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DIFFERENCES OF OPERATORS OF GENERALIZED SZÁSZ TYPE

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ABSTRACT. We derive the approximation of differences of operators. Firstly, we study quantitative estimates for the difference of generalized Szász operators with generalized Szász–Durrmeyer, Szász–Păltănea operators, and generalized Szász–Kantorovich operators. Finally, we obtain the quantitative estimate in terms of the weighted modulus of smoothness for these operators.

1. INTRODUCTION

For $C_\gamma[0,\infty) := \{f \in C[0,\infty) : |f(t)| \leq Me^{\gamma t}$ for some $\gamma > 0, M > 0$, and $t \in [0,\infty)\}$, Miheşan [26] considered a generalization of the well-known Szász operators depending on $\alpha \in \mathbb{R}$ as

$$\mathcal{G}_n^{(\alpha)}(f;x) = \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x) F_{n,k}(f), \quad x \in [0,\infty), \tag{1.1}$$

where $F_{n,k} : D \to \mathbb{R}$ be a positive linear functional defined on a subspace D of $C[0,\infty)$, $\alpha + nx > 0$,

$m_{n,k}^{(\alpha)}(x) = \frac{(\alpha)_k}{k!} \frac{\left(\frac{nx}{\alpha}\right)^k}{\left(1+\frac{nx}{\alpha}\right)^{\alpha+k}}$, and $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1), (\alpha)_0 = 1$.

Kajla [21] studied the local approximation theorem by means of second-order modulus of smoothness, weighted approximation, quantitative Voronovskaya-type theorem of these operators, and rate of convergence for functions having derivatives of bounded variation. Acu-Raşa [9] and Aral, Inoan, and Rasa [10] presented

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some direct results for the difference of operators in order to generalize the problem posed by Lupaş [25] on polynomial differences. Gupta [12] provided a general direct result for the difference of operators and applied the result to Szász type operators. In the literature, many researchers have made significant contributions in this direction. The pioneer works in this direction are due to [11,13,14,18,19]. We refer the reader to some of the related papers [1–8,15–17,22,27].

In the present paper, we compute the approximation of differences of operators. We establish quantitative estimates for the difference of generalized Szász operators with generalized Szász–Durrmeyer, Păltănea operators, and generalized Szász–Kantorovich operators. Also, we obtain the quantitative estimate in terms of weighted modulus of smoothness for these operators.

2. Auxiliary results

In this section, we study certain results, which are necessary to obtain the main results.

Lemma 2.1 ([21,26])**.** *Let $e_i(t)=t^i, i=\overline{0,6}$. For the operators $\mathcal{G}_n^{(\alpha)}(f;x)$, we have*

(i) $\mathcal{G}_n^{(\alpha)}(e_0;x)=1,$

(ii) $\mathcal{G}_n^{(\alpha)}(e_1;x)=x,$

(iii) $\mathcal{G}_n^{(\alpha)}(e_2;x)=x^2+\dfrac{x(nx+\alpha)}{n\alpha},$

(iv) $\mathcal{G}_n^{(\alpha)}(e_3;x)=\dfrac{x^3(1+\alpha)(2+\alpha)}{\alpha^2}+\dfrac{3x^2(1+\alpha)}{n\alpha}+\dfrac{x}{n^2},$

(v) $$\mathcal{G}_n^{(\alpha)}(e_4;x)=\frac{x^4(1+\alpha)(2+\alpha)(3+\alpha)}{\alpha^3}+\frac{6x^3(1+\alpha)(2+\alpha)}{n\alpha^2}+\frac{7x^2(1+\alpha)}{n^2\alpha}+\frac{x}{n^3},$$

(vi) $$\mathcal{G}_n^{(\alpha)}(e_5;x)=\frac{x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}{\alpha^4}+\frac{10x^4(1+\alpha)(2+\alpha)(3+\alpha)}{n\alpha^3}+\frac{25x^3(1+\alpha)(2+\alpha)}{n^2\alpha^2}+\frac{15x^2(1+\alpha)}{n^3\alpha}+\frac{x}{n^4},$$

(vii) $$\mathcal{G}_n^{(\alpha)}(e_6;x)=\frac{x^6(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(5+\alpha)}{\alpha^5}+\frac{15x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)}{n\alpha^4}+\frac{65x^4(1+\alpha)(2+\alpha)(3+\alpha)}{n^2\alpha^3}+\frac{90x^3(1+\alpha)(2+\alpha)}{n^3\alpha^2}+\frac{31x^2(1+\alpha)}{n^4\alpha}+\frac{x}{n^5}.$$

Remark 2.1. For the generalized Szász operators, we have $F_{n,k}(f)=f\left(\frac{k}{n}\right)$ such that $F_{n,k}(e_0)=1, d^{F_{n,k}}:=F_{n,k}(e_1)=\frac{k}{n}$. If we denote $\vartheta_r^{F_{n,k}}=F_{n,k}(e_1-d^{F_{n,k}}e_0)^r, r\in\mathbb{N}$, then by a simple computation, we have

$$\vartheta_r^{F_{n,k}}=F_{n,k}(e_1-d^{F_{n,k}}e_0)^i=0,\quad i=\overline{0,6}.$$

Let $C_B[0,\infty)$ be the class of bounded continuous functions defined on the interval $[0,\infty)$ equipped with norm $||.|| = \sup_{x\in[0,\infty)}$. Let $H_\varphi[0,\infty)$ be the space of all real valued functions on $[0,\infty)$ satisfying the condition $|f(x)| \leq N_f\varphi(x)$, where N_f is a positive constant depending only on f and $\varphi(x) = 1 + x^2$ is a weight function. Let $C_\varphi[0,\infty)$ be the space of all continuous functions in $H_\varphi[0,\infty)$ endowed with the norm

$$\|f\|_\varphi := \sup_{x\in[0,\infty)} \frac{|f(x)|}{\varphi(x)}$$

and

$$C^0_\varphi[0,\infty) := \left\{ f \in C_\varphi[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{\varphi(x)} \quad \text{exists and is finite} \right\}.$$

Let us consider $F_{n,k}, G_{n,k} : D \to \mathbb{R}$ and define the operators

$$U_n(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x)F_{n,k}(f), V_n(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x)G_{n,k}(f).$$

For the class of bounded functions, the author in [12] obtained a general estimate for the difference of two operators.

Theorem 2.1 ([12]). *Let $f^{(j)} \in C_B[0,\infty),\ j = \overline{0,2}$, and let $x \in [0,\infty)$. Then for $n \in \mathbb{N}$, we have*

$$|(U_n - V_n)(f;x)| \leq \frac{||f''||}{2}\beta(x) + \frac{\omega(f'',\delta_1)}{2}(1+\beta(x)) + 2\omega(f,\delta_2),$$

where

$$\beta(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\left(\vartheta_2^{F_{n,k}} + \vartheta_2^{G_{n,k}}\right)$$

and

$$\delta_1^2(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\left(\vartheta_4^{F_{n,k}} + \vartheta_4^{G_{n,k}}\right), \delta_2^2(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\left(d^{F_{n,k}} - d^{G_{n,k}}\right)^2.$$

We apply the weighted modulus of continuity $\Omega(f,\delta)$ defined on $[0,\infty)$ (see [28]). Let

$$\Omega(f,\delta) = \sup_{|h|<\delta, x\in[0,\infty)} \frac{|\ f(x+h) - f(x)\ |}{(1+h^2)(1+x^2)} \text{ for each } C^0_\varphi[0,\infty).$$

Lemma 2.2 ([28]). *Let $f \in C^0_\varphi[0,\infty)$; then*

i) *$\Omega(f;\delta)$ is a monotone increasing function of δ;*
ii) $\lim_{\delta\to 0^+} \Omega(f;\delta) = 0$;
iii) *for each $m \in \mathbb{N}, \Omega(f,m\delta) \leq m\Omega(f;\delta)$;*
iv) *for each $\vartheta \in [0,\infty), \Omega(f;\vartheta\delta) \leq (1+\vartheta)\Omega(f;\delta)$.*

Now we estimate, in the quantitative form, the difference concerning the difference of two operators in the weighted approximation.

Theorem 2.2 ([10]). *Let $f \in C_\varphi[0,\infty)$ with $f \in C_\varphi^0[0,\infty)$; then for $n \in \mathbb{N}$, we have*

$$|(U_n - V_n)(f;x)| \leq \frac{||f''||}{2}\beta(x) + 8\Omega(f'',\delta_1)(1+\beta(x)) + 16\Omega(f,\delta_2)(\gamma(x)+1),$$

where

$$\begin{aligned} \beta(x) &= \sum_{k=0}^{\infty} p_{n,k}(x)\left\{\left(1+\left(d^{F_{n,k}}\right)^2\right)\vartheta_2^{F_{n,k}} + \left(1+\left(d^{G_{n,k}}\right)^2\right)\vartheta_2^{G_{n,k}}\right\}, \\ \gamma(x) &= \sum_{k=0}^{\infty} p_{n,k}(x)\left(1+\left(d^{F_{n,k}}\right)^2\right), \end{aligned}$$

$$\delta_1(x) = \left(\sum_{k=0}^{\infty} p_{n,k}(x)\left\{\left(1+\left(d^{F_{n,k}}\right)^2\right)\vartheta_6^{F_{n,k}} + \left(1+\left(d^{G_{n,k}}\right)^2\right)\vartheta_6^{G_{n,k}}\right\}\right)^{1/4} \leq 1,$$

and

$$\delta_2(x) = \left(\sum_{k=0}^{\infty} p_{n,k}(x)\left(1+\left(d^{F_{n,k}}\right)^2\right)\left(d^{F_{n,k}} - d^{G_{n,k}}\right)^4\right)^{1/4} \leq 1.$$

Now we estimate, in the quantitative form, the difference concerning the difference of two operators in weighted approximation.

Theorem 2.3 ([10]). *Let $f \in C_\varphi[0,\infty)$ with $f \in C_\varphi^0[0,\infty)$. If $d^{F_{n,k}} = d^{G_{n,k}} = d_k$ for all k, then for $n \in \mathbb{N}$, we have*

$$|(U_n - V_n)(f;x)| \leq \frac{||f''||}{2}\beta_1(x) + 8\Omega(f'',\delta_3)(1+\beta_1(x)),$$

where

$$\beta_1(x) = \textstyle\sum_{k=0}^{\infty} p_{n,k}(x)\left\{\left(1+\left(d^{F_{n,k}}\right)^2\right)\left(\vartheta_2^{F_{n,k}} + \vartheta_2^{G_{n,k}}\right)\right\}$$

and

$$\delta_3(x) = \left(\sum_{k=0}^{\infty} p_{n,k}(x)\left\{\left(1+\left(d^{F_{n,k}}\right)^2\right)\left(\vartheta_6^{F_{n,k}} + \vartheta_6^{G_{n,k}}\right)\right\}\right)^{1/4} \leq 1.$$

3. Generalized Szász–Durrmeyer type operators

In [23] Kajla and Acar proposed the Durrmeyer variant of the operators (1.1) as

$$\mathcal{H}_{n,c}^{(\alpha)}(f;x) = \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x) G_{n,k}(f), \tag{3.1}$$

where $m_{n,k}^{(\alpha)}(x)$ is given in (1.1) and

$$G_{n,k}(f) = \int_0^\infty b_{n,k}^c(t) f(t)\,dt, \tag{3.2}$$

with $b^c_{n,k}(t) = \dfrac{c}{B\left(k+1, \frac{n}{c}\right)} \dfrac{(ct)^k}{(1+ct)^{\frac{n}{c}+k+1}}$ $B(k+1,n)$ is the beta function defined by

$B(x,y) = \displaystyle\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x, y > 0$.

We present below the application of Theorem 2.1, that is, the exact estimate for difference of generalized Szász–Durrmeyer and generalized Szász operators.

Theorem 3.1. *Let $f^{(j)} \in C_B[0,\infty), j = \overline{0,2}$ and $x \in [0,\infty)$; then for $n \in \mathbb{N}$ we have*

$$\left|\left(\mathcal{G}^{(\alpha)}_n - \mathcal{H}^{(\alpha)}_{n,c}\right)(f;x)\right| \le \frac{||f''||}{2}\beta(x) + \frac{\omega(f'',\delta_1)}{2}(1+\beta(x)) + 2\omega(f,\delta_2),$$

where

$$\beta(x) = \frac{n^2x^2c(1+\alpha)}{(n-2c)(n-c)^2\alpha} + \frac{nx(2c\alpha+n\alpha)}{(n-2c)(n-c)^2\alpha} + \frac{1}{(n-2c)(n-c)^2}$$

and

$$\begin{aligned}\delta_1^2(x) =& \frac{3c^2(5c+n)n^4x^4(1+\alpha)(2+\alpha)(3+\alpha)}{(n-4c)(n-3c)(n-2c)(n-c)^4\alpha^3} \\ &+ \frac{3nx^3(n+5c)\left(4cn^2(n+c)+6cn^2(n+c)\alpha+2cn^2(n+c)\alpha^2+6c^2n^2(1+\alpha)(2+\alpha)\right)}{(n-4c)(n-3c)(n-2c)(n-c)^4\alpha^2} \\ &+ \frac{3nx^2\left((n+c)(37c^2n\alpha+16cn^2\alpha+n^3\alpha+n\alpha^2(37c^2+16cn+n^2))+7c^2n(5c+n)\alpha(1+\alpha)\right)}{(n-4c)(n-3c)(n-2c)(n-c)^4\alpha^2} \\ &+ \frac{3nx\left(c^2(n+5c)\alpha^2+(n+c)\alpha^2(19c^2+18cn+5n^2)\right)}{(n-4c)(n-3c)(n-2c)(n-c)^4\alpha^2} \\ &+ \frac{3n(2c^2\alpha^2+cn\alpha^2+3n^2\alpha^2)}{(n-4c)(n-3c)(n-2c)(n-c)^4\alpha^2},\end{aligned}$$

$$\delta_2^2(x) = \frac{xc(c+2n)}{n(n-c)^2} + \frac{c^2x^2(1+\alpha)}{(n-c)^2\alpha} + \frac{1}{(n-c)^2}.$$

Proof. We have

$$G_{n,k}(f) = \int_0^\infty b^c_{n,k}(t)t^r dt = \frac{\Gamma[\frac{n}{c}-r]\Gamma[1+k+r]}{c^r\Gamma[1+k]\Gamma[\frac{n}{c}]},$$

implying

$$d^{G_{n,k}} = G_{n,k}(e_1) = \frac{k+1}{(n-c)}.$$

Also, we have

$$\begin{aligned}\vartheta_2^{G_{n,k}} &:= G_{n,k}\left(e_1 - d^{G_{n,k}}e_0\right)^2 \\ &= G_{n,k}(e_2) + \left(\frac{k+1}{n-c}\right)^2 - 2G_{n,k}(e_1)\left(\frac{k+1}{n-c}\right) \\ &= \frac{(k+1)(k+2)}{(n-c)(n-2c)} - \left(\frac{k+1}{n-c}\right)^2 = \frac{(k+1)(n+ck)}{(n-c)^2(n-2c)}\end{aligned}$$

and

$$\begin{aligned}
\vartheta_4^{G_{n,k}} &:= G_{n,k}\left(e_1 - d^{G_{n,k}} e_0\right)^4 \\
&= G_{n,k}(e_4) - 4G_{n,k}(e_3)\left(\frac{k+1}{n-c}\right) + 6G_{n,k}(e_2)\left(\frac{k+1}{n-c}\right)^2 \\
&\quad -4G_{n,k}(e_1)\left(\frac{k+1}{n-c}\right)^3 + G_{n,k}(e_0)\left(\frac{k+1}{n-c}\right)^4 \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{(n-c)(n-2c)(n-3c)(n-4c)} - 4\frac{(k+1)(k+2)(k+3)}{(n-c)(n-2c)(n-3c)}\left(\frac{k+1}{n-c}\right) \\
&\quad +6\frac{(k+1)(k+2)}{(n-c)(n-2c)}\left(\frac{k+1}{n-c}\right)^2 - 4\frac{(k+1)}{(n-c)}\left(\frac{k+1}{n-c}\right)^3 + \left(\frac{k+1}{n-c}\right)^4 \\
&= \frac{3(k+1)(ck+n)\left(n^2(3+k) + nc(k(k+6)+1) + c^2(2+5k(k+1))\right)}{(n-c)^4(n-2c)(n-3c)(n-4c)}
\end{aligned}$$

Applying Lemma 2.1 and Remark 2.1, we have

$$\begin{aligned}
\beta(x) &:= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left(\vartheta_2^{F_{n,k}} + \vartheta_2^{G_{n,k}}\right) \\
&= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\frac{(k+1)(n+ck)}{(n-c)^2(n-2c)} \\
&= \frac{n^2x^2c(1+\alpha)}{(n-2c)(n-c)^2\alpha} + \frac{nx(2c\alpha+n\alpha)}{(n-2c)(n-c)^2\alpha} + \frac{1}{(n-2c)(n-c)^2}
\end{aligned}$$

$$\begin{aligned}
\delta_1^2(x) &= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left(\vartheta_4^{F_{n,k}} + \vartheta_4^{G_{n,k}}\right) = \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\vartheta_4^{G_{n,k}} \\
&= \frac{3c^2(5c+n)n^4x^4(1+\alpha)(2+\alpha)(3+\alpha)}{(n-4c)(n-3c)(n-2c)(n-c)^4\alpha^3} \\
&\quad + \frac{3nx^3(n+5c)\left(4cn^2(n+c) + 6cn^2(n+c)\alpha + 2cn^2(n+c)\alpha^2 + 6c^2n^2(1+\alpha)(2+\alpha)\right)}{(n-4c)(n-3c)(n-2c)(n-c)^4\alpha^2} \\
&\quad + \frac{3nx^2\left((n+c)(37c^2n\alpha + 16cn^2\alpha + n^3\alpha + n\alpha^2(37c^2+16cn+n^2)) + 7c^2n(5c+n)\alpha(1+\alpha)\right)}{(n-4c)(n-3c)(n-2c)(n-c)^4\alpha^2} \\
&\quad + \frac{3nx\left(c^2(n+5c)\alpha^2 + (n+c)\alpha^2(19c^2+18cn+5n^2)\right)}{(n-4c)(n-3c)(n-2c)(n-c)^4\alpha^2} \\
&\quad + \frac{3n(2c^2\alpha^2 + cn\alpha^2 + 3n^2\alpha^2)}{(n-4c)(n-3c)(n-2c)(n-c)^4\alpha^2}
\end{aligned}$$

and

$$\begin{aligned}
\delta_2^2(x) &= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left(d^{F_{n,k}} - d^{G_{n,k}}\right)^2 \\
&= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left[\frac{k}{n} - \left(\frac{k+1}{n-c}\right)\right]^2 = \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\frac{(n^2+k^2c^2+2nkc)}{n^2(n-c)^2} \\
&= \frac{xc(c+2n)}{n(n-c)^2} + \frac{c^2x^2(1+\alpha)}{(n-c)^2\alpha} + \frac{1}{(n-c)^2}.
\end{aligned}$$

Applying Theorem 2.1, we get the required result. □

Theorem 3.2. *Let $f \in C_\varphi[0,\infty)$ with $f'' \in C^0_\varphi[0,\infty)$; then for $n \in \mathbb{N}$, we have*

$$\left|\left(\mathcal{G}_n^{(\alpha)} - \mathcal{H}_{n,c}^{(\alpha)}\right)(f;x)\right| \leq \frac{||f''||}{2}\beta(x) + 8\Omega(f'',\delta_1)\left(1+\beta(x)\right) + 16\Omega(f,\delta_2),$$

where

$$\begin{aligned}
\beta(x) &= \frac{n^4x^4c(1+\alpha)(2+\alpha)(3+\alpha)}{(n-c)^4(n-2c)\alpha^3} \\
&+\frac{n^3x^3\alpha(\alpha^2(n+9c)+3\alpha(n+9c)+2n+18c)}{(n-c)^4(n-2c)\alpha^3} \\
&+\frac{n^2x^2\alpha^2\left(\alpha(c((n-c)^2+19)+6n)+6n+c((n-c)^2+19)\right)}{(n-c)^4(n-2c)\alpha^3} \\
&+\frac{nx(n^3+3c^2n+7n+2c^3+8c)}{(n-c)^4(n-2c)} + \frac{n\alpha^3((n-c)^2+1)}{(n-c)^4(n-2c)\alpha^3},
\end{aligned}$$

$$\delta_1^4(x) = \frac{5c^3x^9n^9(3n^2+80cn+37c^2)}{(n-c)^8(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)\alpha^8} + O\left(\frac{1}{n^3}\right) \leq 1$$

and

$$\begin{aligned}
\delta_2^4(x) &= \frac{n^2x^6c^4(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(5+\alpha)}{(n-c)^6\alpha^5} \\
&+\frac{nx^5c^3(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(17c+4n)}{(n-c)^6\alpha^4} \\
&+\frac{x^4c^2(1+\alpha)(2+\alpha)(3+\alpha)(86c^2+c^4+48nc-2c^3n+6n^2+c^2)}{(n-c)^6\alpha^3} \\
&+\frac{x^3c(1+\alpha)(2+\alpha)(146c^5+6c^5+152c^2n-8c^4n+48cn^2-2c^3n^2+4n^3+4c^2n^3)}{n(n-c)^6\alpha^2} \\
&+\frac{x^2(1+\alpha)(68c^4+7c^6+128c^3n-2c^5n+84c^2n^2-11c^4n^2+20cn^3+n^4+6c^2n^4)}{n^2(n-c)^6\alpha} \\
&+\frac{x(4c^4+c^6+16c^3n+2c^5n+24c^2n^2-c^4n^2+16cn^3-4c^3n^3+3n^4-2c^2n^4+4cn^5)}{n^3(n-c)^6} \\
&+\frac{(1+(n-c)^2)}{(n-c)^6} \leq 1.
\end{aligned}$$

Proof. By a simple calculation, we get

$$\begin{aligned}
\vartheta_6^{G_{n,k}} &:= G_{n,k}\left(e_1 - d^{G_{n,k}}e_0\right)^6 \\
&= G_{n,k}(e_6) - 6G_{n,k}(e_5)\left(\frac{k+1}{n-c}\right) + 15G_{n,k}(e_4)\left(\frac{k+1}{n-c}\right)^2 \\
&\quad -20G_{n,k}(e_3)\left(\frac{k+1}{n-c}\right)^3 + 15G_{n,k}(e_2)\left(\frac{k+1}{n-c}\right)^4 \\
&\quad -6G_{n,k}(e_1)\left(\frac{k+1}{n-c}\right)^5 + G_{n,k}(e_0)\left(\frac{k+1}{n-c}\right)^6
\end{aligned}$$

$$\begin{aligned}
&= \frac{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)}{(n-c)(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)}\\
&\quad - \frac{6(k+1)(k+2)(k+3)(k+4)(k+5)}{(n-c)(n-2c)(n-3c)(n-4c)(n-5c)}\left(\frac{k+1}{n-c}\right)\\
&\quad + \frac{15(k+1)(k+2)(k+3)(k+4)}{(n-c)(n-2c)(n-3c)(n-4c)}\left(\frac{k+1}{n-c}\right)^2\\
&\quad - \frac{20(k+1)(k+2)(k+3)}{(n-c)(n-2c)(n-3c)}\left(\frac{k+1}{n-c}\right)^3\\
&\quad + \frac{15(k+1)(k+2)}{(n-c)(n-2c)}\left(\frac{k+1}{n-c}\right)^4 - \frac{15(k+1)}{(n-c)}\left(\frac{k+1}{n-c}\right)^5 + \left(\frac{k+1}{n-c}\right)^6\\
&= \frac{5k^6(37c^5+80c^4n+3c^3n^2)}{(n-c)^6(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)}\\
&\quad + \frac{5k^5(111c^5+351c^4n+249c^3n^2+9c^2n^3)}{(n-c)^6(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)}\\
&\quad + \frac{5k^4(205c^5+411c^4n+882c^3n^2+293c^2n^3+9cn^4)}{(n-c)^6(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)}\\
&\quad + \frac{5k^3(225c^5+277c^4n+816c^3n^2+920c^2n^3+159cn^4+3n^5)}{(n-c)^6(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)}\\
&\quad + \frac{5k^2(118c^5+229c^4n+205c^3n^2+776c^2n^3+437cn^4+35n^4)}{(n-c)^6(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)}\\
&\quad + \frac{5k(24c^5+116c^4n+23c^3n^2+159c^2n^3+313cn^4+85n^5)}{(n-c)^6(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)}\\
&\quad + \frac{5(24c^4n-2c^3n^2+19c^2n^3+26cn^4+53n^5)}{(n-c)^6(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\beta(x) &= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left\{\left(1+\left(d^{F_{n,k}}\right)^2\right)\vartheta_2^{F_{n,k}} + \left(1+\left(d^{G_{n,k}}\right)^2\right)\vartheta_2^{G_{n,k}}\right\}\\
&= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left(1+\left(\frac{k+1}{n-c}\right)^2\right)\left(\frac{(k+1)(n+ck)}{(n-c)^2(n-2c)}\right)\\
&= \frac{n^4x^4c(1+\alpha)(2+\alpha)(3+\alpha)}{(n-c)^4(n-2c)\alpha^3}\\
&\quad + \frac{n^3x^3\alpha(\alpha^2(n+9c)+3\alpha(n+9c)+2n+18c)}{(n-c)^4(n-2c)\alpha^3}\\
&\quad + \frac{n^2x^2\alpha^2\left(\alpha(c((n-c)^2+19)+6n)+6n+c((n-c)^2+19)\right)}{(n-c)^4(n-2c)\alpha^3}\\
&\quad + \frac{nx(n^3+3c^2n+7n+2c^3+8c)}{(n-c)^4(n-2c)} + \frac{n\alpha^3((n-c)^2+1)}{(n-c)^4(n-2c)\alpha^3},
\end{aligned}$$

$$\begin{aligned}\delta_1^4(x) &= \left(\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left\{\left(1+\left(d^{F_{n,k}}\right)^2\right)\vartheta_6^{F_{n,k}}+\left(1+\left(d^{G_{n,k}}\right)^2\right)\vartheta_6^{G_{n,k}}\right\}\right)\\ &= \frac{5c^3x^9n^9(3n^2+80cn+37c^2)}{(n-c)^8(n-2c)(n-3c)(n-4c)(n-5c)(n-6c)\alpha^8}+O\left(\frac{1}{n^3}\right)\end{aligned}$$

$$\begin{aligned}\delta_2^4(x) &= \left(\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left(1+\left(d^{F_{n,k}}\right)^2\right)\left(d^{F_{n,k}}-d^{G_{n,k}}\right)^4\right)\\ &= \frac{n^2x^6c^4(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(5+\alpha)}{(n-c)^6\alpha^5}\\ &+\frac{nx^5c^3(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(17c+4n)}{(n-c)^6\alpha^4}\\ &+\frac{x^4c^2(1+\alpha)(2+\alpha)(3+\alpha)(86c^2+c^4+48nc-2c^3n+6n^2+c^2)}{(n-c)^6\alpha^3}\\ &+\frac{x^3c(1+\alpha)(2+\alpha)(146c^5+6c^5+152c^2n-8c^4n+48cn^2-2c^3n^2+4n^3+4c^2n^3)}{n(n-c)^6\alpha^2}\\ &+\frac{x^2(1+\alpha)(68c^4+7c^6+128c^3n-2c^5n+84c^2n^2-11c^4n^2+20cn^3+n^4+6c^2n^4)}{n^2(n-c)^6\alpha}\\ &+\frac{x(4c^4+c^6+16c^3n+2c^5n+24c^2n^2-c^4n^2+16cn^3-4c^3n^3+3n^4-2c^2n^4+4cn^5)}{n^3(n-c)^6}\\ &+\frac{(1+(n-c)^2)}{(n-c)^6}.\end{aligned}$$

Using Theorem 2.2, we get the desired result. □

4. Generalized Szász–Păltănea operators

Kajla [20] constructed a new sequence of summation-integral type operators as follows:

$$\mathcal{P}_{n,\rho}^{(\alpha)}(f;x) = \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)L_{n,k,\rho}(f), \tag{4.1}$$

where

$$L_{n,k,\rho}(f)=\int_0^{\infty} s_{n,k}^{\rho}(t)f(t)dt, \quad 1\le k<\infty, \quad L_{n,0,\rho}=f(0),$$

$s_{n,k}^{\rho}(t)=n\rho e^{-n\rho t}\dfrac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}$ and $m_{n,k}^{(\alpha)}(x)$ is defined in (1.1).

We present below the application of Theorem 2.1, that is, the exact estimate for difference of generalized Szász–Păltănea and generalized Szász operators.

Theorem 4.1. *Let $f^{(j)}\in C_B[0,\infty), j=\overline{0,2}$ and $x\in[0,\infty)$; then for $n\in\mathbb{N}$, we have*

$$\left|\left(\mathcal{G}_n^{(\alpha)}-\mathcal{P}_{n,\rho}^{(\alpha)}\right)(f;x)\right|\le\frac{||f''||}{2}\beta(x)+\frac{\omega(f'',\delta_1)}{2}\left(1+\beta(x)\right),$$

where

$$\beta(x)=\frac{x}{n\rho} \text{ and } \delta_1^2(x) = \frac{3x(2+\rho)}{n^3\rho^3}+\frac{3x^2(1+\alpha)}{n^2\rho^2\alpha}.$$

Proof. We have

$$L_{n,k,\rho}(e_r)=\int_0^\infty s_{n,k}^{\rho}(t)t^r dt=\frac{\Gamma(k\rho+r)}{\Gamma(k\rho)(n\rho)^r}, r=\overline{0,6}$$

$$d^{L_{n,k,\rho}}=L_{n,k,\rho}(e_1)=\frac{k}{n}, \vartheta_2^{L_{n,k,\rho}}=L_{n,k,\rho}(e_1-d^{L_{n,k,\rho}}e_0)^2=\frac{k}{n^2\rho}$$

and

$$\begin{aligned}
\vartheta_4^{L_{n,k,\rho}} &:= L_{n,k,\rho}(e_1-d^{L_{n,k,\rho}}e_0)^4\\
&= L_{n,k,\rho}(e_4)-4L_{n,k,\rho}(e_3)\left(\frac{k}{n}\right)+6L_{n,k,\rho}(e_2)\left(\frac{k}{n}\right)^2\\
&\quad -4L_{n,k,\rho}(e_1)\left(\frac{k}{n}\right)^3+L_{n,k,\rho}(e_0)\left(\frac{k}{n}\right)^4\\
&= \frac{3k(2+k\rho)}{n^4\rho^3}.
\end{aligned}$$

Applying Lemma 2.1 and Remark 2.1, we have

$$\begin{aligned}
\beta(x) &= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left(\vartheta_2^{F_{n,k}}+\vartheta_2^{L_{n,k,\rho}}\right)\\
&= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left(\frac{k}{n^2\rho}\right)=\frac{x}{n\rho}
\end{aligned}$$

$$\begin{aligned}
\delta_1^2(x) &= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left(\vartheta_4^{F_{n,k}}+\vartheta_4^{L_{n,k,\rho}}\right)=\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\vartheta_4^{L_{n,k,\rho}}\\
&= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left(\frac{3k(2+k\rho)}{n^4\rho^3}\right)\\
&= \frac{3x(2+\rho)}{n^3\rho^3}+\frac{3x^2(1+\alpha)}{n^2\rho^2\alpha}.
\end{aligned}$$

Applying Theorem 2.1, we get the required result. □

Theorem 4.2. *Let $f\in C_\varphi[0,\infty)$ with $f''\in C_\varphi^0[0,\infty)$; then for $n\in\mathbb{N}$, we have*

$$\left|\left(\mathcal{G}_n^{(\alpha)}-\mathcal{H}_{n,c}^{(\alpha)}\right)(f;x)\right|\le\frac{||f''||}{2}\beta(x)+8\Omega(f'',\delta_1)\left(1+\beta(x)\right),$$

where

$$\beta(x)=\frac{3x^2(1+\alpha)}{n^2\alpha\rho}+\frac{x^3(1+\alpha)(2+\alpha)}{n\alpha^2\rho}+\frac{x(1+n^2)}{n^3\rho^2}$$

and

$$\begin{aligned}
\delta_1^4(x) &= \frac{15x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(5+\alpha)}{n^3\alpha^4\rho^3}\\
&\quad +\frac{10x^4(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(13+15\rho)}{n^4\alpha^3\rho^4}
\end{aligned}$$

$$+\frac{15x^3\left(n^2\rho^2(1+\alpha)(2+\alpha)+(1+\alpha)(2+\alpha)(8+\rho(52+25\rho))\right)}{n^5\alpha^2\rho^5}$$
$$+\frac{5x^2\left((1+\alpha)(18+5\rho)(4+9\rho)+n^2\rho(1+\alpha)(26+9\rho)\right)}{n^6\alpha^4\rho^5}$$
$$+\frac{5x(24+\rho(26+3\rho)\left(1+n^2\right)}{n^7\rho^5}.$$

Proof. We have

$$d^{L_{n,k,\rho}}=L_{n,k,\rho}(e_1)=\frac{k}{n},\quad \vartheta_2^{L_{n,k,\rho}}\ :=\ \frac{k}{n^2\rho}$$

and

$$\begin{aligned}
\vartheta_6^{L_{n,k,\rho}} &:= L_{n,k,\rho}\left(e_1-d^{L_{n,k,\rho}}e_0\right)^6\\
&= L_{n,k,\rho}(e_6)-6L_{n,k,\rho}(e_5)\left(\frac{k}{n}\right)+15G_{n,k}(e_4)\left(\frac{k}{n}\right)^2-20L_{n,k,\rho}(e_3)\left(\frac{k}{n}\right)^3\\
&\quad +15L_{n,k,\rho}(e_2)\left(\frac{k}{n}\right)^4-6L_{n,k,\rho}(e_1)\left(\frac{k}{n}\right)^5+L_{n,k,\rho}(e_0)\left(\frac{k}{n}\right)^6\\
&= \frac{5k\left(24+k\rho(26+3k\rho)\right)}{n^6\rho^5}.
\end{aligned}$$

Applying Lemma 2.1 and Remark 2.1, we have

$$\begin{aligned}
\beta(x) &:= \sum_{k=0}^{\infty}m_{n,k}^{(\alpha)}(x)\left\{\left(1+\left(d^{F_{n,k}}\right)^2\right)\vartheta_2^{F_{n,k}}+\left(1+\left(d^{L_{n,k,\rho}}\right)^2\right)\vartheta_2^{L_{n,k,\rho}}\right\}\\
&= \sum_{k=0}^{\infty}m_{n,k}^{(\alpha)}(x)\left(1+\left(\frac{k}{n}\right)^2\right)\left(\frac{k}{n^2\rho}\right)\\
&= \frac{3x^2(1+\alpha)}{n^2\alpha\rho}+\frac{x^3(1+\alpha)(2+\alpha)}{n\alpha^2\rho}+\frac{x(1+n^2)}{n^3\rho^2},
\end{aligned}$$

$$\begin{aligned}
\delta_1^4(x) &= \left(\sum_{k=0}^{\infty}m_{n,k}^{(\alpha)}(x)\left\{\left(1+\left(d^{F_{n,k}}\right)^2\right)\vartheta_6^{F_{n,k}}+\left(1+\left(d^{L_{n,k,\rho}}\right)^2\right)\vartheta_6^{L_{n,k,\rho}}\right\}\right)\\
&= \frac{15x^5(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(5+\alpha)}{n^3\alpha^4\rho^3}\\
&\quad+\frac{10x^4(1+\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(13+15\rho)}{n^4\alpha^3\rho^4}\\
&\quad+\frac{15x^3\left(n^2\rho^2(1+\alpha)(2+\alpha)+(1+\alpha)(2+\alpha)(8+\rho(52+25\rho))\right)}{n^5\alpha^2\rho^5}\\
&\quad+\frac{5x^2\left((1+\alpha)(18+5\rho)(4+9\rho)+n^2\rho(1+\alpha)(26+9\rho)\right)}{n^6\alpha^4\rho^5}\\
&\quad+\frac{5x(24+\rho(26+3\rho)\left(1+n^2\right)}{n^7\rho^5}.
\end{aligned}$$

Applying Theorem 2.2, we get the required result. $\square$

5. Generalized Szász–Kantorovich operators

Kajla, Araci, and M. Goyal [24] introduced the Kantorovich modification of the operators (1.1) as

$$\begin{aligned}
\mathcal{K}_n^{(\alpha)}(f;x) &= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\mathcal{M}_{n,k}(f)\\
&= (n+1)\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,
\end{aligned}$$

where $M_{n,k}(f) = (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt$.

We study the quantitative estimate for difference of generalized Szász and generalized Szász–Kantorovich operators.

Theorem 5.1. *Let $f^{(j)} \in C_B[0,\infty), j = \overline{0,2}$ and let $x \in [0,\infty)$; then for $n \in \mathbb{N}$, we have*

$$\begin{aligned}
\left|\left(\mathcal{G}_n^{(\alpha)} - \mathcal{K}_n^{(\alpha)}\right)(f;x)\right| &\leq \frac{1}{24(n+1)^2}||f''||\\
&+\frac{1}{2}\,\omega\left(f'', \frac{1}{80(n+1)^4}\right)\left(1+\frac{1}{12(n+1)^2}\right)\\
&+2\omega\left(f, \left(\frac{x^2(1+\alpha)}{(n+1)^2\alpha} + \frac{x(1-n)}{n(n+1)^2} + \frac{1}{4(n+1)^2}\right)\right).
\end{aligned}$$

Proof. Applying Theorem 2.1, we have

$$\begin{aligned}
d^{M_{n,k}} &= M_{n,k}(e_1) = \frac{2k+1}{2(n+1)}\\
\vartheta_2^{M_{n,k}} &:= M_{n,k}\left(e_1 - d^{M_{n,k}}e_0\right)^2\\
&= M_{n,k}(e_2) + \left(\frac{2k+1}{2(n+1)}\right)^2 - 2M_{n,k}(e_1)\left(\frac{2k+1}{2(n+1)}\right)\\
&= \frac{1}{12(n+1)^2}.
\end{aligned}$$

Using Lemma 2.1, we have

$$\begin{aligned}
\beta(x) &= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left(\vartheta_2^{F_{n,k}} + \vartheta_2^{M_{n,k}}\right)\\
&= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\frac{1}{12(n+1)^2} = \frac{1}{12(n+1)^2}.
\end{aligned}$$

Further,

$$\begin{aligned}
\vartheta_4^{M_{n,k}} &:= M_{n,k}\left(e_1 - d^{M_{n,k}}e_0\right)^4\\
&= M_{n,k}(e_4) - 4M_{n,k}(e_3)\left(\frac{2k+1}{2(n+1)}\right) + 6M_{n,k}(e_2)\left(\frac{2k+1}{2(n+1)}\right)^2
\end{aligned}$$

$$-4G_{n,k}(e_1)\left(\frac{2k+1}{2(n+1)}\right)^3+M_{n,k}(e_0)\left(\frac{2k+1}{2(n+1)}\right)^4$$
$$= \frac{1}{80(n+1)^4}.$$

Then using Lemma 2.1, we get

$$\begin{aligned}\delta_1^2(x) &= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left(\vartheta_4^{F_{n,k}}+\vartheta_4^{M_{n,k}}\right)=\sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\vartheta_4^{M_{n,k}}\\ &= \frac{1}{80(n+1)^4}\end{aligned}$$

and

$$\begin{aligned}\delta_2^2(x) &= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left(d^{F_{n,k}}-d^{M_{n,k}}\right)^2\\ &= \sum_{k=0}^{\infty} m_{n,k}^{(\alpha)}(x)\left[\frac{k}{n}-\left(\frac{2k+1}{2(n+1)}\right)\right]^2\\ &= \frac{x^2(1+\alpha)}{(n+1)^2\alpha}+\frac{x(1-n)}{n(n+1)^2}+\frac{1}{4(n+1)^2}.\end{aligned}$$

□

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