Khayyam J. Math. 6 (2020), no. 2, 193–198
DOI: 10.22034/kjm.2020.109816

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ON THE ARENS REGULARITY OF A MODULE ACTION AND ITS EXTENSIONS

SEDIGHEH BAROOTKOOB[1]

Communicated by H.R. Ebrahimi Vishki

Abstract. It is known that if the second dual $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$ is Arens regular, then $\mathcal{A}$ is Arens regular itself. However, the converse is not true, in general. Young gave an example of an Arens regular Banach algebra whose second dual is not Arens regular. Later Pym has polished Young's example for presenting more applicable examples. In this paper, we mimic the methods of Young and Pym to present examples of some Arens regular bilinear maps and module actions whose some extensions are not Arens regular. Finally, some relationships between the topological centers of certain Banach module actions are investigated.

1. Introduction and preliminaries

Let $f : \mathcal{X}\times\mathcal{Y} \to \mathcal{Z}$ be a bounded bilinear map on normed spaces. Following [1], there are two natural extensions f^{***} and f^{t***t} from $\mathcal{X}^{**}\times\mathcal{Y}^{**}$ to $\mathcal{Z}^{**}$, where the adjoint $f^* : \mathcal{Z}^*\times\mathcal{X} \to \mathcal{Y}^*$ of f is defined by

$$\langle f^*(\zeta, x), y\rangle = \langle \zeta, f(x, y)\rangle \quad (x \in \mathcal{X}, y \in \mathcal{Y}, \zeta \in \mathcal{Z}^*)$$

and $f^{**} = (f^*)^*$ and $f^{***} = (f^{**})^*$. The extension f^{t***t} can be defined similarly, in which $f^t : \mathcal{Y}\times\mathcal{X} \longrightarrow \mathcal{Z}$ is the flip map of f defined by $f^t(y, x) = f(x, y)$ ($x \in \mathcal{X}, y \in \mathcal{Y}$).

It can be readily verified that f^{***} is the unique extension of f that is w^*-separately continuous on $\mathcal{X}\times\mathcal{Y}^{**}$, and f^{t***t} is the unique extension of f that is w^*-separately continuous on $\mathcal{X}^{**}\times\mathcal{Y}$. The left and right topological centers of f are defined by

$$Z_\ell(f) = \{\varrho \in \mathcal{X}^{**} : \quad f^{***}(\varrho, \rho) = f^{t***t}(\varrho, \rho) \text{ for every } \rho \in \mathcal{Y}^{**}\},$$

Date: Received: 26 June 2019; Revised: 24 December 2019; Accepted: 1 January 2020.
2010 Mathematics Subject Classification. Primary 46H20; Secondary 46H25.
Key words and phrases. Bilinear map, Banach algebra, module action, Arens regular.

and

$$Z_r(f) = \{\rho \in \mathcal{Y}^{**} : \quad f^{***}(\varrho, \rho) = f^{t***t}(\varrho, \rho) \text{ for every } \varrho \in \mathcal{X}^{**}\},$$

respectively. Clearly, $\mathcal{X} \subseteq Z_\ell(f), \mathcal{Y} \subseteq Z_r(f)$, and $Z_r(f) = Z_\ell(f^t)$.

A bounded bilinear mapping f is said to be Arens regular if $f^{***} = f^{t***t}$. This is equivalent to $Z_\ell(f) = \mathcal{X}^{**}$ as well as $Z_r(f) = \mathcal{Y}^{**}$. The mapping f is said to be left (right) strongly Arens irregular if $Z_\ell(f) = \mathcal{X}$ ($Z_r(f) = \mathcal{Y}$). The same extensions for a triple map were explicitly studied in [10].

If π is the multiplication of a Banach algebra $\mathcal{A}$, then π^{***} and π^{t***t} are the first and second Arens products on $\mathcal{A}^{**}$, respectively. In this case, we write $Z_\ell(\mathcal{A}^{**})$ and $Z_r(\mathcal{A}^{**})$ instead of $Z_\ell(\pi)$ and $Z_r(\pi)$, respectively. A Banach algebra $\mathcal{A}$ is called Arens regular, left strongly Arens irregular, or right strongly Arens irregular if the multiplication π of $\mathcal{A}$ has the corresponding property.

Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and let $\pi_\ell : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$, $\pi_r : \mathcal{X} \times \mathcal{A} \to \mathcal{X}$ be the left and right module actions of $\mathcal{A}$ on $\mathcal{X}$, respectively. Then $\mathcal{X}^*$ is a Banach $\mathcal{A}$-bimodule with module actions π_r^{t*t} and π_ℓ^*. Similarly, for every positive integer n, $\mathcal{X}^{(n)}$ is a Banach $\mathcal{A}$-bimodule with the left module action $\pi_{\ell_n} : \mathcal{A} \times \mathcal{X}^{(n)} \to \mathcal{X}^{(n)}$ and the right module action $\pi_{r_n} : \mathcal{X}^{(n)} \times \mathcal{A} \to \mathcal{X}^{(n)}$, where $\pi_{\ell_n} = \pi_{r_{n-1}}^{t*t}$ and $\pi_{r_n} = \pi_{\ell_{n-1}}^*$. It is clear that $Z_\ell(\pi_{r_{n-2}}) \subseteq Z_\ell(\pi_{r_n})$ and $Z_r(\pi_{\ell_{n-2}}) \subseteq Z_r(\pi_{\ell_n})$. Furthermore, π_{r_n} and π_{ℓ_n} are extensions of $\pi_{r_{n-2}}$ and $\pi_{\ell_{n-2}}$, respectively. For more information on the relation between topological centers of certain module actions see [6, 11].

Arens regularity and strong Arens irregularity of bilinear maps and certain module operations were investigated by many authors; see, for example, [2–9, 11, 13] and the references therein. Our main aim is to continue the same investigations on some extensions of certain bilinear maps and module actions.

For a Banach algebra $\mathcal{A}$, it is clear that if $\mathcal{A}^{**}$ is Arens regular, then so is $\mathcal{A}$. However, the converse is not true in general. Young [14] gave an example of an Arens regular Banach algebra whose the second dual is not Arens regular. Young's example has polished by Pym [12] to give a more applicable example. In this paper, we mimic the methods of Young and Pym to give an example of an Arens regular bilinear map (module action) whose third adjoint is not Arens regular. Finally, we compare the topological centers of certain module actions.

2. Arens regularity of certain direct products of bilinear maps

In [8, Theorem 2.2] (see also [9, Corollary 5.2]), it has been shown that a Banach algebra $\mathcal{A}$ with a bounded approximate identity is reflexive if and only if π^* is Arens regular. The following result studies the same facts for some other right module actions.

Proposition 2.1. *Let $\mathcal{X}$ be a Banach $\mathcal{A}$-module equipped with the module actions π_r and π_ℓ. Then the following statements hold:*

(i) *If π_{ℓ_1} is Arens regular and $\pi_{r_2}(\varrho, \mathcal{A}) = \mathcal{X}^{**}$, for some $\varrho \in \mathcal{X}^{**}$, then $\mathcal{X}$ is reflexive.*

(ii) *If $\mathcal{A}$ is Arens regular and $\pi_{r_2}(\varrho, \mathcal{A}) = \mathcal{X}^{**}$, for some $\varrho \in \mathcal{X}^{**}$, then π_{r_2} is Arens regular.*

(iii) *If $\mathcal{A}$ has a bounded right approximate identity for $\mathcal{X}$, then $\pi_{r_2}^*$ is Arens regular if and only if $\mathcal{X}$ is reflexive.*

Proof. The assertions (i) and (ii) follow from [13, Theorem 2.2, Theorem 3.2], respectively. For (iii), note that $\pi_r^* = \pi_{r_2}^*|_{\mathcal{X}^*\times\mathcal{X}}$. Therefore Arens regularity of $\pi_{r_2}^*$ implies it for π_r^*, so by [11, Proposition 3.6] $\mathcal{X}$ is reflexive. The converse is clear. □

Let $f : \mathcal{X} \times \mathcal{Y} \to Z$ be an Arens regular bilinear map and let $\mathcal{M}, \mathcal{N}$, and $\mathcal{W}$ be closed subspaces of $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ with embedding maps $J_\mathcal{M}$, $J_\mathcal{N}$, and $J_\mathcal{W}$, respectively. Then the restricted bilinear mapping $g : \mathcal{M} \times \mathcal{N} \to \mathcal{W}$ such that $J_\mathcal{W} \circ g = f \circ (J_\mathcal{M} \times J_\mathcal{N})$, or equivalently $g(m, n) = f(m, n) \quad (m \in \mathcal{M}, n \in \mathcal{N})$, is also Arens regular. In particular, every closed subalgebra of an Arens regular Banach algebra is itself Arens regular. Based on the above observation and motivated by the extensive works [12, 14], we provide an Arens regular right module action π_r whose extension π_{r_2} is not Arens regular. In particular, we apply this to show that there exists an Arens regular bilinear map whose third adjoint is not Arens regular. For this, we start with the following definition.

Definition 2.2. Let $\{f_n : \mathcal{X}_n \times \mathcal{Y}_n \to \mathcal{Z}_n\}$ be a uniformly bounded family of bilinear mappings. Then we define the direct product Πf_n and the restricted direct product $\Pi_0 f_n$ of the above family by

$$\begin{array}{rcl} \Pi f_n : \Pi\mathcal{X}_n \times \Pi\mathcal{Y}_n & \to & \Pi\mathcal{Z}_n \\ ((x_n),(y_n)) & \mapsto & (f_n(x_n,y_n)) \end{array}$$

and

$$\begin{array}{rcl} \Pi_0 f_n : \Pi_0\mathcal{X}_n \times \Pi_0\mathcal{Y}_n & \to & \Pi_0\mathcal{Z}_n \\ ((x_n),(y_n)) & \mapsto & (f_n(x_n,y_n)), \end{array}$$

respectively, where

$$\Pi\mathcal{X}_n = \{(x_n) : x_n \in \mathcal{X}_n, \|x\| = \sup \|x_n\| < \infty\}$$

and

$$\Pi_0\mathcal{X}_n = \{(x_n) \in \Pi\mathcal{X}_n : \ \forall\epsilon > 0; \|x_n\| > \epsilon \text{ for only finitely many } i\}.$$

Note that since $(\Pi_0\mathcal{X}_n)^{**} = (\oplus\mathcal{X}_n^*)^* = \Pi\mathcal{X}_n^{**}$ (see [12]) and the bilinear mappings

$$(\Pi_0 f_n)^{***} : (\Pi_0\mathcal{X}_n)^{**} \times (\Pi_0\mathcal{Y}_n)^{**} \to (\Pi_0\mathcal{Z}_n)^{**}$$

and

$$\Pi f_n^{***} : \Pi\mathcal{X}_n^{**} \times \Pi\mathcal{Y}_n^{**} \to \Pi\mathcal{Z}_n^{**}$$

are two extensions of $\Pi_0 f_n$, which are w^*-separately continuous on $(\Pi_0\mathcal{X}_n) \times (\Pi_0\mathcal{Y}_n)^{**}$, by the uniqueness of such extensions, we have $(\Pi_0 f_n)^{***} = \Pi f_n^{***}$. A similar argument also shows that $(\Pi_0 f_n)^{t***t} = \Pi f_n^{t***t}$.

Now if $\{f_n\}$ is a family of Arens regular bilinear mappings, then

$$(\Pi_0 f_n)^{***} = \Pi f_n^{***} = \Pi f_n^{t***t} = (\Pi_0 f_n)^{t***t},$$

and so $\Pi_0 f_n$ is Arens regular.

The latter observation leads us to the next result.

Theorem 2.3. *There exists an Arens regular right module action π_r such that π_{r_2} is not Arens regular.*

Proof. For each integer n, let $\mathbb{Z}_n$ be the cyclic group of order n and let π_n be the pointwise multiplication of the Banach algebra $c_0(\mathbb{Z}_n)$. Then the reflexivity of $c_0(\mathbb{Z}_n)$ implies that the module action $\pi_n^* : l^1(\mathbb{Z}_n) \times c_0(\mathbb{Z}_n) \to l^1(\mathbb{Z}_n)$ is Arens regular and so the right module action $\pi_r = \Pi_0\pi_n^*$ is Arens regular.

On the other hand, Let ρ_n be the pointwise multiplication of $l^\infty(\mathbb{Z}_n)$. Then for the right module action $\rho_r = \Pi\rho_n|_{\Pi_0 c_0(\mathbb{Z}_n)\times \Pi l^\infty(\mathbb{Z}_n)} : \Pi_0 c_0(\mathbb{Z}_n) \times \Pi l^\infty(\mathbb{Z}_n) \to \Pi_0 c_0(\mathbb{Z}_n)$ and the identity $\{1_{\ell^\infty(\mathbb{Z}_n)}\}$ of $\Pi l^\infty(\mathbb{Z}_n)$, we have $\rho_{r_2}\left(\{1_{\ell^\infty(\mathbb{Z}_n)}\}, \{a_n\}\right) = \{a_n\}$, for each $\{a_n\} \in \Pi\ell^\infty(\mathbb{Z}_n)$. Since $\Pi_0 c_0(\mathbb{Z}_n)$ is not reflexive, the part (i) of Proposition 2.1 implies that $\rho_{\ell_1} : \Pi\ell^\infty(\mathbb{Z}_n) \times \oplus\ell^1(\mathbb{Z}_n) \to \oplus\ell^1(\mathbb{Z}_n)$ is irregular and so its extension $\pi_{r_2} : \Pi\ell^\infty(\mathbb{Z}_n) \times \Pi_0 c_0(\mathbb{Z}_n) \to \Pi\ell^\infty(\mathbb{Z}_n)$ is also irregular. □

For each natural number n, consider the Hilbert space $H_n = \mathbb{C}^n$, and let π_n be the multiplication of the operator algebra $K(H_n)$. Then a similar proof as above implies that the right module action $\pi_r = \Pi_0\pi_n^* : \Pi_0 L^1(H_n) \times \Pi_0 K(H_n) \to \Pi_0 L^1(H_n)$ is Arens regular while π_{r_2} is not regular.

From Theorem 2.3, we obtain an Arens regular right module action π_r whose third adjoint π_r^{***} (an extension of π_{r_2}) is not Arens regular.

Corollary 2.4 ([12, 14])**.** *There exists a sequence of Arens regular bilinear mappings $\{f_n\}$ such that Πf_n is not Arens regular.*

Proof. With notations of Theorem 2.3, π_n^* is Arens regular, for each n, while $\pi_r^{***} = \Pi\pi_n^*$ is not Arens regular. □

3. Arens regularity of certain module actions

Let $\mathcal{M}$ and $\mathcal{W}$ be the closed subspaces of the normed spaces $\mathcal{X}$ and $\mathcal{Z}$, respectively, and let $g : \mathcal{M} \times \mathcal{Y} \to \mathcal{W}$ and $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be bounded bilinear maps in which f is an extension of g. Then $Z_r(f) \subseteq Z_r(g)$. Indeed, if $\rho \in Z_r(f), \eta \in \mathcal{M}^{**}, \omega \in \mathcal{W}^*$, $J_{\mathcal{M}} : \mathcal{M} \to \mathcal{X}$, and $J_{\mathcal{W}} : \mathcal{W} \to \mathcal{Z}$ are the embedding maps and $I_{\mathcal{Y}}$ is the identity map on $\mathcal{Y}$, then there is $\zeta \in \mathcal{Z}^*$ such that $J_{\mathcal{W}}^*(\zeta) = \omega$. On the other hand, since f is an extension of g, we have $J_{\mathcal{W}} \circ g = f \circ (J_{\mathcal{M}} \times I_{\mathcal{Y}})$. Therefore

$$\begin{aligned}
\langle g^{***}(\eta,\rho), \omega\rangle &= \langle g^{***}(\eta,\rho), J_{\mathcal{W}}^*(\zeta)\rangle \\
&= \langle J_{\mathcal{W}}^{**}(g^{***}(\eta,\rho)), \zeta\rangle \\
&= \langle (J_{\mathcal{W}} \circ g)^{***}(\eta,\rho)), \zeta\rangle \\
&= \langle (f \circ (J_{\mathcal{M}} \times I_{\mathcal{Y}}))^{***}(\eta,\rho)), \zeta\rangle \\
&= \langle f^{***}(J_{\mathcal{M}}^{**}(\eta), \rho), \zeta\rangle \\
&= \langle f^{t***t}(J_{\mathcal{M}}^{**}(\eta), \rho), \zeta\rangle \\
&= \langle (f \circ (J_{\mathcal{M}} \times I_{\mathcal{Y}}))^{t***t}(\eta,\rho)), \zeta\rangle \\
&= \langle (J_{\mathcal{W}} \circ g)^{t***t}(\eta,\rho)), \zeta\rangle \\
&= \langle J_{\mathcal{W}}^{**}(g^{t***t}(\eta,\rho)), \zeta\rangle
\end{aligned}$$

$$= \langle g^{t***t}(\eta,\rho), J^*_{\mathcal{W}}(\zeta)\rangle$$
$$= \langle g^{t***t}(\eta,\rho), \omega\rangle.$$

Hence $\rho \in Z_r(g)$. In particular, we have the following result for the right module actions.

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{X}$ be a right Banach $\mathcal{A}$-module with the right module action $\pi_r : \mathcal{X} \times \mathcal{A} \to \mathcal{X}$. Then $Z_r(\pi_{r_n}) \subseteq Z_r(\pi_{r_{n-2}})$, for each $n \geq 2$.*

Theorem 3.1 establishes an inclusion relation between the topological centers of π_{r_n} and $\pi_{r_{n-2}}$. It should be noticed that the study of the relation between π_{r_n} and $\pi_{r_{n-2}}$ can be reduced to the same study for the relation between π_{r_2} and π_r. Indeed, for the right module action $\pi_{r_{n-2}} : \mathcal{X}^{(n-2)} \times \mathcal{A} \to \mathcal{X}^{(n-2)}$, if we set $Z = \mathcal{X}^{(n-2)}$ and define $\rho_r : Z \times \mathcal{A} \to Z$ by $\rho_r = \pi_{r_{n-2}}$, then $\rho_{r_2} = \pi_{r_n}$. In the following, we work only on π_r and π_{r_2}.

The following corollary studies the relationship between the right topological center of a Banach algebra $\mathcal{A}$ and the right topological center of a right module action $\pi_r : \mathcal{X} \times \mathcal{A} \to \mathcal{X}$ under some conditions.

Corollary 3.2. *Let π_r^* be onto. Then $Z_r(\pi_{r_2}) \subseteq Z_r(\pi_r) \subseteq Z_r(\mathcal{A}^{**})$.*

Proof. The inclusion $Z_r(\pi_r) \subseteq Z_r(\mathcal{A}^{**})$ follows from [3, Theorem 2.6] and the other follows from Theorem 3.1 for $n = 2$. $\square$

Corollary 3.3. *Let $\pi_r : \mathcal{X} \times \mathcal{A} \to \mathcal{X}$ be the right module action of the Banach algebra $\mathcal{A}$ on the Banach $\mathcal{A}$-module $\mathcal{X}$. Then we have the following assertions:*

(i) *If π_r^* is onto and π_{r_2} is Arens regular, then $\mathcal{A}$ and π_r are Arens regular; see* [13, Theorem 3.1].
(ii) *If π_r^* is onto and $\mathcal{A}$ is right strongly Arens irregular, then π_{r_2} and π_r are right strongly Arens irregular.*

If we consider $\mathcal{A}$ as a Banach $\mathcal{A}$-module equipped with $\pi_r = \pi_\ell = \pi$, the multiplication of $\mathcal{A}$, then it is known that when $\mathcal{A}$ has a bounded right (respectively, left) approximate identity, then $\pi^*_{r_{2k-1}}$ (respectively, $\pi^*_{r_{2k}}$) is onto (see [3, Lemma 2.7]). Using Corollary 3.2, we arrive at the following result.

Corollary 3.4. *Let $\pi_r = \pi$ be the multiplication of the Banach algebra $\mathcal{A}$. Then the following assertions hold:*

(i) *If $\mathcal{A}$ has a bounded right approximate identity and π_{r_3} is Arens regular, then $\mathcal{A}$ is Arens regular.*
(ii) *If $\mathcal{A}$ has a bounded left approximate identity and π_{r_4} is Arens regular, then $\mathcal{A}$ is Arens regular.*

Acknowledgement. The useful comments of the anonymous referees are gratefully acknowledged.

References

1. R. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951) 839–848.

2. N. Arikan, *Arens regularity and reflexivity*, Quart. J. Math. Oxford (2) **32** (1981), no. 128, 383–388.
3. S. Barootkoob, *Topological centers and factorization of certain module actions*, Sahand Com. Math. Anal. (SCMA) **15** (2019) 203–215.
4. S. Barootkoob and H.R. Ebrahimi Vishki, *Lifting derivations and n-weak amenability of the second dual of a Banach algebra*, Bull. Austral. Math. Soc. **83** (2011) 222–129.
5. S. Barootkoob, A.A. Khosravi, and H.R. Ebrahimi Vishki, *The bidual of a Banach algebra associated with a bilinear map*, Boll. Unione Mat. Ital. **12** (2019) 569–574.
6. S. Barootkoob, S. Mohammadzadeh, and H.R. Ebrahimi Vishki, *Topological centers of certain Banach module actions*, Bull. Iranian. Math. Soc. **35** (2009) 25–36.
7. H.G. Dales and A.T.-M. Lau, *The second duals of Beurling algebras*, Mem. Amer. Math. Soc. **177** (2005), no. 836, 191 pp.
8. M. Eshaghi Gordji and M. Filali, *Arens regularity of module actions*, Studia Math. **181** (2007) 237–254.
9. A.A. Khadem-Maboudi and H.R. Ebrahimi Vishki, *Strong Arens irregularity of bilinear mappings and reflexivity*, Banach J. Math. Anal. **6** (2012) 155–160.
10. A.A. Khosravi, H.R. Ebrahimi Vishki and A.M. Peralta, *Aron-Berner extensions of triple maps with application to the bidual of Jordan Banach triple systems*, Linear Algebra Appl. **580** (2019) 436–463.
11. S. Mohammadzadeh and H.R. Ebrahim Vishki, *Arens regularity of module actions and the second adjoint of a derivation*, Bull. Austral. Math. Soc. **77** (2008) 465–476.
12. J. Pym, *Remarks on the second duals of Banach algebras*, J. Nigerian Math. Soc. **2** (1983) 31–33.
13. A. Sahleh and L. Najarpishe, *Arens regularity of bilinear maps and Banach module actions*, Bull. Iranian Math. Soc. **43** (2017) 285–289.
14. N.J. Young, *Periodicity of functionals and representations of normed algebras on reflexive spaces*, Proc. Edinb. Math. Soc. **20** (1976) 99–120.

[1] Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord, Iran.

Email address: `s.barutkub@ub.ac.ir`