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RADICALLY PRINCIPAL RINGS

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Abstract. Let A be a commutative ring. An ideal I of A is radically principal if there exists a principal ideal J of A such that $\sqrt{I} = \sqrt{J}$. The ring A is radically principal if every ideal of A is radically principal. In this article, we study radically principal rings. We prove an analogue of the Cohen theorem, precisely, a ring is radically principal if and only if every prime ideal is radically principal. Also we characterize a zero-dimensional radically principal ring. Finally we give a characterization of polynomial ring to be radically principal.

1. Introduction and preliminaries

A commutative ring A is said to have Noetherian spectrum if A satisfies the ascending chain condition (ACC) on radical ideals. This is equivalent to the condition that A satisfies the ACC on prime ideals and each ideal has only finitely many prime ideals minimal over it. Remark that every Noetherian ring has Noetherian spectrum and the converse is false; see [3]. Many authors studied the property of a commutative ring satisfying the Noetherian spectrum property ([2, 5] and so on). Recall from [4] that, an ideal I of A is said radically of finite type if $\sqrt{I} = \sqrt{J}$ for some finitely generated subideal J of I. In [4], the authors showed that A has Noetherian spectrum if and only if every ideal of A is radically of finite type if and only if every prime ideal of A is radically of finite type. Also Ohm and Pendleton [4] studied the Hilbert basis theorem for a commutative ring satisfying the Noetherian spectrum condition. They showed that a commutative ring A has Noetherian spectrum if and only if the polynomial ring $A[X]$ has Noetherian spectrum.

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Let A be a commutative ring. We say that an ideal I of A is radically principal if there exists a principal ideal J of A such that $\sqrt{I} = \sqrt{J}$. We also define A to be radically principal if each ideal I of A is radically principal. This is equivalent to the condition that every open of Spec$A$ is a principal affine open. Note that if A is a radically principal ring, then A has Noetherian spectrum, and every principal ideal ring is a radically principal ring. In this article, we study rings with this property. We prove the Cohen-type theorem for radically principal property, that is, a ring A is radically principal if and only if every prime ideal is radically principal. Recall that, the avoidance prime theorem says that, if $P \subseteq P_1 \cup \cdots \cup P_n$, where P, P_i are prime ideals, then $P \subseteq P_i$ for some i, this property does not hold for infinite set of ideals P_i. We characterize rings with avoidance property for infinite set of prime ideals. Also we give a characterization of zero-dimensional radically principal rings. Finally we study the radically principal property for the polynomial ring $A[X]$, and we show that the polynomial ring $A[X]$ is radically principal if and only if A is zero-dimensional and has finitely many prime ideals.

2. Radically principal ring

We start this section by introducing the following definition in order to give some results about radically principal rings.

Definition 2.1. Let A be a commutative ring. An ideal I of A is radically principal if the radical of I is a radical of a principal ideal, that is, if there is $a \in A$ such that $\sqrt{I} = \sqrt{(a)}$. The ring A is radically principal if every ideal of A is radically principal.

Remark 2.2. Let I be an ideal of A, and set $U = \text{Spec}A \setminus V(I)$, where $V(I)$ the closed subset $V(I) = \{p \in \text{Spec}A / I \subseteq p\}$. Then I is radically principal if and only if U is a principal affine open of Spec$A$. Indeed, I is radically principal if and only if there exists $a \in A$ such that $\sqrt{I} = \sqrt{(a)}$, which is equivalent to $V(I) = V((a))$, that is $U = \text{Spec}A \setminus V((a))$ is a principal affine open. In particular a ring A is radically principal if and only if every open of Spec$A$ is a principal affine open. Since every principal affine open is quasi-compact, it follows that every radically principal ring has Noetherian spectrum.

Example 2.3. (1) Every principal ideal ring is radically principal.

(2) Let $\mathbb{K}$ be a filed, and set $A = \dfrac{\mathbb{K}[X,Y]}{(X^2, XY, Y^2)} = \mathbb{K}[x,y]$, where $x = \overline{X}$ and $y = \overline{Y}$. Then A is a radically principal ring. Let P be a prime ideal of A. Then $x^2 = y^2 = 0 \in P$, and hence $x, y \in P$, that is $(x,y) \subseteq P$. Since (x,y) is a maximal ideal of A, $P = (x,y)$. Thus the only prime ideal of A is $P = (x,y) = \sqrt{(0)}$. If I is an ideal of A, then either $I = A = \sqrt{(1)}$ or $I \subseteq P$, and in this case $\sqrt{I} = P = \sqrt{(0)}$. This shows that A is radically principal. Note that $P = (x,y)$ is not a principal ideal, hence A is not a principal ideal ring, but it is radically principal.

Proposition 2.4. *Let A be a commutative ring. If I is a radically principal ideal of A, then there is $a \in I$, such that $I = \sqrt{(a)}$.*

Proof. If I is radically principal, then there is $b \in A$ such that $\sqrt{I} = \sqrt{(b)}$. Since $b \in \sqrt{I}$, there exists $n \in \mathbb{N}$ such that $b^n \in I$, and hence $\sqrt{I} = \sqrt{(b)} = \sqrt{(b^n)} = \sqrt{a}$, where $a = b^n$. □

Proposition 2.5. *If I and J are radically principal ideals of A, then so is IJ and $I \cap J$.*

Proof. Let $a, b \in A$ such that $\sqrt{I} = \sqrt{(a)}$ and $\sqrt{J} = \sqrt{(b)}$. Since $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J} = \sqrt{(a)} \cap \sqrt{(b)} = \sqrt{(ab)}$, it follows that IJ and $I \cap J$ are radically principal ideals. □

Proposition 2.6. *Let A be a radically principal ring. For every ideal I of A, it follows that A/I is a radically principal ring.*

Proof. Let J be an ideal of A containing I. Then $J = \sqrt{(a)}$ for some $a \in A$, and hence $\sqrt{J/I} = \sqrt{(\overline{a})}$. it follows that A/I is radically principal. □

The following theorem gives the analogue of the Cohen-type theorem for radically principal rings.

Theorem 2.7. *Let A be a commutative ring. Then A is radically principal if and only if every prime ideal of A is radically principal.*

Proof. There is nothing to show for the direct implication.

Assume that every prime ideal of A is a radically principal ideal. Set $E = \{I/ \ I \text{ not radically principal ideal of } A\}$. We show that $E = \emptyset$. By contradiction, suppose that E is not empty. Now let $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be an increasing chain of elements of E (each ideal of this chain is not radically principal). Set $I = \cup_i I_i$. Clearly I is an ideal of A. If I is radically principal, then $\sqrt{I} = \sqrt{(f)}$ for some $f \in A$. Since $f \in \sqrt{I}$, we have $f^N \in I$, for some $N \in \mathbb{N}$. Hence $f^N \in I_j$ for some j. Thus $f \in \sqrt{I_j}$, it follows that $\sqrt{I_j} = \sqrt{(f)}$, which is a contradiction by the fact that I_j is not radically principal. Now, $I_n \subseteq I$ for all $n \in \mathbb{N}$ and $I \in E$. By Zorn's lemma, E has a maximal element say P. Let us next show that P is a prime ideal. Let $a, b \in A$ such that $ab \in P$, and assume that $a, b \notin P$. Set $P_a = P + (a)$ and $P_b = P + (b)$. Since P is a maximal element of E, $P \subsetneq P_a$, and $P \subsetneq P_b$, it follows that P_a and P_b are radically principal ideals. By Proposition 2.5, $P_a P_b$ is radically principal, but $\sqrt{P_a P_b} = \sqrt{P}$, since $P_a P_b \subseteq P$ and $P^2 \subseteq P_a P_b$. This yields a contradiction. Now P is a prime ideal and not radically principal. It follows that $E = \emptyset$, and every ideal of A is radically principal. □

Corollary 2.8. *Let A be a radically principal ring. If S is a multiplicative subset of A, then $S^{-1}A$ is a radically principal ring.*

Proof. Let P be a prime ideal of $S^{-1}A$. Then $P = S^{-1}p$ for some prime ideal of A. Since $\sqrt{P} = S^{-1}p = S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$, where I is a principal ideal of A such that $p = \sqrt{I}$, P is radically principal. By the previous theorem, $S^{-1}A$ is radically principal. □

Remark 2.9. If A is radically principal, then A_p, where $p \in \operatorname{Spec}(A)$ (respectively A_f, where $f \in A$) is a radically principal ring.

Corollary 2.10. *Let $A_1, \dots, A_m$ be rings, and set $A = A_1 \times \cdots \times A_m$. Then A is radically principal if and only ifA_i is radically principal for every i.*

Proof. If A is a radically principal ring, then $A_i = \dfrac{A}{I_i}$, where $I_i = A_1 \times \cdots \times A_{i-1} \times \{0\} \times A_{i+1} \times \cdots \times A_m$, is a radically principal ring. Conversely, assume that each A_i is a radically principal rings. Let q be a prime ideal of A. Then there is a prime ideal p_i of A_i for some i, such that $p = A_1 \times \cdots \times p_i \times \cdots \times A_m$. Since A_i is a radically principal ring, there is $a_i \in p_i$ such that $p_i = \sqrt{(a_i)}$. It is easy to see that $p = \sqrt{(a)}$, where $a = (1, \dots, a_i, \dots, 1)$. □

Theorem 2.11. *Let A be a commutative ring. The following statements are equivalent:*

(1) *A is a radically principal ring.*
(2) *For every prime ideal P of A, we have $P \nsubseteq \cup_{P \nsubseteq Q} Q$.*

Proof. (1) $\Rightarrow$ (2): Let P be a prime ideal of A, and let $a \in A$ such that $P = \sqrt{(a)}$. If Q is a prime ideal of A with $P \nsubseteq Q$, then $a \notin Q$ since $P = \sqrt{a} \nsubseteq Q$. It follows that $a \notin \cup_{P \nsubseteq Q} Q$. Hence $P \nsubseteq \cup_{P \nsubseteq Q} Q$.

(2) $\Rightarrow$ (1): Let P be a prime ideal of A. Since $P \nsubseteq \cup_{P \nsubseteq Q} Q$, there is $a \in P$ such that $a \notin Q$ whenever $P \nsubseteq Q$. Clearly $\sqrt{a} \subseteq P$. If Q is a prime ideal containing a, then $P \subseteq Q$. Thus $P \subseteq \cap_{a \in Q} Q = \sqrt{a}$. It follows that $P = \sqrt{(a)}$. By Theorem 2.7, A is radically principal. □

For a commutative ring A, it is well known that if P is a prime ideal such that $P \subseteq \cup_{i \in I} P_i$, where P_i are prime ideals and I is finite, then $P \subseteq P_i$ for some $i \in I$. This result does not hold for an infinite set I. The following result characterizes commutative rings with this property.

Corollary 2.12. *Let A be a commutative ring. The following statements are equivalent:*

(1) *A is a radically principal ring.*
(2) *A has the avoidance property, that is, if $P \subseteq \cup_{i \in I} P_i$, where P and P_i are prime ideals, then $P \subseteq P_i$ for some $i \in I$.*

Proof. Immediate from the previous theorem. □

Corollary 2.13. *Let A be a commutative ring. If A has finitely many prime ideals, then A is a radically principal ring.*

Proof. If A has a finitely many prime ideals, then the avoidance property hods. By the previous result, it follows that A is radically principal. □

Remark 2.14. Let V be a valuation domain with finite Krull dimension, $\dim V = n$. Then V has finitely many prime ideals, so radically principal. This is an example of radically principal ring with arbitrary finite Krull dimension.

It is well known that, if A a principal ideal ring, then $\dim A \leq 1$. The previous example shows that, the krull dimension of a radically principal ring is arbitrary. Indeed in the case of Noetherian rings, we have the following result.

Proposition 2.15. *Let A be a commutative ring. If A is radically principal and Noetherian, then* $\dim A \leq 1$.

Proof. Let P be a prime ideal of A. Then $P = \sqrt{(a)}$ for some $a \in A$. It follows that P is a minimal prime ideal over the principal ideal (a). Since A is Noetherian, by Krull's principal ideal theorem, the height of P is less than or equal to 1, so $\dim A \leq 1$. □

3. Zero-dimensional radically principal ring

In this section, we study radically principal rings of small dimension.

Proposition 3.1. *Let A be a zero-dimensional ring. Then A is radically principal if and only if A has finitely many prime ideals.*

Proof. Let A be a zero-dimensional radically principal ring. If A is a radically principal ring, then it has a Noetherian spectrum, by [3, Theorem 1.6], A has finitely many minimal prime ideals. Since A is a zero-dimensional ring, every prime ideal of A is minimal. It follows that A has finitely many prime ideals. For the converse, see Corollary 2.13. □

Theorem 3.2. *Let A be a zero-dimensional ring. Then A is a radically principal ring if and only if A is a product of a zero-dimensional local rings.*

Proof. If A is a product of zero-dimensional local rings, then it is a radically principal ring, since it is a product of a radically principal rings.

Conversely, assume that A is a radically principal ring, by the previous proposition, A has a finitely many prime ideals, say $p_1, \dots, p_r$. For each $1 \leq i \leq r$, $\cap_{j \neq i} p_j \not\subseteq p_i$ (in fact, if $\cap_{j \neq i} p_j \subseteq p_i$, then there exists $j \neq i$ such that $p_j \subseteq p_i$, which is not possible, since $\dim A = 0$). Hence there is $f_i \in \cap_{j \neq i} p_j$ such that $f_i \notin p_i$, that is, $f_i \notin p_i$ and $f_i \in p_j$ for $j \neq i$. For $i \neq j$, we have $f_i f_j \in p_1 \cap \dots \cap p_r$. Since $p_1 \cap \dots \cap p_r = \sqrt{0}$ the nilradical of A, we have that $f_i f_j$ is a nilpotent element. Hence for $i \neq j$, there exists $N_{ij} \in \mathbb{N}$ such that $(f_i f_j)^{N_{ij}} = 0$. Let $N = \max_{i \neq j} N_{ij}$; then we have $(f_i f_j)^N = 0$ whenever $j \neq i$. Now, set $a_i = f_i^N$. Then $a_i \in p_j$ if and only if $i \neq j$ and $a_i a_j = 0$ whenever $i \neq j$. In particular, there is no prime ideal of A containing all a_i, it follows that $(a_1, \dots, a_r) = A$, so $1 = \alpha_1 a_1 + \dots + \alpha_r a_r$, where $\alpha_i \in A$. Set $e_i = \alpha_i a_i$; then $e_i e_j = 0$ if $i \neq j$ and $e_i = e_i(e_1 + \dots + e_r) = e_i^2$. It easy to see that $e_i \in p_j$ for $j \neq i$ and $e_i \notin p_i$ (since $e_i \in p_i$ implies all e_j are in p_i hence $1 \in p_i$). Let $\varphi : A \to A_{e_1} \times \dots \times A_{e_r}$ be the morphism defined by $\varphi(a) = (a, \dots, a)$. Then φ is an isomorphism of rings. For each i, A_{e_i} is a zero-dimensional local ring, by the fact that $A_{e_i} = A_{p_i}$ (p_i is the unique prime ideal of A that does not containing e_i). This shows that A is a product of a zero-dimensional local rings. □

4. The case of the polynomial ring $A[X]$

In this section, we characterize ? when the polynomial ring $A[X]$ is radically principal. We start this section by the case where the ring A is an a integral domain.

Proposition 4.1. *Let A be an integral domain. The following statements are equivalent:*

(1) *$A[X]$ is a radically principal ring.*
(2) *A is a field.*

Proof. If A is a field, then $A[X]$ is a principal domain, which implies that is a radically principal ring.

(1) $\Rightarrow$ (2). Assume that $A[X]$ is a radically principal ring. Let a be a nonzero element of A and let I be the ideal of $A[X]$ generated by a and X. Since $A[X]$ is radically principal, there is $f \in A[X]$ such that $\sqrt{I} = \sqrt{(f)}$. Since $a \in I \subseteq \sqrt{(f)}$, there is $N \in \mathbb{N}$ such that $a^N = gf$, where $g \in A[X]$. Since A is an integral domain, $0 = \deg a^N = \deg f + \deg g$. It follows that f is a nonzero constant. On the other hand, we have $X \in \sqrt{(f)}$, then there is $m \in \mathbb{N}$ such that $X^m = hf$, where $h \in A[X]$. Since f is constant, it follows that, $1 = \alpha_m f$, where α_m is the coefficient of X^m in h. Hence f is an invertible element and so $I = A[X]$, this implies that there exist $R, S \in A[X]$ such that $1 = aS + XR$, in particular $1 = aS(0)$. Thus a is an invertible element. □

The following corollary is an immediate result from the previous proposition. It states that, when A is an integral domain, there is equivalent between principal and radically principal for polynomial rings.

Corollary 4.2. *Let A be an integral domain. Then $A[X]$ is radically principal if and only if $A[X]$ is a principal domain.*

The following theorem gives a characterization for polynomial ring to be radically principal without integral domain hypothesis.

Theorem 4.3. *Let A be a commutative ring. The following statements are equivalent:*

(1) *$A[X]$ is a radically principal ring.*
(2) *A is a zero-dimensional radically principal ring.*
(3) *A is a product of zero-dimensional local rings.*

Proof. (1) $\Rightarrow$ (2) If $A[X]$ is a radically principal ring, then $A[X]/(X)$ is a radically principal ring, so it follows that A is radically principal. Let p be a prime ideal of A. Then $(A/p)[X] = A[X]/p[X]$, it follows that $(A/p)[X]$ is radically principal, by the previous proposition, A/p is a field, so p is a maximal ideal of A. Thus A is a zero-dimensional ring.

(3) $\Rightarrow$ (1) If A is a zero-dimensional local ring, then A has a unique prime ideal, say p, in particular, every element of p is nilpotent. Let Q be a prime ideal of $A[X]$. Clearly $Q \cap A = p$, hence $p[X] \subseteq Q$. Since $A[X]/p[X] = (A/p)[X]$ is a principal ideal domain, there is $f \in Q$ such that $Q/p[X] = (\overline{f})$. Since $f \in Q$, we have $\sqrt{f} \subseteq Q$. Conversely, let $g \in Q$. Then there is $R \in A[X]$ such that $\overline{g} = \overline{Rf}$, so it follows that $g = Rf + S$, where $S \in p[X]$. Since $S \in p[X]$, S is nilpotent, in particular $S \in \sqrt{(f)}$, thus $g = Rf + S \in \sqrt{(f)}$. It follows that $Q = \sqrt{(f)}$. Now, let $A_1, \ldots, A_r$ be zero-dimensional local rings and let $A = A_1 \times \cdots \times A_r$. Clearly $A[X] = A_1[X] \times \cdots \times A_r[X]$. Since for each $1 \leq i \leq r$, A_i is a zero-dimensional

local ring, $A_i[X]$ is a radically principal ring, and hence $A[X]$, as a product of a radically principal rings, is a radically principal ring.

(2) $\Leftrightarrow$ (3) See Theorem 3.2. ☐

We close this section by the following corollary.

Corollary 4.4. *Let $n \geq 1$ and let $X = \mathbb{A}^n = \mathrm{Spec}(A[X_1, ..., X_n])$ the affine space. The following statements are equivalent:*

(1) *Every open of X is a principal affine open.*

(2) *$n = 1$ and A is a zero-dimensional radically principal ring.*

Proof. Straightforward. ☐

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