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SOME CLASSES OF PROBABILISTIC INNER PRODUCT SPACES AND RELATED INEQUALITIES

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Abstract. We give a new definition for probabilistic inner product spaces, which is sufficiently general to encompass the most important class of probabilistic inner product spaces (briefly, PIP spaces). We have established certain classes of PIP spaces and especially, illustrated that how to construct a real inner product from a Menger PIP space. Finally, we have established the analogous of Cauchy–Schwarz inequality in this general PIP spaces.

1. Introduction and preliminaries

A distribution function (briefly, a d.f.) [2, 3, 8, 9] is a function F from the extended real line $\bar{\mathbb{R}} = [-\infty, +\infty]$ into the unit interval $I = [0, 1]$ that is nondecreasing and satisfies $F(-\infty) = 0$ and $F(+\infty) = 1$.

The set of all d.f.'s will be denoted by Δ and the subset of all F's in Δ satisfying $F(0) = 0$ will be denoted by Δ^+. The sets Δ and Δ^+ are partially ordered by the usual point wise partial ordering of functions: ε_∞ is the minimal element of both Δ and Δ^+, $\varepsilon_{-\infty}$ is the maximal element of Δ, and ε_0 is the maximal element of Δ^+.

For every $a \in]-\infty, +\infty[$, the function

$$\varepsilon_a(x) := \begin{cases} 0, & x \in]-\infty, a], \\ 1, & x \in]a, +\infty[, \end{cases}$$

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is a distribution function and the function ε_∞ is defined by

$$\varepsilon_\infty(t) := \begin{cases} 0, & x \in]-\infty, +\infty[, \\ 1, & x = +\infty. \end{cases}$$

The space Δ^+ can be metrized in several ways [9], but we shall here adopt the Sibley metric d_S. Let F and G be d.f.'s, let h be in $]0,1[$, and let $(F,G;h)$ denote the condition:

$$G(x) \leq F(x+h) + h \quad \text{for every } x \in \left]0, \frac{1}{h}\right[.$$

Then the Sibley metric [9] d_S is defined by

$$d_S(F,G) := \inf\{h \in]0,1[: \text{both } (F,G;h) \text{ and } (G,F;h) \text{ hold}\}.$$

In particular, under the usual pointwise ordering of functions, ε_0 is the maximal element of Δ^+.

A triangle function [6,7] is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ that is associative, commutative, and nondecreasing in each place and has ε_0 as identity, this is, for all F, G and H in Δ^+, we have

(TF1): $\tau(\tau(F,G),H) = \tau(F,\tau(G,H))$,
(TF2): $\tau(F,G) = \tau(G,F)$,
(TF3): $F \leq G \Longrightarrow \tau(F,H) \leq \tau(G,H)$,
(TF4): $\tau(F,\varepsilon_0) = \tau(\varepsilon_0,F) = F$.

Moreover, a triangle function is continuous if it is continuous in the metric space (Δ^+, d_S).

Typical continuous triangle functions [7] are

$$\tau_T(F,G)(x) = \sup_{s+t=x} \{T(F(s),G(t))\}$$

and

$$\tau_{T^*}(F,G) = \inf_{s+t=x} \{T^*(F(s),G(t)\}.$$

Here T is a continuous t-norm, that is, a continuous binary operation on $[0,1]$ that is commutative, associative, nondecreasing in each variable and has 1 as identity; T^* is a continuous t-conorm, namely a continuous binary operation on $[0,1]$ which is related to the continuous t-norm T through $T^*(x,y) = 1 - T(1-x, 1-y)$. Let us recall among the triangular function one has the function defined via $T(x,y) = \min(x,y) = M(x,y)$ and $T^*(x,y) = \max(x,y)$ or $T(x,y) = \Pi(x,y) = xy$ and $T^*(x,y) = \Pi^*(x,y) = x + y - xy$.

Definition 1.1 ([7, 12])**.** A probabilistic normed space, which will henceforth be called briefly a PN space, is a quadruple (V, ν, τ, τ^*), where V is a linear space, τ and τ^* are continuous triangle functions, and the mapping $\nu : V \to \triangle^+$ satisfies, for all p and q in V, the conditions

(N1): $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);
(N2): For all $p \in V \quad \nu_{-p} = \nu_p$;
(N3): $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;
(N4): For all $\alpha \in [0,1] \quad \nu_p \leq \tau^*\left(\nu_{\alpha p,}, \nu_{(1-\alpha)p}\right)$.

The function ν is called the probabilistic norm. If (V, ν, τ, τ^*) satisfies the condition, weaker than (N1),

$$\nu_\theta = \epsilon_0,$$

then it is called a probabilistic pseudo-normed space (briefly, a PPN space). If $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t-norm T and its t-conorm T^*, then $(V, \nu, \tau_T, \tau_{T^*})$ is denoted by (V, ν, T) and is a Menger PN space.

Definition 1.2 ([7])**.** A probabilistic normed space of Šerstnev (briefly a Šerstnev space) is a triple (V, ν, τ), where V is a (real or complex) linear space, ν is a mapping from V into $\triangle^+$, and τ is a continuous triangle function and the following conditions are satisfied for all p and q in V:

(N1) $\nu_p = \epsilon_0$ if and only if $p = \theta$ (θ is the null vector in V);

(N2) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;

(Š) For all $\alpha \in \mathbb{R}\backslash\{0\}$, for all $x \in \overline{\mathbb{R}}_+$ $\nu_{\alpha p}(x) = \nu_p\left(\dfrac{x}{\alpha}\right)$.

Notice that condition (Š) implies

(N3) $\nu_{-p} = \nu_p$ for all $p \in V$.

Example 1.3 ([7])**.** Let $(V, \|.\|)$ be a normed space, and define $\nu_p := \varepsilon_{\|p\|}$. Let τ be a triangle function such that

$$\tau\left(\varepsilon_a, \varepsilon_b\right) = \varepsilon_{a+b}$$

for all $a, b \geq 0$ and let τ^* be a triangle function with $\tau \leq \tau^*$. For instance, it suffices to take $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, where T is a continuous t-norm and T^* is its t-conorm. Then (V, ν, τ, τ^*) is a PN space.

Example 1.4 ([7])**.** Let $(V, \|.\|)$ be a real normed space. Define a map $\nu : V \to \triangle^+$ by

$$\nu_p(t) = \begin{cases} \dfrac{t}{t + \|p\|} & \text{if } t \in]0, +\infty[, \\ 1, & t = +\infty, \end{cases}$$

and triangle functions defined by $\Pi_\Pi\left(F, G\right)(t) = F(t)G(t)$ and $\Pi_{\Pi^*}\left(F, G\right)(t) = F(t) + G(t) - F(t)G(t)$. Then $(V, \nu, \Pi_\Pi, \Pi_{\Pi^*})$ is a PN space, called the canonical PN space associated with the normed space $(V, \|.\|)$.

Let $F \in \triangle^+$; then define $F^\wedge(t) = \sup\{u \in (0, +\infty); F(u) < t\}$. The triangle function τ_M is constructed through the left-continuous t-norm M via

$$\tau_M(F, G)(x) := \sup\{M(F(u), G(v)) | \mathrm{Sum}(u, v) = x\}. \tag{1.1}$$

If F and G are strict increasing d.f.'s, then the supremum on the right-hand side of (1.1) is attained precisely when $F(u) = G(v)$. Turning this observation around, we see that for any t in $[0, 1]$, there exist unique values u_t and v_t such that $F(u_t) = G(v_t) = t$ and $\tau_M(F, G)(u_t + v_t) = t$. Inverting, one has

$$[\tau_M(F, G)]^{-1}(t) = u_t + v_t = F^{-1}(t) + G^{-1}(t),$$

whence

$$[\tau_M(F, G)]^{-1} = F^{-1} + G^{-1}. \tag{1.2}$$

Display (1.2) remains valid for any F, G in Δ^+, that is,

$$[\tau_M(F,G)]^\wedge = F^\wedge + G^\wedge, \tag{1.3}$$

from which we have

$$\tau_M(F,G) = [F^\wedge + G^\wedge]^\wedge. \tag{1.4}$$

Equation (1.4) shows that the operation τ_M in Δ^+ is equivalent to point-wise addition on the space of (left-continuous) quasi-inverses. Since the latter operation is simpler than the former, this is a useful result, applied in what follows.

Next, since $F^\wedge$ and $G^\wedge$ are nondecreasing, we may write

$$F^\wedge(x) + G^\wedge(x) = \inf\{\mathrm{Sum}(F^\wedge(u), G^\wedge(v)) | M(u,v) = x\}. \tag{1.5}$$

The expressions on the right-hand sides of (1.1) and (1.5) are dual in the sense that each may be obtained from the other as follows: Interchange M and Sum, interchange sup and inf, and replace functions by their quasi-inverses. Furthermore, (1.3) and (1.4) show that each expression is the quasi-inverse of the other. These observations, together with the definition of $\tau_{T,L}$ by

$$\tau_{T,L}(F,G)(x) = \sup\{T(F(u), G(v)) | L(u,v) = x\},$$

suggest that the foregoing relationships remain valid when M is replaced by any continuous t-norm T and Sum by any L.

Schweizer and Sklar [9] introduced the first definition of probabilistic inner product spaces (i.e., briefly PIP spaces) in their book “Probabilistic Metric Spaces” [9], but the biggest challenge for us and other mathematicians was how to give a concise, logical and functional definition for PIP spaces. In 1986, Zhang [1] proposed a modified definition of PIP spaces and established some convergence theorems, the Schwarz inequality, and the orthogonal properties of PIP spaces. Zhang defined the probabilistic inner product in the following way.

Definition 1.5 ([1])**.** A probabilistic inner product space is a triplet $(S, \mathcal{F}, \delta)$, where S is a real vector space, δ is a t-norm, and $\mathcal{F}$ is a mapping from $S \times S \to \Delta^+$ ($\mathcal{F}_{p,q}(x)$ denotes the value of $\mathcal{F}_{p,q}$ at $x \in \mathbb{R}$) satisfying the following conditions: For all $p, q, r \in S$ and $\alpha \in \mathbb{R}$

(P-1) $\mathcal{F}_{p,q}(0) = 0$;
(P-2) $\mathcal{F}_{p,q} = \mathcal{F}_{q,p}$;
(P-3) $\mathcal{F}_{p,p}(x) = \varepsilon_0(t)$ for all $x \in \mathbb{R}$ if and only if $p = 0$;
(P-4) $\mathcal{F}_{p,q}(x) := \begin{cases} \mathcal{F}_{p,q}\left(\dfrac{t}{\alpha}\right), & \alpha < 0, \\ \varepsilon_0(x), & \alpha = 0, \\ 1 - \mathcal{F}_{p,q}\left(\dfrac{t}{\alpha}+\right), & x > 0, \end{cases}$

where $\mathcal{F}_{p,q}\left(\dfrac{t}{\alpha}+\right)$ is the right-hand limit of $\mathcal{F}_{p,q}$ at $\dfrac{t}{\alpha}$;

(P-5) $\mathcal{F}_{p+q,r}(x) = \sup\limits_{u+v=x; u,v\in\mathbb{R}} \Delta\left(\mathcal{F}_{p,r}(u), \mathcal{F}_{q,r}(v)\right).$

Many authors tried to develop various analogue results in PIP space with the definition of (1.1), but they could not achieve it to a possible extent. Motivated by this definition and the idea used by Zhang [1], the authors of this paper define the generalized probabilistic inner product spaces with the help of a t-norm and a t-conorm.

Definition 1.6 ([7]). A copula is a function $C : [0,1]^2 \to [0,1]$ that satisfies the following conditions:

(C1): $C(0,t) = C(t,0) = 0$ and $C(1,t) = C(t,1) = t$ for every $t \in [0,1]$;
(C2): C is 2-increasing, that is, for all s, s', t and t' in $[0,1]$, with $s \leq s'$ and $t \leq t'$, we have

$$C(s',t') - C(s',t) - C(s,t') + C(s,t) \geq 0.$$

Definition 1.7 ([7]). For every $F \in \Delta$, the d.f. $\overline{F} \in \Delta$ is defined as

$$\overline{F}(t) := \ell^-(1 - F(t))$$

for every $t \in \mathbb{R}$. Note that $\overline{\overline{F}} = F$ for every $F \in \Delta$ and that $F = \overline{F}$ if and only if F is symmetric.

Lemma 1.8 ([7]). *For a d.f.* $F \in \Delta, \tau_M(F, \bar{F}) = \epsilon_0$ *if and only if* $F = \epsilon_c$ *for some* $c \in \mathbb{R}$

Theorem 1.9 ([7]). *For a pair* (V, ν) *that satisfies conditions* (N1) *and* (N2), *the following statements are equivalent:*

(a) (V, ν) *satisfies also condition* (Š);
(b) *For all* $p \in V$ *and for all* $\alpha \in [0,1]$, *we have*

$$\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p}).$$

2. Probabilistic inner product spaces

In this section, we have redefined the probabilistic inner product space given by Alsina, and Schweizer, Sempi, and Sklar [4]. This new class of probabilistic inner product spaces will help us to find various analogous results in PIP spaces from our classical inner product spaces and we provide many examples.

The first thing to notice when going from real inner product to probabilistic inner product is that since real inner products can assume negative values, we need to deal with distribution functions (d.f.'s) in Δ rather than with d.f.'s confined to the subspace Δ^+.

Definition 2.1 ([7]). A multiplication on Δ is a binary operation τ on Δ that is commutative, associative, nondecreasing on each place, and whose restriction to Δ^+ is a triangle function.

Multiplications of particular interest to us in this paper are the extensions of the functions τ_T and τ_S [7,9] defined on $\Delta \times \Delta$ by

$$\tau_T(F,G)(x) := \sup_{u+v=x} T[F(u), G(v)]$$

and

$$\tau_S(F,G)(x) := l^-\left(\inf_{u+v=x} S[F(u), G(v)]\right),$$

respectively. Here T is a continuous t-norm and S is a continuous t-conorm and for any $F \in \Delta, l^- F$ is the left continuous normalization of F, that is, $l^- F(x) = F(x^-)$ for every $x \in \mathbb{R}$.

Definition 2.2. A probabilistic inner product space (briefly, PIP space) is a quadruple $(V, \mathcal{G}, \tau, \tau^*)$, where V is a real linear space, τ and τ^* are multiplications on Δ such that $\tau \leq \tau^*$ and $\mathcal{G}$ is a mapping from $V \times V$ into Δ such that, if $\mathcal{G}_{p,q}$ denotes the value of $\mathcal{G}$ at the point (p, q) (i.e., given any point $x \in \mathbb{R}$, the value $\mathcal{G}_{p,q}(x)$ is interpreted as the probability that the inner product of p and q is less than x) and if the function $\nu : V \to \Delta^+$ is defined via:

The probabilistic version of usual case where the norm satisfies $\|p\|^2 = \langle p, p \rangle$ leads to axiom (P0), because

$$\nu_p(x) = P(\|p\| < x) = P\left(\langle p, p \rangle < x^2\right) = G_{p,p}(x^2).$$

(P0) $\nu_p(x) := \begin{cases} \mathcal{G}_{p,p}(x^2), & x > 0, \\ 0, & x \leq 0. \end{cases}$

For all $p, q, r \in V$, the following conditions hold:

(P1) $\mathcal{G}_{p,p} \in \Delta^+$ and $\mathcal{G}_{0,0} = \varepsilon_0$, where θ is the null vector in V;
(P2) $\mathcal{G}_{p,p} \leq \varepsilon_0$ if $p \neq \theta$;
(P3) $\mathcal{G}_{\theta,p} = \varepsilon_0$;
(P4) $\mathcal{G}_{p,q} = \mathcal{G}_{q,p}$;
(P5) $\mathcal{G}_{p,q}(x) = \bar{\mathcal{G}}_{p,q}(x) = l^-(1 - \mathcal{G}_{p,q}(-x))$ for all $x \in \mathbb{R}$;
(P6) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;
(P7) $\nu_p \leq \tau^*\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)$ for every $\alpha \in I = [0, 1]$;
(P8) $\tau(\mathcal{G}_{p,r}, \mathcal{G}_{q,r}) \leq \mathcal{G}_{p+q,r} \leq \tau^*(\mathcal{G}_{p,r}, \mathcal{G}_{q,r})$;
(P9) If $\{x_n\}$ converges to x and $\{y_n\}$ converges to y, then $\mathcal{G}_{x_n,y_n}$ converges to $\mathcal{G}_{x,y}$.

If $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t-norm T and its associated t-conorm T^*, then $(V, \mathcal{G}, \tau_T, \tau_{T^*})$ is a Menger PIP space, which we denote by $(V, \mathcal{G}, T)$. If $\tau^* = \tau_M$ and equality holds in (P7), then $(V, \mathcal{G}, \tau_T, \tau_M)$ is a Šerstnev PIP space. If (P1) and (P3)–(P8) are satisfied, then $(V, \mathcal{G}, \tau_T, \tau_{T^*})$ is a probabilistic pseudo-inner product space.

It is immediate that $(V, \nu, \tau_T, \tau_{T^*})$ is a probabilistic norm and we shall refer to ν as a probabilistic norm derived from a probabilistic inner product $\mathcal{G}$. Note that, if $(V, \mathcal{G}, \tau_T, \tau_M)$ is a Šerstnev PIP space, then in view of the fact that $\nu_{-p} = \nu_p$, and (P7) may be replaced by

$$\nu_{\lambda p}(x) = \nu_p\left(\frac{x}{|\lambda|}\right),$$

for all λ and x in $\mathbb{R}$, where by convention, $\nu_p\left(\frac{x}{0}\right) = \varepsilon_0(x)$.

Definition 2.3. Let $(V, \mathcal{G}, \tau, \tau^*)$ be a PIP space. Then, a sequence $\{x_n\}$ in X is said to converge to $x \in X$ if for every $\varepsilon > 0$ and for every $\alpha \in (0, 1]$, there exists $k \in \mathbb{N}$ such that $\mathcal{G}_{x_n - x, x_n - x}(\varepsilon) > 1 - \alpha$.

Definition 2.4. Let $(V, \mathcal{G}, \tau, \tau^*)$ be a PIP space. Then a sequence $\{x_n\}$ in V is said to be a Cauchy sequence in X if for every $\varepsilon > 0$ and for every $\alpha \in (0, 1]$, there exists $k \in \mathbb{N}$ such that $\mathcal{G}_{x_n - x_m, x_n - x_m}(\varepsilon) > 1 - \alpha$ when $m, n \geq k$.

Definition 2.5. A PIP space $(V, \mathcal{G}, \tau, \tau^*)$ is said to be probabilistic Hilbert space if every Cauchy sequence converges in V.

The following theorems establish a new class of probabilistic inner product spaces.

Theorem 2.6. *Let $(\mathbb{R}^n, \langle ., . \rangle)$ be an inner product space. Then the quadruple $(\mathbb{R}^n, \mathcal{G}, \tau_\Pi, \tau_{\Pi^*})$, where*

$$\mathcal{G}_{p,q}(x) := \begin{cases} 0, & x < 0, \\ e^{-\sqrt{\langle p,q \rangle}}, & x \in]0, +\infty[, \\ 1, & x = +\infty, \end{cases}$$

and

$$\nu_p(x) := \begin{cases} 0, & x \leq 0, \\ e^{-\|p\|}, & x \in]0, +\infty[\ , \\ 1, & x = +\infty, \end{cases}$$

is a Menger probabilistic inner product space for $\tau_\Pi = \Pi_\Pi$ and $\tau_{\Pi^*} = \Pi_{\Pi^*}$.

Proof. We have $\mathcal{G}_{p,q}(x) = e^{-\sqrt{\langle p,q \rangle}}$ implies $\nu_p(x) = e^{-\sqrt{\langle p,p \rangle}} = e^{-\|p\|}$ for $x \in]0, +\infty[$.

(P6) One can see that, $\nu_{p+q} = e^{-\|p+q\|} \geq e^{-\|p\| - \|q\|}$ and $\Pi_\Pi(\nu_p, \nu_q) = e^{-\|p\|}e^{-\|q\|} = e^{-\|p\| - \|q\|}$ implies that $\nu_{p+q} \geq \Pi_\Pi(\nu_p, \nu_q)$.

(P7) We have to show that $\nu_p \leq \tau^*\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)$ for every $\alpha \in I = [0, 1]$. Therefore, $\nu_p = e^{-\|p\|} \leq e^{-\alpha\|p\|} + e^{-(1-\alpha)\|p\|} - e^{-\|p\|}$ if and only if $2e^{-\|p\|} \leq e^{-\alpha\|p\|} + e^{-(1-\alpha)\|p\|}$. Multiplying both sides by $e^{\|p\|}$, we get $2 \leq e^{\alpha\|p\|} + e^{(1-\alpha)\|p\|}$, which is true.

(P8) We have $\mathcal{G}_{p+q,r} = e^{-\sqrt{\langle p+q,r \rangle}} \geq e^{-\sqrt{\langle p,r \rangle}}e^{-\sqrt{\langle q,r \rangle}}$, which is true, because $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ in $\mathbb{R}^+$.

Now it is necessary to see that $\mathcal{G}_{p+q,r} \leq \Pi_{\Pi^*}(\mathcal{G}_{p,r}, \mathcal{G}_{q,r})$, and one has

$$\mathcal{G}_{p+q,r} = e^{-\sqrt{\langle p+q,r \rangle}} \leq e^{-\sqrt{\langle p,r \rangle}} + e^{-\sqrt{\langle q,r \rangle}} - e^{-\left(\sqrt{\langle p,r \rangle} + \sqrt{\langle q,r \rangle}\right)}$$

implies that

$$e^{-\sqrt{\langle p,r \rangle + \langle q,r \rangle}} + e^{-\sqrt{\langle p,r \rangle} - \sqrt{\langle q,r \rangle}} \leq e^{-\sqrt{\langle p,r \rangle}} + e^{-\sqrt{\langle q,r \rangle}}.$$

Multiplying both sides by $e^{\sqrt{\langle p,r \rangle + \langle q,r \rangle}}$, we get

$$\begin{aligned} 1 + e^{-\sqrt{\langle p,r \rangle} - \sqrt{\langle q,r \rangle} + \sqrt{\langle p,r \rangle + \langle q,r \rangle}} &\leq e^{-\sqrt{\langle p,r \rangle} + \sqrt{\langle p,r \rangle + \langle q,r \rangle}} + e^{-\sqrt{\langle q,r \rangle} + \sqrt{\langle p,r \rangle + \langle q,r \rangle}} \\ &\leq e^{-\sqrt{\langle p,r \rangle}} + e^{-\sqrt{\langle q,r \rangle}}, \end{aligned}$$

which is true.

Finally, as a consequence, the quadruple $(\mathbb{R}^n, \mathcal{G}, \tau_\Pi, \tau_{\Pi^*})$ is a Menger PIP space with a real inner product. □

The next theorem illustrates how we can induce a real inner product from a Menger probabilistic inner product. Then one can say that every Menger PIP space is a real inner product space.

Theorem 2.7. *Let $(V, \mathcal{G}, M)$ be a Menger PIP space. Then V is a real inner product space. That is, there exists a real inner product $\langle ., . \rangle : V \times V \to \mathbb{R}$ such that $\mathcal{G}_{p,q} = \varepsilon_{\langle p,q \rangle}$ for all $p, q \in V$.*

Proof. Since $\tau_M = \tau_{M^*}$, it follows from P(8) that, for all $p.q, r \in V$,

$$\mathcal{G}_{p+q,r} = \tau_M \left(\mathcal{G}_{p,r}, \mathcal{G}_{q,r} \right).$$

Letting $q = -p$ and using P(3) and P(5) yield

$$\varepsilon_0 = \tau_M \left(\mathcal{G}_{p,r}, \bar{\mathcal{G}}_{p,r} \right).$$

By Lemma 1.8, for each distribution function $F \in \Delta, \tau_M \left(F, \bar{F} \right) = \varepsilon_0$ if and only if $F = \varepsilon_c$ for some $c \in \mathbb{R}$. Applying Lemma 1.8, one has $\mathcal{G}^{\wedge}_{p,r} = c$.

Now, define $\langle ., . \rangle : V \times V \to \mathbb{R}$ by $\langle p, q \rangle = \mathcal{G}^{\wedge}_{p,q}$ for all $p, q \in V$. Notice that by duality, one has

$$\langle p + q, r \rangle = \mathcal{G}^{\wedge}_{p+q,r} = \left[\tau_M \left(\mathcal{G}_{p,r}, \mathcal{G}_{q,r} \right) \right]^{\wedge} = \mathcal{G}^{\wedge}_{p,r} + \mathcal{G}^{\wedge}_{q,r} = \langle p, r \rangle + \langle q, r \rangle.$$

This completes the proof. □

Theorem 2.8. *The quadruple $(\mathbb{R}^2, \langle ., . \rangle, F, M)$ is a simple space generated by $(\mathbb{R}^2, \langle ., . \rangle)$ and F is a d.f. different from ε_0 and ε_∞. Moreover $\mathbb{R}^2$ is a Menger PIP space under M and a Šerstnev space. For $p, q, r \in \mathbb{R} \setminus \{0\}$, define*

$$\nu_p(t) := F\left(\frac{t}{\sqrt{\langle p, p \rangle}} \right) \text{ and } \mathcal{G}_{p,q}(x) = F\left(\frac{x}{\sqrt{p, q}} \right) \text{ for } x > 0 \text{ and } \nu_\theta = \varepsilon_0.$$

Proof. (P6) By the property of duality, we have

$$\begin{aligned} \left[\tau_M \left(\nu_p, \nu_q \right) \right]^{\wedge} &= \nu_p^{\wedge} + \nu_q^{\wedge} \\ &= \|p\| F^{\wedge} + \|q\| F^{\wedge} \\ &= \left(\|p\| + \|q\| \right) F^{\wedge} \\ &\geq \|p + q\| F^{\wedge} = \nu^{\wedge}_{p+q}. \end{aligned}$$

Finally one has $\nu_{p+q} \geq \tau_M \left(\nu_p, \nu_q \right)$. Since, $p = \alpha p + (1 - \alpha) p$, by (P6), one has $\nu_p \geq \tau_M \left(\nu_{\alpha p}, \nu_{(1-\alpha)p} \right)$ for $\alpha \in [0, 1]$.

From (P6) and (P7), we have $\nu_p(t) = \tau_M \left(\nu_{\alpha p}, \nu_{(1-\alpha)p} \right)$. As a consequence, $(\mathbb{R}^2, \langle ., . \rangle, F, M)$ is a Šerstnev space.

P(8) One can see that

$$\mathcal{G}^{\wedge}_{p+q,r}(t) = \left[F\left(\frac{t}{\sqrt{\langle p + q, r \rangle}} \right) \right]^{\wedge} = \sqrt{\langle p + q, r \rangle} \leq \left(\sqrt{\langle p, r \rangle} + \sqrt{\langle q, r \rangle} \right) F^{\wedge}.$$

So, $\mathcal{G}_{p+q,r}(t) \geq \tau_M \left(\mathcal{G}_{p,r}, \mathcal{G}_{q,r} \right)$ and $\mathcal{G}_{p+q,r}(t) \leq \tau_{M^*} \left(\mathcal{G}_{p,r}, \mathcal{G}_{q,r} \right)$. Since $\tau_M = \tau_{M^*}$, finally in (P8) the equality holds. □

Theorem 2.9. *Let* $(\mathbb{R}^2, \mathcal{G}, \tau, \Pi_M)$, *where* $\mathbb{R}^2$ *is a real vector space,* τ *and* Π_M *are multiplications in* Δ *such that* $\tau \leq \Pi_M$, *and* $\mathcal{G}$ *is a mapping from* $\mathbb{R}^2 \times \mathbb{R}^2$ *in to* Δ *such that* $\mathcal{G}_{p,q}$ *denotes the value of* $\mathcal{G}$ *at the pair* (p, q), *and if the function* $\nu : \mathbb{R}^2 \times \mathbb{R}^2 \to \Delta$ *defined via,*

$$\nu_p(x) := \begin{cases} \mathcal{G}_{p,p}(x^2), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Then, for all $p, q, r \in \mathbb{R}^2$, *the following results hold:*

(1) $(\mathbb{R}^2, \mathcal{G}, \tau, \Pi_M)$ *is a PIP space with* $\mathcal{G} : \mathbb{R}^2 \times \mathbb{R}^2 \to \Delta$ *by* $(p, q) \longmapsto \mathcal{G}_{p,q}$ *and* $\mathcal{G}_{p,q} := \varepsilon_{\frac{\beta+\sqrt{\langle p,q\rangle}}{\beta}}, \beta \in (0, +\infty)$, *and let* τ *be the triangle function such that* $\tau(\varepsilon_c, \varepsilon_d) \leq \varepsilon_{c+d}$ *for* $c, d > 0$.

(2) $(\mathbb{R}^2, \mathcal{G}, \tau, \Pi_M)$ *is neither a topological vector space nor a Šerstnev space.*

Proof. (1) (P8) We have

$$\begin{aligned} \mathcal{G}_{p+q,r}(t) &= \varepsilon_{\frac{\beta+\sqrt{\langle p,q\rangle}}{\beta}} \\ &= \varepsilon_{\frac{\beta+\sqrt{\langle p,r\rangle+\langle q,r\rangle}}{\beta}} \\ &= \varepsilon_{\frac{\beta+\sqrt{\langle p,r\rangle+\langle q,r\rangle}}{\beta}} \\ &\geq \varepsilon_{\frac{\beta+\sqrt{\langle p,q\rangle}}{\beta}+\frac{\beta+\sqrt{\langle q,r\rangle}}{\beta}} \\ &= \tau(\mathcal{G}_{p,r}, \mathcal{G}_{q,r}). \end{aligned}$$

We need to check $\mathcal{G}_{p+q,r} \leq \Pi_M(\mathcal{G}_{p,r}, \mathcal{G}_{q,r})$. We know that, $\Pi_M \gg \tau$, and as a consequence $\Pi_M \gg \tau$, in Δ^+. It is immediate from the above case. Moreover, $\tau(\mathcal{G}_{p,r}, \mathcal{G}_{q,r}) \leq \Pi_M(\mathcal{G}_{p,r}, \mathcal{G}_{q,r})$, and finally $(\mathbb{R}^2, \mathcal{G}, \tau, \Pi_M)$ is a PIP space.

(2) $(\mathbb{R}^2, \nu, \tau, \Pi_M)$ is neither a topological vector space nor a Šersternev space. It follows from [9, Lemma 7.2.13] that

$$\begin{aligned} \tau_M\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right) &= \tau_M\left(\varepsilon_{\frac{\beta+\alpha\|p\|}{\beta}}, \varepsilon_{\frac{\beta+(1-\alpha)\|p\|}{\beta}}\right) \\ &= \varepsilon_{\frac{2\beta+\|p\|}{\beta}} \\ &\leq \varepsilon_{\frac{\beta+\|p\|}{\beta}} = \nu_p. \end{aligned}$$

As a consequence, for every $t \in \left[1 + \frac{\|p\|}{\beta}, 2 + \frac{\|p\|}{\beta}\right]$, one has

$$\tau_M\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)(t) < \nu_p(t),$$

so that the space considered is not a Šersternev space. In other hand, let $\{\alpha_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} \alpha_n = 0$. Then, for every $p \neq \theta$, we have

$$\lim_{n\to+\infty} \nu_{\alpha_n p} = \lim_{n\to+\infty} \varepsilon_{\frac{\beta+\|p\|}{\beta}} = \varepsilon_1 \neq \varepsilon_0.$$

Therefore, $(\mathbb{R}^2, \nu, \tau, \Pi_M)$ is not a topological vector space. □

2.1. **EN spaces.** EN spaces, shortly to be defined, provides an important class of PN spaces. Their importance derives from the role they play in the study of convergence of random variables.

Definition 2.10 ([7])**.** Let $(\Omega, \mathcal{A}, P)$ be a probability space, let $(V, \langle .,. \rangle)$ be a real inner product space, and let S be a set of functions from Ω in to V. Then $(S, \mathcal{G})$ is an EN-space with base $(\Omega, \mathcal{A}, P)$ and target $(V, \langle .,. \rangle)$ if the following conditions holds:

(i) S under pointwise addition and scalar multiplication, is a real linear space. The zero element in S is a constant function θ given by $\theta(\omega) = n$ for all $\omega \in \Omega$, where n is the null vector in V.

(ii) For all $p, q \in V$ and for all $x \in \mathbb{R}$, the set $\{\omega \in \Omega : \langle p(\omega), q(\omega) \rangle < x\}$ belongs to $\mathcal{A}$, that is, the composite function $\langle p, q \rangle$ from Ω in to $\mathbb{R}$ defined by $\langle p, q \rangle(\omega) = \langle p(\omega), q(\omega) \rangle$ is P-measurable, or, in other words, it is a real random variable.

(iii) For all $p, q \in V, \mathcal{G}(p, q)$ is the distribution function of $\langle p, q \rangle$, that is, for all $x \in \mathbb{R}$,

$$\mathcal{G}(p, q)(x) = P\{\omega \in \Omega : \langle p(\omega), q(\omega) \rangle < x\}.$$

If for any $p \in V, \langle p, q \rangle = 0$ a.s. only if $p = \theta$, then $(V, \mathcal{G})$ is a canonical EN-space (E-normed space).

Theorem 2.11. *If $(V, \mathcal{G})$ is an EN-space, then $(V, \mathcal{G}, \tau_W, \tau_{W^*})$ is a pseudo-PIP space. If $(V, \mathcal{G})$ is a canonical EN space, then $(V, \mathcal{G}, \tau_W, \tau_{W^*})$ is a PIP space, that is, $(V, \mathcal{G}, W)$ is a Menger PIP space.*

Proof. The properties from (P1),(P3),(P4), and (P5) are immediate, as is (P2) when $(V, \mathcal{G})$ is canonical.

Next, it follows from the definition of EN spaces that (V, ν) is an E-normed space. As shown in [9], such a space is a pseudo-PN space in the sense of Šerstnev in which $\tau = \tau_W$. Condition (P6) is just the triangle inequality for this space. Since conditions $\nu_{-p} = \nu_p$ and $\nu_p = \tau_M \left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)$ for all $p \in V$ and $\alpha \in (0, 1)$, taken together are equivalent to Šerstnev condition,

$$\nu_{\alpha p}(x) = \nu_p \left(\frac{x}{|\alpha|} \right) \text{ for all } p \in V \text{ and } \alpha \in \mathbb{R}.$$

(P7) holds with $\tau = \tau_M$ and since, $\tau_M = \tau_{M^*}$, a fortiory with $\tau = \tau_{W^*}$.

$$\begin{aligned} \mathcal{G}_{p+q,r}(x) &= P\{\omega \in \Omega : \langle p(\omega) + q(\omega, r(\omega)) \rangle < x\} \\ &= P\{\omega \in \Omega : \langle p(\omega), r(\omega) \rangle + \langle q(\omega), r(\omega) \rangle < x\}. \end{aligned}$$

Thus $\mathcal{G}_{p+q,r}$ is the distribution function of the sum of the random variables $\langle p, r \rangle$ and $\langle q, r \rangle$.

(P6) For all $p, q, r \in V$, and for every $t > 0$, let $u, v \in [0, +\infty)$ be such that $u + v = t$. Define the sets A, B, and C by $A = \{\omega \in \Omega : \|p(\omega)\| < u\}$, $B = \{\omega \in \Omega : \|q(\omega)\| < v\}$, and $C = \{\omega \in \Omega : \|p(\omega) + q(\omega)\| < t\}$. Since the norm $\|.\|$, satisfies the triangle inequality, it follows that $A \cup B \subset C$, so that $P(C) \geq P(a \cup B) \geq W\left(P(A), P(B)\right)$.

For every $p \in V$ and for every $t > 0$, define a mapping $\nu : V \to \Delta^+$ via

$$\nu_p(t) := P\{\omega \in \Omega : \|p(\omega)\| < t\}, \tag{2.1}$$

where P is a probability measure in Ω. Therefore, one has $P(A) = \nu_p(u), P(B) = \nu_q(v)$ and $P(C) = \nu_{p+q}(t)$, so that $\nu_{p+q}(t) \geq W\left(\nu_p(u), \nu_q(u)\right)$, and hence

$$\nu_{p+q}(t) = \sup\{W\left(\nu_p(u), \nu_q(v)\right); u + v = t\} = \tau_W\left(\nu_p, \nu_q\right)(t).$$

For every $p \in V$ and for every $t > 0$, one has from equation (2.1) and with $\alpha \in [0, 1]$ that

$$\begin{aligned}
\tau_M\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right) &= \sup\{\nu_{\alpha p}(u) \wedge \nu_{(1-\alpha)p}(t-u) : u \in [0,1]\} \\
&= \sup_{u\in[0,t]}\{P\{|\alpha|\|p(\omega)\| < u\} \wedge P\{|1-\alpha|\|p(\omega)\| < t-u\}\} \\
&= P\left(\|p\| < \sup_{u\in[0,t]}\{\frac{u}{|\alpha|} \wedge \frac{t-u}{|1-\alpha|}\}\right).
\end{aligned}$$

Taking into account, $\dfrac{u}{\alpha} \leq \dfrac{t-u}{1-\alpha}$ if and only if $u \leq \alpha t$, one obtain, for every $t > 0$,

$$\tau_M\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)(t) = \nu_p(t),$$

so that (V, ν, τ_W, τ_M) is a PPN space. Therefore, by virtue of Theorem 1.9, when the PN space (V, ν) is canonical, it is a Šerstnev space under τ_W. It remains to establish (P8).

(P8) Let $C_{\langle p,r\rangle,\langle q,r\rangle}$ be the copula of these random variables, so that $C_{\langle p,r\rangle,\langle q,r\rangle}\left(\mathcal{G}_{p,r}, \mathcal{G}_{q,r}\right)$ is their joint d.f. Then (see [4, 10, 11]), we have

$$\mathcal{G}_{p+q,r} = \sigma_{C_{\langle p,r\rangle,\langle q,r\rangle}(\mathcal{G}_{p,r},\mathcal{G}_{q,r})},$$

where, for any pair of d.f.'s, F and G and any copula C,

$$\sigma_C(F, G) = \iint_{u+v<x} dC\left(F(u), G(v)\right).$$

Next (see [9–11]), for any copula C and for any pair of d.f.'s, F and G,

$$\tau_W(F, G) \leq \sigma_C(F, G) \leq \tau_{W^*}(F, G).$$

This yields (P8), with $\tau = \tau_W$ and $\tau^* = \tau_{W^*}$, and completes the proof. □

Now we prove the analogous of the Cauchy–Schwarz inequality in PIP spaces.

Theorem 2.12. *Let $(S, \mathcal{G}, \tau, \tau^*)$ be a PIP space. For every $p, q \in S$, then*

$$\mathcal{G}_{p,q} \geq \tau\left(\nu_p, \nu_q\right).$$

Proof. For $p, q \in S$ and $t, s \in \mathbb{R}^+$, we have

$$\begin{aligned}
\mathcal{G}_{tp+sq,p+q} &\geq \tau\left(\mathcal{G}_{tp+sq,p}, \mathcal{G}_{tp+sq,q}\right) \\
&\geq \tau\left(\tau\left(\mathcal{G}_{tp,p}, \mathcal{G}_{p,sq}\right), \tau\left(\mathcal{G}_{tp,q}, \mathcal{G}_{sq,q}\right)\right) \\
&\geq \tau\left(\tau\left(\mathcal{G}_{p,p} \circ \frac{j}{t}, \mathcal{G}_{p,q} \circ \frac{j}{s}\right), \tau\left(\mathcal{G}_{p,q} \circ \frac{j}{t}, \mathcal{G}_{q,q} \circ \frac{j}{s}\right)\right) \\
&\geq \tau\left(\tau\left(\mathcal{G}_{p,p} \circ \frac{j}{t}, \mathcal{G}_{q,q} \circ \frac{j}{s}\right), \tau\left(\mathcal{G}_{p,q} \circ \frac{j}{t}, \mathcal{G}_{p,q} \circ \frac{j}{s}\right)\right).
\end{aligned}$$

Since the left side d.f. is in Δ, by letting $\alpha = \max\{t, s\}$, we have

$$\tau\left(\tau\left(\nu_p \circ \frac{j^{1/2}}{\alpha}, \nu_q \circ \frac{j^{1/2}}{\alpha}\right), \mathcal{G}_{p,q} \circ \frac{j}{\alpha}\right) \in \Delta,$$

by the associative property of τ functions. Since

$$\tau\left(\nu_p \circ \frac{j^{1/2}}{\alpha}, \nu_q \circ \frac{j^{1/2}}{\alpha}\right) \in \Delta^+,$$

we conclude that

$$\tau\left(\nu_p \circ \frac{j^{1/2}}{\alpha}, \nu_q \circ \frac{j^{1/2}}{\alpha}\right) \leq \mathcal{G}_{p,q} \circ \frac{j}{\alpha},$$

that is,

$$\tau\left(\nu_p, \nu_q\right) \leq \mathcal{G}_{p,q}.$$

□

Corollary 2.13. *The Cauchy–Schwarz inequality leads to an expression in which only the probabilistic norms in p and q, ν_p and ν_q intervene.*

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