

## STRONG RAINBOW COLORING OF UNICYCLIC GRAPHS

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**ABSTRACT.** A path in an edge-colored graph is called a *rainbow path*, if no two edges of the path are colored the same. An edge-colored graph  $G$ , is *rainbow-connected* if any two vertices are connected by a rainbow path. A rainbow-connected graph is called strongly rainbow connected if for every two distinct vertices  $u$  and  $v$  of  $V(G)$ , there exists a rainbow path  $P$  from  $u$  to  $v$  that in the length of  $P$  is equal to  $d(u, v)$ . The notations  $rc(G)$  and  $src(G)$  are the smallest number of colors that are needed in order to make  $G$  rainbow connected and strongly rainbow connected, respectively. In this paper, we find the exact value of  $rc(G)$ , where  $G$  is a unicyclic graph. Moreover, we determine the upper and lower bounds for  $src(G)$ , where  $G$  is a unicyclic graph, and we show that these bounds are sharp.

### 1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, all graphs are finite, undirected, and connected. We refer the reader to the book [1] for graph-theoretical notation and terminology not described here. A path  $P$  in an edge-colored graph  $G$ , where adjacent edges may be colored the same, is a *rainbow path* if no two edges of it are colored the same. An *edge-connected* graph  $G$ , whose adjacent edges may have the same color, is rainbow-connected if every two vertices are connected by a rainbow path. The *rainbow connection number* of a connected graph  $G$ , denoted  $rc(G)$ , is the smallest number of colors that are needed in order to make  $G$  rainbow connected.

For any two vertices  $u$  and  $v$  of  $G$ , a *rainbow  $u$ - $v$  geodesic* in  $G$  is a rainbow path from  $u$  to  $v$  by the length  $d(u, v)$ . A graph  $G$  is strongly rainbow connected if there exists a rainbow  $u$ - $v$  geodesic for every pair of distinct vertices  $u$  and  $v$

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in  $G$ . In this case, the edge coloring  $c : E(G) \rightarrow \{1, 2, \dots, n\}$ ,  $n \in \mathbb{N}$ , is called a *strong rainbow coloring* of  $G$ . Similarly we define the strong rainbow connection number of a connected graph  $G$ , denoted by  $\text{src}(G)$ , as the smallest number of colors that are needed in order to make  $G$  strong rainbow connected. A strong rainbow coloring of  $G$  using  $\text{src}(G)$  colors is called a *minimum strong rainbow coloring* of  $G$ .

Chartrand et al. [3] found that the numbers  $\text{rc}(G)$  and  $\text{src}(G)$  are equal to the size of the graph  $G$  if and only if  $G$  is a tree. Therefore, this relevant question arises: What is the relation between  $\text{rc}(G)$ ,  $\text{src}(G)$ , and the size of  $G$ , where  $G$  is not a tree? Li and Sun [7] showed that there is no graph  $G$  with  $\text{src}(G) = m - 1$ . Moreover, they characterized all graphs that satisfy  $\text{src}(G) = m - 2$  in which  $m$  is the size of graph.

Although the rainbow connection number is obtained for some specific graphs, it is proved that for a given graph  $G$ , deciding if  $\text{rc}(G) = 2$  is already NP-complete; see [6]. More generally, it was shown in [6] that for any fixed  $k \geq 2$ , deciding if  $\text{rc}(G) = k$  is NP-complete. Therefore giving the precise value of  $\text{rc}(G)$  and  $\text{src}(G)$  for a given arbitrary graph  $G$  is almost impossible. Nowadays, finding the upper bound for  $\text{rc}(G)$  and describing the relation between  $\text{rc}(G)$  and other parameters of graph, specially the  $k$ -connectivity [2, 4, 5], radius, bridge, minimum degree, and order of graph are investigated.

In this paper, we concentrate on determining  $\text{rc}(G_r)$  and  $\text{src}(G_r)$ , where  $G_r$  is a *unicyclic* graph defined as follows.

**Definition 1.1.** Suppose that  $T_1, T_2, \dots, T_r$  are trees with the roots  $w_1, \dots, w_r$ , respectively, and let  $C_n$  be a cycle with the vertices  $v_1, v_2, \dots, v_n$  ( $n \geq r$ ). We consider  $G_r$  as a unicyclic graph such that  $\{i_1, i_2, \dots, i_r\} \subseteq \{1, 2, \dots, n\}$  and  $w_j$  is identified to  $v_{i_j}$  for every  $1 \leq j \leq r$ . In addition, we define  $\alpha_i$  as the number of leaves of  $T_i$  minus one. We show the sum of all  $\alpha_i$ ,  $1 \leq i \leq r$ , by  $\alpha$ .

In Section 2, we find the exact value of  $G_1$  and  $G_2$ . In Section 3, we obtain the sharp upper and lower bounds for  $\text{src}(G_r)$ , where  $G_r$  has an odd cycle. Finally, in Section 4, we look at unicyclic graphs with even cycle, and we set the sharp upper and lower bounds for  $\text{src}(G_r)$ , depending on the size of trees that are attached to the cycle  $C$ .

## 2. RAINBOW CONNECTION NUMBER

For finding  $\text{rc}(G_1)$  and  $\text{rc}(G_2)$ , first we need the following two lemmas, which show that if  $c$  is a rainbow coloring of  $G_r$ , then each color of trees is used on the cycle at most once and the maximum number of colors of  $T_i$  that we can use on the cycle, is  $\alpha_i$ .

**Lemma 2.1.** *Let  $G_r$  be a unicyclic graph, and let  $\{c_1, \dots, c_t\}$  be the colors of a rainbow coloring of  $G_r$ . Then each color of  $E(T_i)$ ,  $1 \leq i \leq r$ , is used at most twice on  $G_r$ .*

*Proof.* Let  $e_1 = u_1v_1$  with color  $c_1$  be an edge in  $E(T_j)$ , and let  $e_2 = u_2v_2$  and  $e_3 = u_3v_3$  be another edges of  $G_r$  with color  $c_1$ . It is obvious that  $e_2$  and  $e_3$  are

in the cycle of  $G_r$ . Now, delete the edges  $e_2$  and  $e_3$  from  $G_r$  (do not remove their vertices). Thus, we have  $G_r \setminus \{e_2, e_3\} \simeq H_1 \cup H_2$ , where  $H_1$  and  $H_2$  are connected and  $w_j \in V(H_2)$ . Without loss of generality, assume that  $d(w_j, u_1) > d(w_j, v_1)$  on  $G_r$ . In this case, we do not have any rainbow path between  $u_1$  and vertices of  $H_1$ , which is a contradiction.  $\square$

*Remark 2.2.* Let  $c$  be a rainbow coloring of  $G_r$ , and let  $e \in E(T_i)$  be an edge that its color is used on the cycle. If  $P$  is the path  $w_i - v_1 - \dots - v_h$  such that  $v_1, \dots, v_h$  are in  $V(T_i)$  and  $P$  includes the edge  $e$ , then none of the colors of  $E(P) \setminus \{e\}$  are used in the cycle.

**Lemma 2.3.** *The maximum number of colors of  $T_i$ , which can be used in the cycle of  $G_r$ , is equal to  $\alpha_i$  ( $1 \leq i \leq r$ ).*

*Proof.* Let the edges of  $T_i$  be colored by  $c_j$ ,  $1 \leq j \leq m_i$ , which corresponds to the edges  $e_j$ , respectively. Assume that the colors  $c_{i_1}, \dots, c_{i_t}$  are the maximum number of colors of  $T_i$  that are used in the cycle of  $G_r$ . It is enough to show that  $t \leq \alpha_i$ .

If all edges  $e_{i_1}, \dots, e_{i_t}$  are leaves, then we have nothing to prove. Suppose that  $e_{i_1} = u_1 v_1$  is not a leaf of  $T_i$  and that  $d(u_1, w_i) \leq d(v_1, w_i)$ . Now, remove the edge  $e_{i_1}$  from  $T_i$ , and so we have two trees  $T_{i_1}$  and  $T_{i_2}$  such that  $w_i \in V(T_{i_2})$ . Therefore, we can say that  $T_{i_1}$  is isomorphic to a path with root  $v_1$ . Otherwise, Remark 2.2 implies that  $T_{i_1}$  has at least two leaves like  $e_1$  and  $e_2$  that their colors are not in the set  $\{c_{i_1}, \dots, c_{i_t}\}$ . Now, remove the color  $c_{i_1}$  from the set and add the colors  $c_1$  and  $c_2$  to it, which is a contradiction. Thus, each edge of  $e_{i_j}$ ,  $1 \leq j \leq t$ , is either a leaf or corresponds to a leaf of tree, and we have  $t \leq \alpha_i$ .  $\square$

**Definition 2.4.** Let  $k$  be a positive integer. Define the function  $H_k$  as follows:

$$H_k(s) = \begin{cases} 1, & k > s, \\ 0, & k \leq s. \end{cases}$$

**Theorem 2.5.** *Let  $G_1$  be a unicyclic graph. Then  $\text{rc}(G_1) = m_1 + \lceil \frac{k - \alpha_1}{2} \rceil H_k(\alpha_1)$ , where  $k$  is the length of  $C$ .*

*Proof.* We know that  $m_1$  colors are needed for coloring the tree. By Lemma 2.3, maximum colors of tree that we can use in the cycle, is  $\alpha_1$ ; on the other hand, each color of tree is used at most once in  $C$ . If  $k \leq \alpha_1$ , then we can color the edges of cycle with the color of leaves and  $\text{rc}(G_1) = m_1$ ; otherwise if  $k > \alpha_1$ , then we can color  $\alpha_1$  edges of  $C$ . Thus, there are  $k - \alpha_1$  edges without color. Since each color is used on  $G_1$  at most twice, we need  $\lceil \frac{k - \alpha_1}{2} \rceil$  new colors.  $\square$

*Remark 2.6.* We must put the colors of leaves of  $T_1$  in the cycle  $C$  (for example, an even cycle with length of  $n$ ) in the last theorem by this method that we are going to illustrate. At first, we denominate the edges of cycle  $C$ , starting from  $w_1$  in the cycle clockwise by  $e_i$  ( $1 \leq i \leq \frac{n}{2}$ ), and the edges of  $C$ , starting anticlockwise from  $w_1$  in the cycle  $C$ , are denoted by  $e'_i$  ( $1 \leq i \leq \frac{n}{2}$ ). Set the priority order

of putting the color of leaves in the color of edges in the cycle  $C$ , by finding the greater index of  $i$  ( $1 \leq i \leq \frac{n}{2}$ ) in  $\{e_i, e'_i\}$ .

**Theorem 2.7.** *Let  $G_2$  be a unicyclic graph, and let  $\alpha_2 \geq \alpha_1$ . Then*

$$\text{rc}(G_2) = \min\{m_1 + m_2 + s + \lceil \frac{k - \alpha - 2s}{2} \rceil H_k(\alpha + 2s), m_1 + m_2 + \lceil \frac{k - \alpha_2}{2} \rceil H_k(\alpha_2)\},$$

where  $k$  is the length of cycle and  $s$  is the length of minimum path between  $w_1$  and  $w_2$ .

*Proof.* For a rainbow coloring of  $G_2$ , we need at least  $m_1 + m_2$  colors for  $E(T_1) \cup E(T_2)$ . Now, consider the following cases:

**Case1.** There is at least one color of each tree in  $E(C)$ . Since the maximum number of colors  $T_i$ , which we can use on  $C$ , is  $\alpha_i$  ( $i = 1, 2$ ), we have at most  $\alpha$  colors of  $T_i$ 's in the cycle, by Lemma 2.3. Suppose that  $P_1$  and  $P_2$  are two paths between  $w_1$  and  $w_2$  in  $G_2$  such that  $s = l(P_1) \leq l(P_2)$ . From the fact that, each tree has at least one color on the cycle, the colors of leaves, which are used in the cycle, have to be in  $P_1$  or  $P_2$ . Since  $l(P_1) \leq l(P_2)$ , we assume that these colors are in  $P_2$  (Note that if  $l(P_2) < \alpha$ , then we cannot use all colors of leaves). Therefore, we need  $s$  new colors for coloring the edges of  $P_1$ . It is not difficult to see that in

this case we have  $\text{rc}(G) \leq m_1 + m_2 + s + \lceil \frac{k - \alpha - 2s}{2} \rceil H_k(\alpha + 2s)$ . Now, we show that the inequality is sharp. Clearly, we need  $m_1 + m_2 + s$  colors for the rainbow coloring of  $T_i$ 's and  $P_1$ ,  $i = 1, 2$ . First assume  $k > \alpha + 2s$ . Since each color of leaves or the edges  $P_1$  is used twice, by Lemma 2.1, we need at least  $\lceil \frac{k - \alpha - 2s}{2} \rceil$  new colors for the edges  $P_2$ . Thus,  $\text{rc}(G) = m_1 + m_2 + s + \lceil \frac{k - \alpha - 2s}{2} \rceil$ . Second, assume that  $k \leq \alpha + 2s$ . In this case, we have  $\text{rc}(G) = m_1 + m_2 + s$ .

**Case 2.** The colors of one tree are used in the cycle. Since  $\alpha_1 \leq \alpha_2$ , we use  $\alpha_2$  colors of tree  $T_2$  on  $C$ . In this case, we have a unicyclic with one tree  $T_2$ . Thus, Theorem 2.5 implies that

$$\text{rc}(G_2) = m_1 + m_2 + \frac{k - \alpha_2}{2} H_k(\alpha_2).$$

□

### 3. UNICYCLIC GRAPH WITH ODD CYCLE

Throughout this section, we assume that  $G_r$  is a connected unicyclic graph with  $r$  trees and odd cycle with order  $2k + 1$ . Since we need  $\sum_{i=1}^r m_i$  distinct colors for all edges of trees, we want to know how many colors of trees we can use on  $C$  and to find the strongly rainbow connection number.

**Lemma 3.1.** *If  $c$  is a strongly rainbow coloring of  $G_r$ , then at most one color of  $T_i$  is used on  $C$ .*

*Proof.* Suppose by contradiction that  $c_1$  and  $c_2$  are, respectively, two distinct colors of  $e_1 = u_1v_1$  and  $e_2 = u_2v_2$  of  $T_i$ , such that they are used on  $C$ . Without loss of generality, assume that  $d(u_1, w_i) \leq d(v_1, w_i)$ . It is not difficult to see that

there is a unique edge  $e = uv$  of  $C$  such that  $d(u, w_i) = d(v, w_i) = k$ . Since we use the colors  $c_1$  and  $c_2$  on  $C$ , there is a distinct edge from  $e$  on  $C$  like  $e' = u'v'$  with color  $c_1$  or  $c_2$ . Without loss of generality, assume that  $e'$  is colored by  $c_1$  and that  $d(u_1, w_i) \leq d(v_1, w_i)$ . From the fact that  $C$  has an odd length, we have either  $d(u', w_i) < d(v', w_i)$  or  $d(u', w_i) > d(v', w_i)$ . If the former case happens, then there is no  $v_1-v'$  geodesic path. For the latter case, consider the vertices  $v_1$  and  $u'$ .  $\square$

From the proof of Lemma 3.1, if we use one color of  $E(T_i)$  in the cycle of  $G_r$ , then there is only one edge for that color, which is introduced above. Let the edges of  $C$  be colored by each specific color of  $T_i$ 's for  $1 \leq i \leq r$ . Then we can extend  $C$  to a strongly rainbow connected graph, because, for every integers  $i \neq j$ , if  $u \in V(T_i)$  and  $v \in V(T_j)$ , then there are no repetitive colors of trees on the shortest path between  $u$  and  $v$ . Thus, for an arbitrary graph  $G_r$ , to find  $\text{src}(G_r)$ , it is enough to consider a cycle of length  $2k + 1$  with  $r$  different coloring edges that belong to colors of leaves originally, and to extend  $C$  by minimum new colors to a strongly rainbow connected graph.

**Definition 3.2.** Let  $H_r$  be a cycle of length  $2k + 1$  with  $r$  distinguished edge colors such that the induced subgraph on these colored edges forms a path of length  $r$  ( $r > 0$ ), and let  $H'_r$  be a cycle of length  $2k + 1$  with  $r$  different edge colors such that these colors decompose edges  $C_{2k+1}$  to four paths  $P_1, P_2, P_3$ , and  $P_4$  such that the induced subgraph on colored edges is  $P_1 \cup P_2$ ,  $V(P_1) \cap V(P_2) = \emptyset$ , and  $|l(P_1) - l(P_2)| \leq 1$ . Similarly the induced subgraph, which edges do not have any color, is  $P_3 \cup P_4$ ,  $V(P_3) \cap V(P_4) = \emptyset$ , and  $|l(P_3) - l(P_4)| \leq 1$ .

**Theorem 3.3.** *Let  $H$  be the cycle  $C_{2k+1}$  with  $r$  distinct coloring edges ( $1 \leq r \leq 2k + 1$ ). Then*

- (i)  $\text{src}(H)$  is maximum, if  $H \simeq H_r$ ;
- (ii)  $\text{src}(H)$  is minimum, if  $H \simeq H'_r$ .

*Proof.* First we find the maximum  $\text{src}(H)$ . Since  $\text{src}(C_{2k+1}) = k + 1$ , then  $\text{src}(H) \leq k$ . Suppose  $r \leq k$ . In this sense,  $H_r$  has a path with length  $k$  without any color; thus, we need  $k$  new colors for coloring the path. Thus,  $\text{src}(H_r) = k$  and the maximum case happens. If  $r > k$ , then  $H_r$  has a path with length  $2k - r + 1$ , which needs  $2k - r + 1$  new colors for its own. In this case, the graph  $H_r$  is colored by  $2k + 1$  distinct colors, which is maximum.

Second, we show that  $H'_r$  needs minimum colors. If  $r$  is even, then  $l(P_1) = l(P_2) = \frac{r}{2}$ . Without loss of generality, assume that  $l(P_3) = k - \frac{r}{2}$  and that  $l(P_4) = k - \frac{r}{2} + 1$ . Since  $l(P_4) \leq k$ , we need  $k - \frac{r}{2} + 1$  new colors for  $E(P_4)$ . Locate  $k - \frac{r}{2} + 1$  colors on  $P_4$ , and put these colors for  $E(P_3)$  such that the result is a strongly rainbow coloring. Thus, the minimum  $\text{src}(H)$  happens and equals  $\lceil \frac{2k-r+1}{2} \rceil$ . If  $r$  is odd, then assume  $l(P_1) = \frac{r-1}{2}$  and  $l(P_2) = \frac{r+1}{2}$ . Hence,  $l(P_3) = l(P_4) = \frac{2k-r+1}{2}$ , and similarly,  $\text{src}(H) = \frac{2k-r+1}{2}$ .  $\square$

The following corollary is an immediate result of Theorem 3.3, which shows upper and lower bounds for a strongly rainbow connection of unicyclic graphs with odd cycle.

**Corollary 3.4.** *Let  $G_r$  be an arbitrary unicyclic graph with a cycle of length  $2k + 1$ , and let  $T_1, \dots, T_r$  be  $r$  trees of it.*

- *If  $r \leq k$ , then*

$$\sum_{i=1}^r m_i + \lceil \frac{2k - r + 1}{2} \rceil \leq \text{src}(G_r) \leq \sum_{i=1}^r m_i + k. \tag{3.1}$$

- *If  $k < r \leq 2k$ , then*

$$\sum_{i=1}^r m_i + \lceil \frac{2k - r + 1}{2} \rceil \leq \text{src}(G_r) \leq \sum_{i=1}^r m_i + 2k - r + 1. \tag{3.2}$$

#### 4. UNICYCLIC GRAPH WITH EVEN CYCLE

Throughout this section, we assume that  $G_r$  is a connected unicyclic graph with  $r$  trees that are attached to an even cycle  $C$  of order  $2k$ . Let  $T$  be the set of all trees in  $G_r$ , let  $A = \{T_1, T_2, \dots, T_z\}$  be the set of  $z$  trees from  $G_r$  that are not paths, and let  $B$  be the set of trees that are paths. Therefore  $T = A \cup B$ . In this paper, the variable  $z$  denotes the number of trees that are not paths, and  $T_{z+1}, \dots, T_r$  are paths.

**Lemma 4.1.** *Let  $c$  be a strongly rainbow coloring of  $G_r$ ; then at most two colors of  $E(T_i)$  are used on  $c$ .*

*Proof.* Since  $l(C) = 2k$ , for each  $1 \leq i \leq r$ , there is a vertex  $u \in V(C)$  such that  $d(u, w_i) = k$ . Suppose that  $N_i = \{uu', uu''\}$ , where  $u', u'' \in V(C)$ . Similar to the argument of the proof of Lemma 3.1, the colors of  $E(T_i)$  have just two choices on  $C$ , which are  $uu'$  and  $uu''$ . If  $T_i$  is a path, then we can use just one color of  $E(T_i)$ . Otherwise, we can use two colors of  $E(T_i)$ .  $\square$

From the proof of Lemma 4.1, there are two fixed options on  $C$  for colors of each tree, but depending on the position of trees on the cycle, sometimes we can use at most one color of each tree on  $C$ .

**Lemma 4.2.** *Let  $r$  be a positive integer, and let  $k \geq 3$ . Then  $G_r$  cannot use maximum colors of  $E(T_i)$  on  $C$  if and only if one of the following cases occurs:*

- Case 1.**  $T_i$  is not a path and there is  $T_j$  ( $j \neq i$ ) such that  $d(w_i, w_j) = k$ .
- Case 2.**  $T_i$  is not a path and there is  $T_j$  ( $j \neq i$ ) such that  $T_j$  is not path and  $d(w_i, w_j) = 1$ .

*Proof.* Let  $i$  be a fixed integer. Then  $1 \leq d(w_i, w_j) \leq k$  for every  $1 \leq j \leq r$  ( $j \neq i$ ). Now, Consider two cases:

**Case 1.** We have  $2 \leq d(w_i, w_j) \leq k$  ( $j \neq i$ ) for every  $j$ . In this case,  $N_i \cap N_j = \emptyset$ . If  $2 \leq d(w_i, w_j) \leq k - 1$ , then we can select maximum colors of  $E(T_i)$  for the edges of  $N_i$ . Moreover, if  $d(w_i, w_j) = k$ , then we can use just two colors of  $E(T_i) \cup E(T_j)$  depending on the following cases:

- (i) Both  $T_i$  and  $T_j$  are paths.
- (ii) Both  $T_i$  and  $T_j$  are not paths.
- (iii)  $T_i$  is not a path, but  $T_j$  is a path, or conversely.

For (i), we can use one color from each tree in  $C$ . Since there are two  $w_i w_j$ -paths with same length, two colors have to utilize in one of  $w_i w_j$ -paths to be a strongly rainbow coloring. If (ii) happens, then it should be used two colors of one tree or one color of each tree (The latter case is more efficient). The last item is similar to (ii).

**Case 2.** We have  $d(w_i, w_j) = 1$  for some  $j \neq i$ ,  $1 \leq j \leq r$ . Clearly,  $|N_i \cap N_j| = 1$  and we can use maximum three colors of  $E(T_i) \cup E(T_j)$ . Thus, when both  $T_i$  and  $T_j$  are not paths, we miss one color of the four colors. □

For achieving  $\text{src}(G_r)$ , we need  $\sum_{i=1}^r m_i$  colors for trees. Hence, it is important to find the position of trees, and then color the edges of  $C$  corresponding to each tree. The maximum number of colors of trees, which we can use on  $C$ , is denoted by  $\lambda(G_r)$ . In this case, finding  $\text{src}(G_r)$  is equivalent to considering a cycle of length  $2k$  with  $\lambda(G_r)$  colored edges corresponding to trees for  $1 \leq \lambda(G_r) \leq 2r$ .

**Lemma 4.3.** *Let  $T = A$  and let  $|A| = k$ , in which  $k$  is an even positive integer and  $l(C) = 2k$ . Then  $\lambda(G_r) \leq 2k - 1$ .*

*Proof.* Since  $k$  is even, we cannot use all  $T_1, \dots, T_k$  on  $C$  such that  $d(w_i, w_j) \neq 1, k$  ( $1 \leq i, j \leq k$ ). Define  $G_r^1$  as follows: Assign  $T_1, \dots, T_{k-1}$  on  $C$  such that  $d(w_i, w_j) \neq 1, k$ . Now, put  $T_k$  on  $C$  such that  $d(w_k, w_i) \neq k$ . In this sense,  $\lambda(G_r) = 2k - 1$ . □

If  $T = A$  and  $|A| = k$ , in which  $k$  is an even positive integer and  $l(C) = 2k$ , then we denote  $G_r$  by  $G'_k$ . In the following theorems, we find the maximum and minimum of  $\lambda(G_r)$  for each  $G_r$  with  $r$  fixed trees, where  $G_r$  is not isomorphic to  $G'_k$ .

**Lemma 4.4.** *Suppose that  $\mathcal{T}$  is the set of all unicyclic graphs  $G_r$  with  $r$  trees, in which  $G_r$  is not isomorphic to  $G'_k$ . Then  $\max_{G_r \in \mathcal{T}} (\lambda(G_r)) = \min\{z + r, 2k\}$ .*

*Proof.* It is straightforward to see that  $\max_{G_r \in \mathcal{A}} (\lambda(G_r)) = \min\{z + r, 2k\} \leq z + r$ , by Lemma 4.2. We show that the sharp case happens. Define a unicyclic graph with  $r$  trees as follows: If  $z < k$ , then put  $z$  trees  $T_1, T_2, \dots, T_z$  such that  $d(w_i, w_j) \neq 1, k$  for every  $1 \leq i, j \leq m$ . Since  $z < k$ , it is possible. If  $e = uv \in E(C)$  is not colored by the first  $z$  trees, then there exist vertices  $u', v' \in V(C)$  such that  $d(u, u') = d(v, v') = k$  and at least one of  $u'$  and  $v'$  is distinct from  $w_i$ ,  $1 \leq i \leq z$ . Thus, put  $T_{z+1}$  on this vertex. By following this process, we put trees on  $C$  such that the maximum number of colors is used. If  $z \geq k$  and  $k$  is odd, then we can put  $k$  trees alternatively on  $C$  such that  $d(w_i, w_{i+1}) = 2$  and we use  $2k$  colors on the cycle. If  $z \geq k$  and  $k$  is even, then put  $k - 1$  trees  $T_1, \dots, T_{k-1}$  on  $C$  such that  $d(w_i, w_j) \neq 1, k$ . Clearly, we can utilize  $2k - 2$  colors of these trees on  $C$ . Since  $G_r$  is not isomorphic to  $G'_k$ , there exist  $T_k$  and  $T_{k+1}$ , yet. Similar to the above argument, there are two vertices on  $C$  that if we put the two trees there, then all edges are colored. □

**Lemma 4.5.** *Let  $G_r$  be an arbitrary unicyclic graph with  $r$  trees. Then  $\min\{r + 1, 2k\} \leq \lambda(G_r)$ , if  $T = A$ , and  $|A|$  is an odd number. Otherwise,  $\min\{r, 2k\} \leq \lambda(G_r)$ .*

*Proof.* If  $r = z$  and  $z$  is odd, then put  $z$  trees on  $C$  such that  $d(w_i, w_{i+1}) = 1$ , say  $G$ . In this case,  $\lambda(G) = \min\{r + 1, 2k\}$ . Obviously, we cannot use less than  $r + 1$  colors, because by Lemma 4.2, each tree loses at most one color, and it happens if for each tree  $T_i$ , there is a tree  $T_j$  such that  $d(w_i, w_j) = k$ . Since  $z$  is odd, it does not occur. If  $z$  is even, then define the graph  $H$  by putting trees  $T_1, T_2, \dots, T_z$  on  $C$  such that  $d(w_i, w_{\frac{z}{2}+i}) = k$  for  $1 \leq i \leq \frac{z}{2}$ . Now, assign  $r - z$  paths  $T_{z+1}, \dots, T_r$  on vertices of  $C$  that are distinguished from  $w_1, \dots, w_z$ . Thus,  $\lambda(H) = \min\{r, 2k\}$ . Since each tree does not lose all colors on  $C$ , this number is minimum. If  $r \neq z$  and  $z$  is odd, then define the graph  $H$  as above by the difference that  $d(w_i, w_{\frac{z+1}{2}+i}) = k$  for  $1 \leq i \leq \frac{z+1}{2}$ .  $\square$

The following theorem is similar to Theorem 3.3, and so we state it without proof.

**Theorem 4.6.** *Let  $H$  be cycle  $C_{2k}$  with  $r$  distinct coloring edges ( $1 \leq r \leq 2k$ ). Then*

- (i)  $\text{src}(C_{2k})$  is maximum, if  $H \simeq H_r$ .
- (ii)  $\text{src}(C_{2k})$  is minimum, if  $H \simeq H'_r$ .

**Definition 4.7.** Suppose that some edges of the graph  $G_r$  have colors. We call an edge of cycle  $C$ , where  $l(C) = 2k$ , as *independent edge*, if  $e = uv$  does not have any color, and the edges  $u_1u_2$ ,  $u_2v_2$ , and  $v_1v_2$  have colors, where  $d(u, v_2) = d(v, u_2) = k$ . We denote an independent edge by *IE*. If every edge of path  $P$  is IE, then it is called *the IE path* or *IE* briefly.

*Remark 4.8.* Suppose that  $r$  edges of cycle  $C$  with length of  $2k$  are colored by  $r$  different colors and that another  $2k - r$  new colors are independent edges. In this case, we need  $2k - r$  new colors for  $C$  to be a strongly rainbow coloring.

In the following theorems, we determine an upper bound for  $\text{src}(G_r)$ , where  $r$  is a positive integer.

**Theorem 4.9.** *Let  $r$  be a positive integer, and let  $z$  be an even number. Then*

- (i) *if  $r \geq z \geq k$  and  $k$  is even number, then*

$$\text{src}(G_r) \leq \sum_{i=1}^r m_i + 2k - r;$$

- (ii) *if  $r + 2 \geq z \geq k$  and  $k$  is an odd number, then*

$$\text{src}(G_r) \leq \sum_{i=1}^r m_i + 2k - r;$$

- (iii) *otherwise,*

$$\text{src}(G_r) \leq \sum_{i=1}^r m_i + \min\{k, 2k - r - 1\}.$$



*Proof.* Let  $H$  be a graph such that the trees  $T_1, \dots, T_r$  are located on  $C$  consecutively. In this case,  $\text{src}(H) = \sum_{i=1}^r m_i + 2k - r - 1$ . It is obvious if  $r < k$ , then  $\text{src}(G_r) \leq \text{src}(H)$ . Suppose that  $r \geq k$ . We show that there exists a graph  $H$  such that  $\text{src}(H) = \sum_{i=1}^r m_i + 2k - r$ . Theorem 4.2 and Remark 4.8 imply that  $H$  has to satisfy in the following conditions:

- (i) For each  $T_i \in A$ , there is  $T_j \in T, j \neq i$  such that  $d(w_i, w_j) = k$ .
- (ii) After using the colors of trees on  $C$ , each uncolored edge is  $IE$ .

If we put  $r$  trees on  $C$  randomly and color the correspond edges of trees, then uncolored edges create a path  $P_i, 1 \leq i \leq t$ , where  $t$  is minimum. Now we can say the following statements:

- (1) If  $l(P_i) = 1$ , then the edge of  $P_i$  is not  $IE$ .
- (2) If  $l(P_i) = 2$ , then  $E(P_i)$  can be  $IE$ . Hence we have to put trees and their colorings like Figure 1.

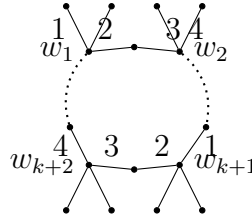


FIGURE 1.  $w_1w_2$ -path is  $IE$ .

- (3) If  $l(P_i) \geq 3$ , then none of the edges  $E(P_i)$  are  $IE$ .

Therefore, we have to locate  $r$  trees on  $C$  such that the number of  $P_i$ 's with  $l(P_i) = 2$  is maximum. Thus, we put trees like Figure 2.

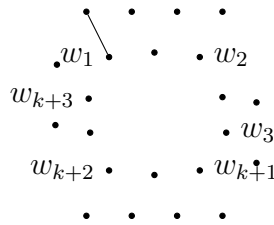


FIGURE 2. Graph  $H$  with  $\text{src}(H) = \sum_{i=1}^r m_i + 2k - r$

Since we are looking for the minimum colors from trees for  $G_r$ , we use two colors from one tree and no color from the opposite tree. Hence it creates  $2k - r$  edges without any coloring consecutively and  $r$  edges with color consecutively. It is obvious that at most two edges of  $r$  colors could be  $IE$ . Since  $r - z$  remaining trees are path, we put one path on  $C$  after  $w_{k+1}$  and one path on the opposite side on  $C$  before  $w_1$ . Therefore all of the coloring edges are  $IE$ .  $\square$

By Lemma 4.5, we have the following theorem, which we omit its proof.

**Theorem 4.10.** *Let  $T = A$ , and let  $|A|$  be an odd number. Then for the graph  $G_r$ , it follows that*

$$\text{src}(G_r) \leq \sum_{i=1}^r m_i + \min\{k, 2k - r - 1\}.$$

Now we find a lower bound for  $G_r$ , where  $r$  is a positive number.

**Theorem 4.11.** *For a positive integer  $r$ ,*

(i) *if  $T = A$  and  $|A|$  is an even number, then*

$$\sum_{i=1}^r m_i + \max\{0, k - z + 1\} \leq \text{src}(G_r);$$

(ii) *if  $T = A$  and  $|A|$  is an odd number, then*

$$\sum_{i=1}^r m_i + \max\{0, k - z\} \leq \text{src}(G_r);$$

(iii) *if  $T \neq A$ , then*

$$\sum_{i=1}^r m_i + \max\{0, k + \lceil \frac{-r - z}{2} \rceil\} \leq \text{src}(G_r).$$

*Proof.* First assume that  $r = z = k$  and that  $k$  is even. Now, define  $G_r^1$  as the proof of Lemma 4.3. Then we need  $\sum_{i=1}^r m_i$  colors for edges of trees and one new color for the cycle  $C$ . Thus,  $\text{src}(G_r^1) = \sum_{i=1}^r m_i + 1$ . If  $z > k$ , then we can color the edges of  $C$  by colors of trees. If  $z < k$ , then we can use  $2z$  colors of trees on  $C$ , by Lemma 4.4. Since we cannot construct the graph  $H'_r$  and all trees have at least two leaves, we have to define  $G_r$  as follows: Put  $T_1, \dots, T_z$  on  $C$  such that  $d(w_i, w_{\frac{z}{2}+i}) = k - 1$ ,  $d(w_2, w_3) = k - 1$ , and  $d(w_j, w_{j+1}) = 2$  for  $1 \leq i \leq \frac{z}{2}, 1 \leq j \leq z, j \neq \frac{z}{2}$ . In this case, we create a cycle with four disjoint paths  $P_1, P_2, P_3$ , and  $P_4$  such that  $P_1$  and  $P_2$  are colored by tree's colors and edges of  $P_3$  and  $P_4$  are uncolored. Clearly  $|l(P_1) - l(P_2)| = 2$ , and we need  $\lceil \frac{2k-2z+2}{2} \rceil$  new colors for the edges  $C$ . It is not difficult to see that there is no graph like  $G_r$  such that  $\text{src}(G_r) = k - z$ .

For the second case, if  $z \geq k$ , then we can color the edges of  $C$  by trees. If  $z < k$ , then assign  $T_1, \dots, T_{z-1}$  as above and put  $T_z$  on  $C$  such that  $d(w_z, w_{\frac{z}{2}}) = 2$ . Hence, in this case, the strongly rainbow connection number is  $\lceil \frac{2k-2z}{2} \rceil$ , which is minimum.

For the last case, first put  $T_1, T_2, \dots, T_z$  like above. If  $z$  is an odd number, then put  $T_{z+1}$  on  $C$  such that  $d(w_{z+1}, w_z) = 1$  and  $d(w_{z+1}, w_{z-1}) = k - 2$ . Put  $T_{z+2}$  on  $C$  such that  $d(w_{z+2}, w_{z+1}) = k$ . Put  $T_{z+3}, T_{z+5}, \dots$  consecutively on  $C$  after  $T_{z+1}$  and similarly  $T_{z+4}, T_{z+6}, \dots$  after  $T_{z+2}$ . If  $z$  is even, then put  $T_1, T_2, \dots, T_z$  as above. Now, put  $T_{z+1}$  and  $T_{z+2}$  continuously on  $C$  such that  $d(w_{z+1}, w_{\frac{z}{2}}) = 2$

and  $d(w_{z+2}, w_{\frac{z}{2}}) = 3$ . Thus we have two disjoint uncolored paths on the cycle  $C$ , which have the equal length.

For other trees, put  $T_{z+3}, T_{z+5}, \dots$  consecutively after  $T_{z+2}$  and  $T_{z+4}, T_{z+6}, \dots$  such that  $d(w_{2i-1}, w_{2i}) = k$  for each  $\frac{z}{2} + 2 \leq i$ . It is not difficult to see that we need  $\max\{0, \lceil \frac{2k-r-z}{2} \rceil\}$  new colors for the uncolored edges of  $C$ , whether  $z$  is even or odd.

□

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