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APPROXIMATION FOR THE BERNSTEIN OPERATOR OF MAX-PRODUCT KIND IN SYMMETRIC RANGE

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Abstract. In the approximation theory, polynomials are particularly positive linear operators. Nonlinear positive operators by means of maximum and product were introduced by B. Bede. In this paper, the max-product of Bernstein operators for symmetric ranges are introduced and some upper estimates of approximation error for some subclasses of functions are obtained. Also, we investigate the shape-preserving properties.

1. Introduction and preliminaries

In the theory of approximation, researchers investigated many operators within approximation of a continuous function by a sequence of linear positive operators. These operators are defined by the means of the addition and multiplication of the reals and all of them are linear operators. Up to now, in [1, 4, 11, 12, 13, 14], researchers studied approximation theory for linear operators, which has an admitted interest in the past decades. Bede, Coroianu, and Gal Sb introduced the nonlinear positive operators by means of discrete linear approximating operators. In [2, 3, 6, 9, 16, 17], "max-product kind operators" were presented by using maximum in the name of sum in usual linear operators and gave a Jackson-type error estimate in terms of modulus of continuity.

The Bernstein polynomials $B_n(x)$ were introduced by Bernstein (see [7]) as follows:

$$B_n(x) = B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \tag{1.1}$$

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which is called a Bernstein polynomial of order n of the function $f(x)$, where $f(x)$ is defined on the closed interval $[0,1]$. The general case of the Bernstein operator of max-product kind has given so much interest. In this paper, we study Bernstein operator of max-product kind in symmetric range. The Bernstein polynomials in symmetric range are defined by

$$C_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}+\frac{x}{2}\right)^k\left(\frac{1}{2}-\frac{x}{2}\right)^{n-k}f\left(2\frac{k}{n}-1\right), \tag{1.2}$$

where $x\in[-1,1]$, $f\in C[-1,1]$, and $n\in\mathbb{N}$ by Çilo [8] under the supervision of Izgi. They investigated that these operators given in (1.2) are linear positive in symmetric range and provide the Korovkin theorem conditions. Also they indicated that (1.2) are smooth convergence on the range of $[-1,1]$.

In this section, we indicate some general notations and definitions, which will be used in this study. Operations "$\vee$" (maximum) and "$\cdot$" (product) are considered over the set of positive reals, and $(\mathbb{R}_+,\vee,\cdot)$ is called as a max-product algebra.

Let $I\subset\mathbb{R}$ be a finite or infinite interval, and set

$$CB_+(I)=\{f:I\longrightarrow\mathbb{R}_+;f\quad continous\quad and\quad bounded\quad on\quad I\}.$$

The general form of discrete max-product-type approximation operators

$$L_n(f)(x)=\bigvee_{i=0}^{n}K_n(x,x_i)f(x_i),$$

$$L_n(f)(x)=\bigvee_{i=0}^{\infty}K_n(x,x_i)f(x_i),$$

where $n\in\mathbb{N}$, $f\in CB_+(I)$, $K_n(.,x_i)\in CB_+(I)$, and $x_i\in I$, for all i. These operators are nonlinear positive operators satisfying the pseudo-linearity property

$$L_n(\alpha.f\vee\beta.g)(x)=\alpha.L_n(f)(x)\vee\beta.L_n(g)(x),$$

for all $\alpha,\beta\in\mathbb{R}_+$ and $f,g:I\to\mathbb{R}_+$. Additionally, the max-product operators are positive homogeneous, in other words, $L_n(\lambda f)=\lambda L_n(f)$ for all $\lambda\geq 0$.

Now, we present the following general results to be useful later in the study.

Lemma 1.1 (see [5]). *Let $I\subset\mathbb{R}$ be a bounded or unbounded interval, let $f\in CB_+(I)$, and let $L_n:CB_+(I)\to CB_+(I)$, $n\in\mathbb{N}$, be a sequence of operators satisfying the following properties:*

(1) *If $f,g\in CB_+(I)$ satisfy $f\leq g$, then $L_n(f)\leq L_n(g)$ for all $n\in\mathbb{N}$;*
(2) *$L_n(f+g)\leq L_n(f)+L_n(g)$ for all $f,g\in CB_+(I)$.*

Then for all $f,g\in CB_+(I)$, $n\in\mathbb{N}$ and $x\in I$, we get

$$|L_n(f)(x)-L_n(g)(x)|\leq L_n(|f-g|)(x).$$

Remark 1.2. One can see that max-product type operators satisfy the conditions (1), (2) in Lemma 1.1. In fact, for $\alpha=1,\beta=1$, they satisfy a pseudo-linearity, which is stronger than the above condition (2).

2. Main results

In this section, we define the nonlinear Bernstein operators for symmetric ranges of max-product type as below:

$$C_n^{(M)}(f;x)=\frac{\bigvee_{k=0}^{n} p_{n,k}(x) f(2\frac{k}{n}-1)}{\bigvee_{k=0}^{n} p_{n,k}(x)},\quad n\in\mathbb{N},$$

where

$$p_{n,k}(x)=\binom{n}{k}\left(\frac{1}{2}+\frac{x}{2}\right)^{k}\left(\frac{1}{2}-\frac{x}{2}\right)^{n-k}.$$

Also $f:[-1,1]\to\mathbb{R}$ is a continuous function and the operators $C_n^{(M)}(f)(x)$ are positive and continuous on $[-1,1]$. Note that, $C_n^{(M)}(f)(x)$ operators satisfy the pseudo-linearity property and these operators also are positive homogeneous. Since $C_n^{(M)}(f)(-1)-f(-1)=C_n^{(M)}(f)(1)-f(1)=0$ for all n, throughout the paper, we can suppose that $-1<x<1$.

Now, we need the following notations and lemmas for the proofs of the main results. For each $k,j\in\{0,1,2,\dots,n\}$ and $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$, we obtain the following structure:

$$M_{k,n,j}(x)=\frac{p_{n,k}(x)\left|2\frac{k}{n}-1-x\right|}{p_{n,j}(x)},\quad m_{k,n,j}(x)=\frac{p_{n,k}(x)}{p_{n,j}(x)}.$$

If $k\geq j+1$, then

$$M_{k,n,j}(x)=\frac{p_{n,k}(x)\left(2\frac{k}{n}-1-x\right)}{p_{n,j}(x)},\tag{2.1}$$

and if $k\leq j-1$, then

$$M_{k,n,j}(x)=\frac{p_{n,k}(x)\left(x-2\frac{k}{n}+1\right)}{p_{n,j}(x)}.\tag{2.2}$$

Furthermore, for each $k,j\in\{0,1,2,\dots,n\}$, $k\geq j+2$, and $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$, we get the following

$$\overline{M}_{k,n,j}(x)=\frac{p_{n,k}(x)\left(2\frac{k}{n+1}-1-x\right)}{p_{n,j}(x)},\tag{2.3}$$

and for each $k,j\in\{0,1,2,\dots,n\}$, $k\leq j-2$, and $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$, we get the following

$$\widehat{M}_{k,n,j}(x)=\frac{p_{n,k}(x)\left(x-2\frac{k}{n+1}+1\right)}{p_{n,j}(x)}.\tag{2.4}$$

Lemma 2.1. *Let $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$. Then*

(1) *for all $k,j\in\{0,1,2,\dots,n\}$, $k\geq j+2$, we have*

$$\overline{M}_{k,n,j}(x)\leq M_{k,n,j}(x)\leq 3\overline{M}_{k,n,j}(x);$$

(2) *for all $k,j\in\{0,1,2,\dots,n\}$, $k\leq j-2$, we have*

$$M_{k,n,j}(x)\leq \widehat{M}_{k,n,j}(x)\leq 6M_{k,n,j}(x).$$

Proof. (1) By (2.1) and (2.3), it is clear that $\overline{M}_{k,n,j}(x) \le M_{k,n,j}(x)$. On the other hand,

$$\begin{aligned}\frac{M_{k,n,j}(x)}{\overline{M}_{k,n,j}(x)} =& \frac{2\frac{k}{n} - 1 - x}{2\frac{k}{n+1} - 1 - x} \le \frac{2\frac{k}{n} - 1 - \frac{2j}{n+1} + 1}{2\frac{k}{n+1} - 1 - \frac{2j+2}{n+1} + 1} \\ =& \frac{kn + k - nj}{n(k-j-1)} = \frac{k-j}{k-j-1} + \frac{k}{n(k-j-1)} \le 3,\end{aligned}$$

which proves (1). By using (2.2) and (2.4), it is obvious that $M_{k,n,j}(x) \le \widehat{M}_{k,n,j}(x)$. Additionally,

$$\begin{aligned}\frac{\widehat{M}_{k,n,j}(x)}{M_{k,n,j}(x)} =& \frac{x - 2\frac{k}{n+1} + 1}{x - 2\frac{k}{n} + 1} \le \frac{\frac{2j+2}{n+1} - 1 - \frac{2k}{n+1} + 1}{2\frac{j}{n+1} - 1 - 2\frac{k}{n} + 1} \\ =& \frac{n(j+1-k)}{nj - nk - k} \le \frac{(n+1)(j+1-k)}{nj - nk - n} \\ =& \frac{n+1}{n} \cdot \frac{j+1-k}{j-k-1} \le 2\frac{j+1-k}{j-k-1} \\ =& 2\left(1 + \frac{2}{j-k-1}\right) \le 6.\end{aligned}$$

□

Lemma 2.2. *For all $k, j \in \{0, 1, 2, \dots, n\}$ and $x \in \left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right]$, we get*

$$m_{k,n,j}(x) \le 1.$$

Proof. For the proof of the above lemma, we consider two cases:

$$(a)\ k \ge j, \quad \text{and} \quad (b)\ k \le j.$$

Case (a). Let $k \ge j$. Because $g(x) = \frac{1-x}{1+x}$ is nonincreasing on $\left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right]$, it follows that

$$\begin{aligned}\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)} =& \frac{k+1}{n-k} \cdot \frac{1-x}{1+x} \ge \frac{k+1}{n-k} \cdot \frac{1 - \frac{2j+2}{n+1} + 1}{1 + \frac{2j+2}{n+1} - 1} \\ =& \frac{k+1}{n-k} \cdot \frac{n-j}{j+1} \ge \frac{k+1}{n-k} \cdot \frac{n-k}{j+1} = \frac{k+1}{j+1} \ge 1,\end{aligned}$$

which indicates

$$m_{j,n,j}(x) \ge m_{j+1,n,j}(x) \ge m_{j+2,n,j}(x) \ge \dots \ge m_{n,n,j}(x).$$

Case (b). Let us take the case $k \le j$. We have

$$\begin{aligned}\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)} =& \frac{n-k+1}{k} \cdot \frac{1+x}{1-x} \ge \frac{n-k+1}{k} \cdot \frac{1 + \frac{2j}{n+1} - 1}{1 - \frac{2j}{n+1} + 1} \\ =& \frac{n-k+1}{k} \cdot \frac{j}{n-j+1} \ge \frac{n-k+1}{k} \cdot \frac{k}{n-k+1} = 1,\end{aligned}$$

which implies

$$m_{j,n,j}(x) \ge m_{j-1,n,j}(x) \ge m_{j-2,n,j}(x) \ge \dots \ge m_{0,n,j}(x).$$

□

Lemma 2.3. *Let* $x \in \left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right]$.

(1) *If* $k \in \{j+2, j+3, \ldots, n-1\}$ *is such that* $k - \sqrt{k+1} \geq j$, *then* $\overline{M}_{k,n,j}(x) \geq \overline{M}_{k+1,n,j}(x)$.

(2) *If* $k \in \{1, 2, \ldots, j-2\}$ *is such that* $k + \sqrt{k} \leq j$, *then* $\widehat{M}_{k,n,j}(x) \geq \widehat{M}_{k-1,n,j}(x)$.

Proof. (1) Let $k \in \{j+2, j+3, \ldots, n-1\}$ with $k - \sqrt{k+1} \geq j$. Then we have

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)} = \frac{k+1}{n-k} \cdot \frac{1-x}{1+x} \cdot \frac{2\frac{k}{n+1} - 1 - x}{2\frac{k+1}{n+1} - 1 - x}.$$

Since the function $\mu(x) = \frac{1-x}{1+x} \cdot \frac{2k-(n+1)-(n+1)x}{2(k+1)-(n+1)-(n+1)x}$ is nonincreasing, it follows that

$$\mu(x) \geq \mu\left(\frac{2j+2}{n+1} - 1\right) = \frac{n-j}{j+1} \cdot \frac{k-j-1}{k-j}$$

for all $x \in \left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right]$. Then, since the condition $k - \sqrt{k+1} \geq j$ implies $(k+1) \cdot (k-j-1) \geq (j+1)(k-j)$ and

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)} \geq \frac{k+1}{n-k} \cdot \frac{n-j}{j+1} \cdot \frac{k-j-1}{k-j} \geq 1,$$

we get $\overline{M}_{k,n,j}(x) \geq \overline{M}_{k+1,n,j}(x)$.

(2) We have

$$\frac{\widehat{M}_{k,n,j}(x)}{\widehat{M}_{k-1,n,j}(x)} = \frac{n-k+1}{k} \cdot \frac{1+x}{1-x} \cdot \frac{x - \frac{2k}{n+1} + 1}{x - 2\frac{k-1}{n+1} + 1}.$$

Since the function $\eta(x) = \frac{1+x}{1-x} \cdot \frac{(n+1)x-2k+(n+1)}{(n+1)x-2(k-1)+(n+1)}$ is nondecreasing, it follows that

$$\eta(x) \geq \eta\left(\frac{2j}{n+1} - 1\right) = \frac{j}{n+1-j} \cdot \frac{j-k}{j-k+1}$$

for all $x \in \left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right]$. Then since the condition $k + \sqrt{k} \leq j$ implies $j(j-k) \geq k(j-k+1)$, we have

$$\frac{\widehat{M}_{k,n,j}(x)}{\widehat{M}_{k-1,n,j}(x)} \geq \frac{n-k+1}{k} \cdot \frac{j}{n+1-j} \cdot \frac{j-k}{j-k+1} \geq 1.$$

□

Lemma 2.4. *Let us indicate* $p_{n,k}(x) = \binom{n}{k}\left(\frac{1}{2} + \frac{x}{2}\right)^k \left(\frac{1}{2} - \frac{x}{2}\right)^{n-k}$, *then*

$$\bigvee_{k=0}^{n} p_{n,k}(x) = p_{n,j}(x) \quad \text{for all } x \in \left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right], \quad j = 0, 1, \ldots, n.$$

In addition,

$$\bigvee_{k=0}^{n} p_{n,k}(x) = p_{n,0}(x) \quad \text{for all } x \in [0,1]$$

and

$$\bigvee_{k=0}^{n} p_{n,k}(x) = p_{n,n}(x) \quad \textit{for all } x \in [-1, 0].$$

Proof. Firstly, we demonstrate for fixed $n \in \mathbb{N}$ and $0 \leq k \leq k+1 \leq n$, that

$$0 \leq p_{n,k+1}(x) \leq p_{n,k}(x) \quad \text{if and only if} \quad x \in \left[0, \frac{2k+2}{n+1} - 1\right].$$

Let us take the following inequality:

$$0 \leq \binom{n}{k+1} \left(\frac{1}{2} + \frac{x}{2}\right)^{k+1} \left(\frac{1}{2} - \frac{x}{2}\right)^{n-k-1} \leq \binom{n}{k} \left(\frac{1}{2} + \frac{x}{2}\right)^{k} \left(\frac{1}{2} - \frac{x}{2}\right)^{n-k}.$$

After some simplifications and using this equality $\binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1}$, we can reduce the above inequality to

$$0 \leq x \leq \frac{2k+2}{n+1} - 1.$$

By taking $k = 0, 1, \ldots, n$ in the inequality above, we get

$$p_{n,1}(x) \leq p_{n,0}(x), \quad \text{if and only if} \quad x \in \left[0, \frac{2}{n+1} - 1\right],$$
$$p_{n,2}(x) \leq p_{n,1}(x), \quad \text{if and only if} \quad x \in \left[0, \frac{4}{n+1} - 1\right],$$
$$p_{n,3}(x) \leq p_{n,2}(x), \quad \text{if and only if} \quad x \in \left[0, \frac{6}{n+1} - 1\right],$$

and

$$p_{n,k+1}(x) \leq p_{n,k}(x), \quad \text{if and only if} \quad x \in \left[0, \frac{2k+2}{n+1} - 1\right],$$

and finally

$$p_{n,n-2}(x) \leq p_{n,n-3}(x), \quad \text{if and only if} \quad x \in \left[0, \frac{2n-4}{n+1} - 1\right],$$
$$p_{n,n-1}(x) \leq p_{n,n-2}(x), \quad \text{if and only if} \quad x \in \left[0, \frac{2n-2}{n+1} - 1\right],$$
$$p_{n,n}(x) \leq p_{n,n-1}(x), \quad \text{if and only if} \quad x \in \left[0, \frac{2n}{n+1} - 1\right].$$

On the other hand, because

$$p_{n,k}(x).p_{n,k+1}(x) = \binom{n}{k+1}\binom{n}{k} \frac{1}{2^2 n} (1+x)^{2k+1} (1-x)^{2(n-k)-1} > 0$$

and from all these inequalities given above, reasoning by recurrence, we obtain

$$\text{if}\quad x\in\left[0,\frac{2}{n+1}-1\right], \text{then}\quad p_{n,k}(x)\le p_{n,0}(x), \text{for all}\quad k=0,1,\dots,n;$$

$$\text{if}\quad x\in\left[\frac{2}{n+1}-1,\frac{4}{n+1}-1\right], \text{then}\quad p_{n,k}(x)\le p_{n,1}(x), \text{for all}\quad k=0,1,\dots,n;$$

$$\text{if}\quad x\in\left[\frac{4}{n+1}-1,\frac{6}{n+1}-1\right], \text{then}\quad p_{n,k}(x)\le p_{n,2}(x), \text{for all}\quad k=0,1,\dots,n;$$

and in general

$$\text{if}\quad x\in\left[\frac{2n}{n+1}-1,1\right], \text{then}\quad p_{n,k}(x)\le p_{n,n}(x), \text{for all}\quad k=0,1,\dots,n.$$

□

3. Degree of approximation by $C_n^{(M)}(f)(x)$

In this section, we give the main results about the nonlinear Bernstein operator of max-product kind for the symmetric range defined in Section 1.

Theorem 3.1. *If $f:[-1,1]\to\mathbb{R}_+$ is a continuous function, then the following inequality holds*

$$\left|C_n^{(M)}(f)(x)-f(x)\right|\le 24\omega_1\left(f;\frac{1}{\sqrt{n+1}}\right),\quad \text{for all } n\in\mathbb{N},\quad x\in[-1,1],$$

where

$$\omega_1(f;\delta)=\sup\{|f(x)-f(y)|;x,y\in[-1,1],|x-y|\le\delta\}.$$

Proof. Since $C_n^{(M)}(e_0)(x)=1$, by using the Shisha–Mond theorem given for nonlinear max-product type operators in [2, 5], we get

$$\left|C_n^{(M)}(f)(x)-f(x)\right|\le\left(1+\frac{1}{\delta_n}C_n^{(M)}(\varphi_x)(x)\right)\omega_1(f;\delta_n), \tag{3.1}$$

where $\varphi_x(t)=|t-x|$. Therefore it is enough to estimate only the following term:

$$E_n(x)=C_n^{(M)}(\varphi_x)(x)=\frac{\bigvee_{k=0}^n p_{n,k}(x)\left|2\frac{k}{n}-1-x\right|}{\bigvee_{k=0}^n p_{n,k}(x)}.$$

Let $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$, where $j\in\{0,1,\dots,n\}$ is fixed and arbitrary. By Lemma 2.4, we have

$$E_n(x)=\bigvee_{k=0}^n M_{k,n,j}(x).$$

Since for $j=0$, we get $E_n(x)\le 2/n$ for all $x\in[-1,\frac{1-n}{n+1}]$, we may suppose that $j\in\{1,\dots,n\}$. We will find an upper estimate for each $M_{k,n,j}(x)$, where $j\in\{0,1,\dots,n\}$ is fixed, $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$, and $k\in\{0,1,\dots,n\}$. The proof will be divided into 3 cases:

$$\text{(a) } k\in\{j-1,j,j+1\},\quad \text{(b) } k\ge j+2\quad\text{and}\quad \text{(c) } k\le j-2.$$

Case (a). If $k=j$ then $M_{j,n,j}(x)=\left|2\frac{j}{n}-1-x\right|$. Since $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$, it follows immediately that $M_{j,n,j}(x)\leq 2/(n+1)$.

If $k=j+1$, then $M_{j+1,n,j}(x)=m_{j+1,n,j}(x)\cdot\left(2\frac{j+1}{n}-1-x\right)$. From Lemma 2.2, we have $m_{j+1,n,j}(x)\leq 1$, which implies

$$\begin{aligned}M_{j+1,n,j}(x)\leq&2\frac{j+1}{n}-1-x\leq 2\frac{j+1}{n}-1-2\frac{j}{n+1}+1\\=&2\frac{n+j+1}{n(n+1)}\leq 6/(n+1).\end{aligned}$$

If $k=j-1$, then $M_{j-1,n,j}(x)=m_{j-1,n,j}(x)\cdot\left(x-2\frac{j-1}{n}-1\right)$. From Lemma 2.2, we have $m_{j-1,n,j}(x)\leq 1$, which implies

$$\begin{aligned}M_{j-1,n,j}(x)\leq&x-2\frac{j-1}{n}-1\leq 2\frac{j+2}{n+1}-1-2\frac{j-1}{n}+1\\=&2\frac{2n-(j-1)}{n(n+1)}\leq 4/(n+1).\end{aligned}$$

Case (b). Subcase (b.1). Let $k-\sqrt{k+1}<j$; then

$$\begin{aligned}\overline{M}_{k,n,j}(x)=&m_{k,n,j}(x)\left(2\frac{k}{n+1}-1-x\right)\\\leq&2\frac{k}{n+1}-1-x\leq 2\frac{k}{n+1}-1-2\frac{j}{n+1}+1\\\leq&2\frac{k}{n+1}-2\frac{k-\sqrt{k+1}}{n+1}=2\frac{\sqrt{k+1}}{n+1}\leq\frac{2}{\sqrt{n+1}}.\end{aligned}$$

Subcase (b.2). Let $k-\sqrt{k+1}\geq j$. Since the function $g(x)=x-\sqrt{x+1}$ is nondecreasing on the interval $[-1,1]$, it follows that there exists $\overline{k}=\{0,1,2,\dots,n\}$ of the maximum value, such that $\overline{k}-\sqrt{\overline{k}+1}<j$. Then for $k_1=\overline{k}+1$, we get $k_1-\sqrt{k_1+1}\geq j$ and

$$\begin{aligned}\overline{M}_{\overline{k}+1,n,j}(x)=&m_{\overline{k}+1,n,j}(x)\left(2\frac{\overline{k}+1}{n+1}-1-x\right)\\\leq&2\frac{\overline{k}+1}{n+1}-1-2\frac{j}{n+1}+1\\\leq&2\frac{\overline{k}+1}{n+1}-2\frac{\overline{k}-\sqrt{\overline{k}+1}}{n+1}\\=&\frac{2(\sqrt{\overline{k}+1}+1)}{n+1}\leq\frac{4}{\sqrt{n+1}}.\end{aligned}$$

Furthermore, we have $k_1\geq j+2$, this is a consequence of the fact that g is nondecreasing and it is easy to see that $g(j+1)<j$. By Lemma 2.3(1), it follows that $\overline{M}_{\overline{k}+1,n,j}(x)\geq\overline{M}_{\overline{k}+2,n,j}(x)\geq\cdots\geq\overline{M}_{n,n,j}(x)$.

We obtain $\overline{M}_{k,n,j}(x)\leq\frac{4}{\sqrt{n+1}}$ for any $k\in\left\{\overline{k}+1,\overline{k}+2,\dots,n\right\}$. Thus in Subcases (b.1) and (b.2), we have $\overline{M}_{k,n,j}(x)\leq\frac{4}{\sqrt{n+1}}$. Hence, from Lemma 2.1(1), we have $M_{k,n,j}(x)\leq\frac{12}{\sqrt{n+1}}$.

Case (c). Subcase (c.1). Let $k+\sqrt{k}\geq j$. Then

$$\begin{aligned}\widehat{M}_{k,n,j}(x) =& m_{k,n,j}(x)\left(x-2\frac{k}{n+1}+1\right)\\ \leq& \frac{2j+2}{n+1}-1-2\frac{k}{n+1}+1\\ \leq& \frac{2(k+\sqrt{k}+1)}{n+1}-2\frac{k}{n+1}=\frac{2(\sqrt{k}+1)}{n+1}\\ \leq& \frac{2(\sqrt{n}+1)}{n+1}\leq\frac{4}{\sqrt{n+1}}.\end{aligned}$$

Subcase (c.2). Now let $k+\sqrt{k}<j$ and let $\widetilde{k}=\{0,1,2,\dots,n\}$ be the minimum value such that $\widetilde{k}+\sqrt{\widetilde{k}}\geq j$. Then $k_2=\widetilde{k}-1$ satisfies $k_2+\sqrt{k_2}<j$ and

$$\begin{aligned}\widehat{M}_{\widetilde{k}-1,n,j}(x) =& m_{\widetilde{k}-1,n,j}(x)\left(x-2\frac{\widetilde{k}-1}{n+1}+1\right)\\ \leq& \frac{2j+2}{n+1}-1-2\frac{\widetilde{k}-1}{n+1}+1\\ \leq& \frac{2(\sqrt{\widetilde{k}}+2)}{n+1}\leq\frac{6}{\sqrt{n+1}}.\end{aligned}$$

Also, in this case, we have $j\geq 2$, which implies $k_2\leq j-2$. By Lemma 2.3(2), we get $\widehat{M}_{\widetilde{k}-1,n,j}(x)\geq\widehat{M}_{\widetilde{k}-2,n,j}(x)\geq\dots\geq\widehat{M}_{0,n,j}(x)$.

Hence, we obtain

$$\widehat{M}_{k,n,j}(x)\leq\frac{6}{\sqrt{n+1}}\quad\text{for any}\quad k\leq j-2\quad\text{and}\quad x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right].$$

Therefore in Subcases (c.1) and (c.2), we have $\widehat{M}_{k,n,j}(x)\leq\frac{6}{\sqrt{n+1}}$. Hence from Lemma 2.1(2), we have $M_{k,n,j}(x)\leq\frac{6}{\sqrt{n+1}}$. Consequently, collecting all the above estimates, we obtain

$$M_{k,n,j}(x)\leq\frac{12}{\sqrt{n+1}}\quad\text{for all } x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right],\ k=\{0,1,2,\dots,n\},$$

which implies that

$$E_n(x)\leq\frac{12}{\sqrt{n+1}}\quad\text{for all } x\in[-1,1],n\in\mathbb{N},$$

and indicating $\delta_n=\frac{12}{\sqrt{n+1}}$ in (3.1), we get the estimate

$$\left|C_n^{(M)}(f)(x)-f(x)\right|\leq 24\omega_1\left(f;\frac{1}{\sqrt{n+1}}\right),\quad\text{for all } n\in\mathbb{N},x\in[-1,1].$$

□

Now, we want to show a better order of approximation for subclasses of function f. For demonstrating a better approximation, let us consider the functions $f_{k,n,j} : \left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right] \to \mathbb{R}$ defined as

$$f_{k,n,j}(x) = m_{k,n,j}(x) f\left(2\frac{k}{n} - 1\right) = \frac{\binom{n}{k}}{\binom{n}{j}} \left(\frac{1+x}{1-x}\right)^{k-j} f\left(2\frac{k}{n} - 1\right). \tag{3.2}$$

Hence, for any $j \in \{0, 1, 2, \ldots, n\}$ and $x \in \left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right]$, we can write

$$C_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x). \tag{3.3}$$

Lemma 3.2. *Let us take $f : [-1, 1] \to [0, \infty)$ such that*

$$C_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\} \quad \text{for all } x \in \left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right].$$

Then

$$\left|C_n^{(M)}(f)(x) - f(x)\right| \leq 4\omega_1\left(f; \frac{1}{n}\right),$$

where $\omega_1(f; \delta) = \sup\{|f(x) - f(y)|; x, y \in [-1, 1], |x - y| \leq \delta\}$.

Proof. For the proof, we need the following two cases:

Case (a). Let $x \in \left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right]$ be fixed such that $C_n^{(M)}(f)(x) = f_{j,n,j}(x)$. By applying simple calculations, we get $-\frac{2}{n+1} \leq x - 2\frac{j}{n} + 1 \leq \frac{2}{n+1}$ and $f_{j,n,j}(x) = f\left(2\frac{j}{n} - 1\right)$, it follows that

$$\left|C_n^{(M)}(f)(x) - f(x)\right| = \left|f\left(2\frac{j}{n} - 1\right) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n+1}\right). \tag{3.4}$$

Case (b). Let $x \in \left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right]$ be fixed such that $C_n^{(M)}(f)(x) = f_{j+1,n,j}(x)$.

Subcase (b.1). If $C_n^{(M)}(f)(x) \leq f(x)$, then $f_{j,n,j}(x) \leq f_{j+1,n,j}(x) \leq f(x)$, and we have

$$\begin{aligned}\left|C_n^{(M)}(f)(x) - f(x)\right| &= |f_{j+1,n,j}(x) - f(x)| = f(x) - f_{j+1,n,j}(x) \\ &\leq f(x) - f_{j,n,j}(x) = f(x) - f\left(2\frac{j}{n} - 1\right) \\ &\leq 2\omega_1\left(f; \frac{1}{n+1}\right).\end{aligned}$$

Subcase (b.2). Let $C_n^{(M)}(f)(x) > f(x)$; then

$$\begin{aligned}\left|C_n^{(M)}(f)(x) - f(x)\right| &= f_{j+1,n,j}(x) - f(x) \\ &= m_{j+1,n,j}(x) f\left(2\frac{j+1}{n} - 1\right) - f(x) \\ &\leq f\left(2\frac{j+1}{n} - 1\right) - f(x).\end{aligned}$$

Because

$$0 \leq 2\frac{j+1}{n} - 1 - x \leq 2\frac{j+1}{n} - 1 - 2\frac{j}{n+1} + 1$$
$$= 2\left(\frac{j}{n(n+1)} + \frac{n+1}{n(n+1)}\right) < \frac{4}{n},$$

then

$$f\left(2\frac{j+1}{n} - 1\right) - f(x) \leq 4\omega_1\left(f; \frac{1}{n}\right).$$

□

Lemma 3.3. *Let $f : [-1, 1] \to [0, \infty)$ such that*

$$C_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j-1,n,j}(x)\} \quad \textit{for all } x \in \left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right].$$

Then

$$\left|C_n^{(M)}(f)(x) - f(x)\right| \leq 4\omega_1\left(f; \frac{1}{n}\right).$$

Proof. For the proof, we need two cases:

Case (a). Let $C_n^{(M)}(f)(x) = f_{j,n,j}(x)$. When as Lemma 3.2, we have

$$\left|C_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n+1}\right).$$

Case (b). Let $x \in \left[\frac{2j}{n+1} - 1, \frac{2j+2}{n+1} - 1\right]$ be fixed such that $C_n^{(M)}(f)(x) = f_{j-1,n,j}(x)$. Here, we have two cases:

Subcase (b.1). Let $C_n^{(M)}(f)(x) \leq f(x)$. Then following the proof of Lemma 3.2, we get

$$C_n^{(M)}(f)(x) - f(x) \leq 2\omega_1\left(f; \frac{1}{n+1}\right).$$

Subcase (b.2). If $C_n^{(M)}(f)(x) > f(x)$ and using the following inequality:

$$0 \leq x - 2\frac{j-1}{n} + 1 \leq \frac{2(j+2)}{n+1} - 1 - \frac{2(j-1)}{n} + 1$$
$$= 2\left(\frac{-j}{n(n+1)} + \frac{1}{n+1} + \frac{1}{n}\right) < \frac{4}{n},$$

we obtain

$$\begin{aligned}
\left|C_n^{(M)}(f)(x) - f(x)\right| &= f_{j-1,n,j}(x) - f(x) \\
&= m_{j-1,n,j}(x) f\left(2\frac{j-1}{n} - 1\right) - f(x) \\
&\leq f\left(2\frac{j-1}{n} - 1\right) - f(x) \\
&\leq 4\omega_1\left(f; \frac{1}{n}\right).
\end{aligned}$$

□

Lemma 3.4. *Let* $f:[-1,1]\to[0,\infty)$ *such that*

$$C_n^{(M)}(f)(x)=\max\{f_{j-1,n,j}(x),f_{j,n,j}(x),f_{j+1,n,j}(x)\}$$

for all $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$. *Then*

$$\left|C_n^{(M)}(f)(x)-f(x)\right|\leq 4\omega_1\left(f;\frac{1}{n}\right).$$

Proof. Let $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$. If $C_n^{(M)}(f)(x)=f_{j,n,j}(x)$ or $C_n^{(M)}(f)(x)=f_{j+1,n,j}(x)$, then $C_n^{(M)}(f)(x)=\max\{f_{j,n,j}(x),f_{j+1,n,j}(x)\}$ and from Lemma 3.2, we have

$$\left|C_n^{(M)}(f)(x)-f(x)\right|\leq 4\omega_1\left(f;\frac{1}{n}\right).$$

If $C_n^{(M)}(f)(x)=f_{j-1,n,j}(x)$, then $C_n^{(M)}(f)(x)=\max\{f_{j,n,j}(x),f_{j-1,n,j}(x)\}$ and from Lemma 3.3, we have

$$\left|C_n^{(M)}(f)(x)-f(x)\right|\leq 4\omega_1\left(f;\frac{1}{n}\right).$$

□

Lemma 3.5. *Let us take the function* $f:[-1,1]\to[0,\infty)$ *is concave. Then the following two properties hold:*

(1) *The function* $g:(-1,0)\cup(0,1]\to[0,\infty)$, $g(x)=f(x)/|x|$ *is nonincreasing.*
(2) *The function* $h:[-1,0)\cup(0,1)\to[0,\infty)$, $h(x)=f(x)/1-x$ *is nondecreasing.*

Proof. (1) Let us take $x,y\in(-1,0)\cup(0,1]$ such that $x\leq y$. Then

$$f(x)=f\left(\frac{x}{y}\cdot y+\frac{y-x}{y}\cdot 0\right)\geq\frac{x}{y}f(y)+\frac{y-x}{y}f(0)\geq\frac{x}{y}f(y),$$

which implies $f(x)/|x|\geq f(y)/|y|$.

(2) Let $x,y\in[-1,0)\cup(0,1)$ such that $x\geq y$. Then

$$f(x)=f\left(\frac{1-x}{1-y}\cdot y+\frac{x-y}{1-y}\cdot 1\right)\geq\frac{1-x}{1-y}f(y)+\frac{x-y}{1-y}f(1)\geq\frac{1-x}{1-y}f(y),$$

which implies $f(x)/(1-x)\geq f(y)/(1-y)$. □

Corollary 3.6. *The function* $f:[-1,1]\to[0,\infty)$ *is a concave function and for all* $x\in[-1,1]$, *it follows that*

$$\left|C_n^{(M)}(f)(x)-f(x)\right|\leq 4\omega_1\left(f;\frac{1}{n}\right).$$

Proof. Let us take $x\in[-1,1]$ and let $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$ for $j\in\{0,1,2,\ldots,n\}$. If $k\geq j$, for $k\in\{0,1,2,\ldots,n\}$, then

$$\begin{aligned}f_{k+1,n,j}(x)=&\frac{\binom{n}{k+1}}{\binom{n}{j}}\left(\frac{1+x}{1-x}\right)^{k+1-j}f\left(2\frac{k+1}{n}-1\right)\\=&\frac{\binom{n}{k}}{\binom{n}{j}}\frac{n-k}{k+1}\left(\frac{1+x}{1-x}\right)^{k-j}\frac{1+x}{1-x}f\left(2\frac{k+1}{n}-1\right).\end{aligned}$$

From Lemma 3.5(1), we have

$$f\left(2\frac{k+1}{n}-1\right)/\left(2\frac{k+1}{n}-1\right)\leq\left(\frac{k+1}{k}\right)f\left(2\frac{k}{n}-1\right),$$

and since $\frac{1+x}{1-x}\leq\frac{j+1}{n-j}$, we can write that

$$f\left(2\frac{k+1}{n}-1\right)/\left(2\frac{k+1}{n}-1\right)\leq f\left(2\frac{k}{n}-1\right)/\left(2\frac{k}{n}-1\right).$$

Therefore, we obtain

$$\begin{aligned}f_{k+1,n,j}(x)=&\frac{\binom{n}{k}}{\binom{n}{j}}\frac{n-k}{k+1}\left(\frac{1+x}{1-x}\right)^{k-j}\frac{j+1}{n-j}\frac{k+1}{k}f\left(2\frac{k}{n}-1\right)\\=&f_{k,n,j}(x)\frac{n-k}{n-j}\frac{j+1}{k}.\end{aligned}$$

By using the above inequality for $k\geq j+1$, we have $f_{k,n,j}(x)\geq f_{k+1,n,j}(x)$. Hence

$$f_{k+1,n,j}(x)\geq f_{k+2,n,j}(x)\geq\cdots\geq f_{n,n,j}(x). \tag{3.5}$$

When $k\leq j$, then

$$\begin{aligned}f_{k-1,n,j}(x)=&\frac{\binom{n}{k-1}}{\binom{n}{j}}\left(\frac{1+x}{1-x}\right)^{k-1-j}f\left(2\frac{k-1}{n}-1\right)\\=&\frac{\binom{n}{k}}{\binom{n}{j}}\frac{k}{n-k+1}\left(\frac{1+x}{1-x}\right)^{k-j}\frac{1-x}{1+x}f\left(2\frac{k-1}{n}-1\right).\end{aligned}$$

Since

$$f\left(2\frac{k}{n}-1\right)/\left(2-2\frac{k}{n}\right)\geq f\left(2\frac{k-1}{n}-1\right)/\left(2-2\frac{k-1}{n}\right)$$

which comes from Lemma 3.5, we have

$$f\left(2\frac{k}{n}\right)\geq\left(\frac{n-k}{n-k+1}\right)f\left(2\frac{k-1}{n}-1\right).$$

Because $\frac{1+x}{1-x}\leq\frac{n+1-j}{j}$, we get

$$\begin{aligned}f_{k-1,n,j}(x)=&\frac{\binom{n}{k}}{\binom{n}{j}}\frac{k}{n-k+1}\left(\frac{1+x}{1-x}\right)^{k-j}\frac{n+1-j}{j}\frac{n-k+1}{n-k}f\left(2\frac{k}{n}-1\right)\\=&f_{k,n,j}(x)\frac{k}{j}\frac{n+1-j}{n-k}.\end{aligned}$$

By the above inequality, for $k\leq j-1$, we get

$$f_{j-1,n,j}(x)\geq f_{j-2,n,j}(x)\geq\cdots\geq f_{0,n,j}(x). \tag{3.6}$$

Now, by using (3.5) and (3.6), we get

$$C_n^{(M)}(f)(x)=\max\left\{f_{j-1,n,j}(x),f_{j,n,j}(x),f_{j+1,n,j}(x)\right\},$$

and consequently from Lemma 3.4, the corollary proof is completed. $\square$

Remark 3.7. The max-product Bernstein operators in a symmetric range given in Corollary 3.6, which indicates the approximation order is $4\omega_1\left(f;\frac{1}{n}\right)$, have better approximation properties than its linear form given in [8], which give the approximation error as a $2\omega_1\left(f;\frac{1}{\sqrt{n}}\right)$.

4. Shape preserving properties

In this section,we present some shape-preserving properties for max-product type Bernstein operators in symmetric range. Firstly, for any $k,j\in\{0,1,2,\dots,n\}$, let us take the functions $f_{k,n,j}:\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]\to\mathbb{R}$ defined by

$$\begin{aligned} f_{k,n,j}(x) =& m_{k,n,j}(x) f\left(2\frac{k}{n}-1\right) \\ =& \frac{\binom{n}{k}}{\binom{n}{j}}\left(\frac{1+x}{1-x}\right)^{k-j} f\left(2\frac{k}{n}-1\right). \end{aligned}$$

Therefore we can write

$$C_n^M(f)(x)=\bigvee_{k=0}^{n} f_{k,n,j}(x),$$

for any $j\in\{0,1,2,\dots,n\}$ and $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$.

Lemma 4.1. *Let $f:[-1,1]\to[0,\infty)$ be a nondecreasing function; then for any $k,j\in\{0,1,2,\dots,n\}$, $k\le j$ and $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$, we get $f_{k,n,j}(x)\ge f_{k-1,n,j}(x)$.*

Proof. By the proof of Lemma 2.2, Case (b), and for $k\le j$, it follows that $m_{k,n,j}(x)\ge m_{k-1,n,j}(x)$. Because f is nondecreasing, we get $f\left(2\frac{k}{n}-1\right)\ge f\left(2\frac{k-1}{n}-1\right)$, so we have

$$m_{k,n,j}(x) f\left(2\frac{k}{n}-1\right)\ge m_{k-1,n,j}(x) f\left(2\frac{k-1}{n}-1\right).$$

This implies that $f_{k,n,j}(x)\ge f_{k-1,n,j}(x)$. □

Corollary 4.2. *Let $f:[-1,1]\to[0,\infty)$ be a nonincreasing function; then for any $k,j\in\{0,1,2,\dots,n\}$, $k\ge j$ and $x\in\left[\frac{2j}{n+1}-1,\frac{2j+2}{n+1}-1\right]$, we get $f_{k,n,j}(x)\ge f_{k+1,n,j}(x)$.*

Proof. By the proof of Lemma 2.2, Case (a), and for $k\ge j$, it follows that $m_{k,n,j}(x)\ge m_{k+1,n,j}(x)$. Since f is nonincreasing, we get $f\left(2\frac{k}{n}-1\right)\ge f\left(2\frac{k+1}{n}-1\right)$. Hence, we have

$$m_{k,n,j}(x) f\left(2\frac{k}{n}-1\right)\ge m_{k+1,n,j}(x) f\left(2\frac{k+1}{n}-1\right).$$

This implies that $f_{k,n,j}(x)\ge f_{k+1,n,j}(x)$. □

Theorem 4.3. *If $f:[-1,1]\to[0,\infty)$ is a nondecreasing function, then $C_n^M f(x)$ is nondecreasing.*

Proof. Because $C_n^M f(x)$ is continuous on $[-1,1]$, it is sufficient to prove that on each subinterval of the form $\left[\frac{2j}{n+1}-1, \frac{2j+2}{n+1}-1\right]$, with $j \in \{0,1,2,\dots,n\}$, $C_n^M f(x)$ is nondecreasing. Let us take $j \in \{0,1,2,\dots,n\}$ and $x \in \left[\frac{2j}{n+1}-1, \frac{2j+2}{n+1}-1\right]$. Because f is nondecreasing from Lemma 4.1, we have

$$f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq \cdots \geq f_{0,n,j}(x).$$

For all $x \in \left[\frac{2j}{n+1}-1, \frac{2j+2}{n+1}-1\right]$, it follows that

$$C_n^M(f)(x) = \bigvee_{k=j}^{n} f_{k,n,j}(x).$$

Clearly for $k \geq j$, the function $f_{k,n,j}(x)$ is nondecreasing and $C_n^M(f)(x)$ can be written as the maximum of nondecreasing. □

Corollary 4.4. *If $f : [-1,1] \to [0,\infty)$ is a nonincreasing function, then $C_n^M f(x)$ is nonincreasing.*

Proof. By the hypothesis, f is continuous on $[-1,1]$. Because $C_n^M f(x)$ is continuous on $[-1,1]$, it is sufficient to prove that on each subinterval of the form $\left[\frac{2j}{n+1}-1, \frac{2j+2}{n+1}-1\right]$, with $j \in \{0,1,2,\dots,n\}$, $C_n^M f(x)$ is nonincreasing. Let us take $j \in \{0,1,2,\dots,n\}$ and $x \in \left[\frac{2j}{n+1}-1, \frac{2j+2}{n+1}-1\right]$. Because f is nonincreasing from Corollary 4.2, we have

$$f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq \cdots \geq f_{n,n,j}(x).$$

For all $x \in \left[\frac{2j}{n+1}-1, \frac{2j+2}{n+1}-1\right]$, it follows that

$$C_n^M(f)(x) = \bigvee_{k=0}^{j} f_{k,n,j}(x).$$

Clearly that for $k \leq j$ the function $f_{k,n,j}(x)$ is nonincreasing and $C_n^M(f)(x)$ can be written as the maximum of nonincreasing.

□

Definition 4.5 (see [10])**.** Let us take $f : [0,\infty) \to \mathbb{R}$ continuous on $[0,\infty)$. Then f is quasi-convex on $[0,\infty)$ if it satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\} \quad \text{for all } x, y \in [0,\infty) \quad \text{and} \quad \lambda \in [0,1].$$

Remark 4.6. By [15], the continuous function f is quasi-convex on the bounded interval$[-1,1]$, which means that there exists a point $c \in [-1,1]$ such that f is nonincreasing on $[-1,c]$ and nondecreasing on $[c,1]$.

Corollary 4.7. *If $f[-1,1] \to [0,\infty)$ is continuous and quasi-convex on $[-1,1]$, then $C_n^{(M)} f(x)$ is quasi-convex on $[-1,1]$ for all $n \in \mathbb{N}$.*

Proof. We know that a continuous function f is quasi-convex on $[-1,1]$ if there exists a point $c \in [-1,1]$ such that f is nonincreasing on $[-1,c]$ and nondecreasing on $[c,1]$.

If f is a nonincreasing (resp. nondecreasing) function on $[-1,1]$, then by Corollary 4.4 (resp. Theorem 4.3) for all $n \in \mathbb{N}$, $C_n^{(M)} f(x)$ is nonincreasing (resp. nondecreasing) on $[-1,1]$.

Let us suppose that there exists a point $c \in (-1,1)$ such that f is nonincreasing on $[-1,c]$ and nondecreasing on $[c,1]$. Now we define the functions F and G as follows:

$$F, G : [-1,1] \to [0,\infty),$$
$$F(x) = f(x) \quad \text{for all } x \in [-1,c], \quad F(x) = f(c) \quad \text{for all } x \in [c,1],$$
$$G(x) = f(c) \quad \text{for all } x \in [-1,c], \quad G(x) = f(x) \quad \text{for all } x \in [c,1].$$

It is clear that F is nonicreasing and continuous on $[-1,1]$, G is nondecreasing and continuous on $[-1,1]$, and $f(x) = \max\{F(x), G(x)\}$, for all $x \in [-1,1]$.

In addition, because $C_n^{(M)}(f)(x)$ is pseudo-linear, we can write for all $x \in [-1,1]$

$$C_n^{(M)}(f)(x) = \max\left\{C_n^{(M)}(F)(x), C_n^{(M)}(G)(x)\right\}.$$

Therefore by the Corollary 4.4 and Theorem 4.3, $C_n^{(M)}(F)(x)$ is nonincreasing and continuous on $[-1,1]$ and $C_n^{(M)}(G)(x)$ is nondecreasing and continuous on $[-1,1]$. Now, we have two cases:

(a) $C_n^{(M)}(F)(x)$ and $C_n^{(M)}(G)(x)$ do not intersect each other,

(b) $C_n^{(M)}(F)(x)$ and $C_n^{(M)}(G)(x)$ intersect each other.

Case (a). For all $x \in [-1,1]$, we have

$$\max\left\{C_n^{(M)}(F)(x), C_n^{(M)}(G)(x)\right\} = C_n^{(M)}(F)(x)$$

or

$$\max\left\{C_n^{(M)}(F)(x), C_n^{(M)}(G)(x)\right\} = C_n^{(M)}(G)(x),$$

which precisely proves that $C_n^{(M)}(f)(x)$ is quasi-convex on $[-1,1]$.

Case (b). If $C_n^{(M)}(F)(x)$ and $C_n^{(M)}(G)(x)$ intersect each other, then there exists a point $c \in [-1,1]$ such that $C_n^{(M)}(f)(x)$ is nonincreasing on $[-1,c]$ and nondeccreasing on $[c,1]$. This implies that $C_n^{(M)}(f)(x)$ is quasi-convex on $[-1,1]$. □

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