langle c\rangle$. Now by some tedious calculations, one can see that

$$(a^sb^t)^{G(p,m,n)} = \{a^sb^tc^k \mid 0 \leq k \leq p-1\} = a^sb^tH, \tag{2.8}$$

where $H = \langle c\rangle$, $0 \leq s \leq p^m - 1$, $0 \leq t \leq p^n - 1$, and p does not divide simultaneously s and t. For our main proof, now consider three cases as follows:

(1) *$p \mid s$ and $p \nmid t$.* Suppose $s = rp$, where r is an integer. Since $a^p \in Z(G(p,m,n))$, for each positive integer t, we have $(a^{r_1p}b^{t_1})(a^{r_2p}b^{t_2}) = (a^{r_2p}b^{t_2})(a^{r_1p}b^{t_1})$. Hence all conjugacy classes as $(a^{rp}b^t)$ are adjacent together. On the other hand, if $0 \leq t \leq p^n - 1$, $p \nmid t$, and $0 \leq r \leq p^{m-1} - 1$, then the number of conjugacy classes is $p^{m-1}(p^n - p^{n-1})$. Since these conjugacy classes are not adjacent with other conjugacy classes, the commuting conjugacy class graph is isomorphic to $K_{p^{m-1}(p^n-p^{n-1})}$.

(2) *$p \nmid s$ and $p \mid t$.* Suppose $t = rp$, where r is an integer. Since $b^p \in Z(G(p,m,n))$, for each positive integer s, we have $(a^{s_1}b^{r_1p})(a^{s_2}b^{r_2p}) = (a^{s_2}b^{r_2p})(a^{s_1}b^{r_1p})$. Hence all conjugacy classes in the form (a^sb^{rp}) are adjacent together. On the other hand, if $0 \leq s \leq p^m - 1$, $p \nmid s$, and $0 \leq r \leq p^{n-1} - 1$, then there are $p^{n-1}(p^m - p^{m-1})$ conjugacy classes and these conjugacy classes are not adjacent with other classes. Therefore, the commuting conjugacy class graph of the group is isomorphic to complete graph $K_{p^{n-1}(p^m-p^{m-1})}$.

(3) *$p \nmid s$ and $p \nmid t$.* By Equation (2.7) and the fact that $c \in Z(G(p,m,n))$, for every u, we have

$$(a^sb^t)^u = a^{us}b^{ut}c^{-st\frac{u(u-1)}{2}}. \tag{2.9}$$

Thus, for all integers s and t, $(a^sb^t)(a^sb^t)^u = (a^sb^t)^u(a^sb^t)$ if and only if $(a^sb^t)(a^{us}b^{ut}) = (a^{us}b^{ut})(a^sb^t)$. Our assumption implies $a^{p^m} = b^{p^n} = 1$. Without loss of generality, we assume that $n \leq m$. Since $0 \leq u \leq p^m - 1$ and $p \nmid u$, there are $p^{m-n}(p^n - p^{n-1})$ conjugacy classes of this type. These conjugacy classes are adjacent to each other, and we have to count the number of these cliques. Since $(s,p) = 1$, $(s,p^m) = 1$ and so there are x_s and y_s such that $sx_s + p^my_s = 1$. By Equation (2.9) and the fact that $a^{p^m} = 1$, we have

$$\begin{aligned}(a^sb^t)^{x_s} &= a^{sx_s}b^{tx_s}c^{-st\frac{x_s(x_s-1)}{2}}\\ &= a^{1-p^my_s}b^{tx_s}c^{-st\frac{x_s(x_s-1)}{2}}\\ &= ab^{tx_s}c^{-st\frac{x_s(x_s-1)}{2}}.\end{aligned} \tag{2.10}$$

By Equations (2.8) and (2.10) and the fact that $H = \langle c\rangle \leq Z(G(p,m,n))$, we have

$$\begin{aligned}((a^sb^t)^{x_s})^{G(p,m,n)} &= (ab^{tx_s}c^{-st\frac{x_s(x_s-1)}{2}})^{G(p,m,n)}\\ &= (ab^{tx_s})^{G(p,m,n)}c^{-st\frac{x_s(x_s-1)}{2}}\\ &= ab^{tx_s}Hc^{-st\frac{x_s(x_s-1)}{2}}\\ &= ab^{tx_s}H = (ab^{tx_s})^{G(p,m,n)}.\end{aligned}$$

Since $0 \leq t \leq p^n - 1$ and $p \nmid t$, there are $p^n - p^{n-1}$ such cliques. Therefore, the commuting conjugacy class graph is isomorphic to $(p^n - p^{n-1})K_{p^{m-n}(p^n-p^{n-1})}$.

We now apply our calculations to prove that the commuting conjugacy class graph of $G(p,m,n)$ is isomorphic to $K_{p^{m-1}(p^n-p^{n-1})} \cup K_{p^{n-1}(p^m-p^{m-1})} \cup (p^n - p^{n-1})K_{p^{m-n}(p^n-p^{n-1})}$. $\square$

3. Commuting conjugacy class graph of CA-groups

In Section 2, the graph structure of five classes of CA-groups are given. In this section, we analyze these examples to obtain the structure of commuting conjugacy class graph of a CA-group. For the sake of completeness, we mention here a result, which is crucial in the proof of our main result. Let G be a finite CA-group. Then by [2, Remark 2.1(4)], for all $a, b \in G \setminus Z(G)$, either $C_G(a) = C_G(b)$ or $C_G(a) \cap C_G(b) = Z(G)$.

Lemma 3.1. *Let G be a CA-Group. The noncentral conjugacy classes x^G and y^G of G are adjacent in $\Gamma(G)$ if and only if $C_G(x)$ and $C_G(y)$ are conjugate in G.*

Proof. Suppose that the noncentral conjugacy classes x^G and y^G are adjacent in $\Gamma(G)$. There are $t \in x^G$ and $s \in y^G$ such that $ts = st$. This implies that $t \in C_G(t) \cap C_G(s) \neq Z(G)$ and since G is a CA-group, $C_G(t) = C_G(s)$. On the other hand, $t \in x^G$ and $s \in y^G$ imply that there are $g_1, g_2 \in G$ such that $t = g_1^{-1}xg_1$ and $s = g_2^{-1}yg_2$. Since $C_G(t) = C_G(s)$, we have $g_1^{-1}C_G(x)g_1 = C_G(g_1^{-1}xg_1) = C_G(g_2^{-1}yg_2) = g_2^{-1}C_G(y)g_2$, which proves that $C_G(x) = (g_2g_1^{-1})^{-1}C_G(y)(g_2g_1^{-1})$, as desired.

Conversely, let x^G and y^G be noncentral conjugacy classes of G such that $C_G(x)$ and $C_G(y)$ are conjugate. Suppose that $C_G(x) = g^{-1}C_G(y)g$ and $s = g^{-1}yg$. Since $g^{-1}C_G(y)g = C_G(g^{-1}yg) = C_G(s)$, we have $xs = sx$ and so x^G and $s^G = y^G$ are adjacent in $\Gamma(G)$, which proves the lemma. □

Lemma 3.2. *Let G be a CA-group and let $x \in G \setminus Z(G)$. Then $C_G(x) \lhd G$ if and only if $x^G \subseteq C_G(x)$.*

Proof. If $C_G(x) \lhd G$, then obviously $x^G \subseteq C_G(x)$. Conversely, we assume that $x^G \subseteq C_G(x)$ and that $g \in G$ is arbitrary. Then $g^{-1}xg \in x^G \cap C_G(g^{-1}xg) \subseteq C_G(x) \cap C_G(g^{-1}xg)$ and so $C_G(x) \cap C_G(g^{-1}xg) \neq Z(G)$. Since G is a CA-group, we have $C_G(x) = C_G(g^{-1}xg) = g^{-1}C_G(x)g$, which proves that $C_G(x) \lhd G$. □

For a finite group G, we define $Cent(G) = \{C_G(x) \mid x \in G\}$. Consider the equivalence relation $\sim$ on $Cent(G) \setminus \{G\}$ by $C_G(x) \sim C_G(y)$ if and only if $C_G(x)$ and $C_G(y)$ are conjugate in G. Set $A(G) = \frac{Cent(G)\setminus\{G\}}{\sim}$ is the set of all equivalence classes of $\sim$.

Theorem 3.3. *If G is a CA-group, then $\Gamma(G) = \bigcup_{\frac{C_G(x)}{\sim} \in A(G)} K_{n_{\frac{C_G(x)}{\sim}}}$, where*

$$n_{\frac{C_G(x)}{\sim}} = \frac{|C_G(x)| - |Z(G)|}{[N_G(C_G(x)) : C_G(x)]}.$$

Proof. Suppose that $x \in G \setminus Z(G)$, that k is the number of conjugates of $C_G(x)$, and that $s = |x^G \cap C_G(x)|$. It is clear that $|x^G| = ks = s[G : N_G(C_G(x))]$. By Poincaré's theorem, $|x^G| = [G : C_G(x)] = [G : N_G(C_G(x))][N_G(C_G(x)) : C_G(x)]$ and so $s = [N_G(C_G(x)) : C_G(x)]$. If $y \in C_G(x) \setminus ((x^G \cap C_G(x)) \cup Z(G))$, then $yx = xy$. Since G is a CA-group, $C_G(x) = C_G(y)$, and by Lemma 3.1, x^G and y^G are adjacent in $\Gamma(G)$. Define $n_{\frac{C_G(x)}{\sim}} = \frac{|C_G(x)|-|Z(G)|}{s}$. Then $n_{\frac{C_G(x)}{\sim}} = \frac{|C_G(x)|-|Z(G)|}{[N_G(C_G(x)):C_G(x)]}$. Note that $n_{\frac{C_G(x)}{\sim}}$ is the number of noncentral conjugacy classes of G, which is contained in $C_G(x)$. Since G is a CA-group, we have $\Gamma(G) = \bigcup_{\underset{\sim}{C_G(x)} \in A(G)} K_{n_{\underset{\sim}{C_G(x)}}}$. $\square$

We can apply this theorem to obtain the commuting conjugacy class graphs of the dihedral group D_{2n}, the semi-dihedral group SD_{8n}, the dicyclic group T_{4n}, and the groups V_{8n}, $U_{(m,n)}$, $G(p,m,n)$, but our proofs given in Section 2 are simpler than the proofs that can be obtained by Theorem 3.3.

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