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A SURVEY ON OSTROWSKI TYPE INEQUALITIES RELATED TO POMPEIU'S MEAN VALUE THEOREM

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Abstract. In this paper we survey some recent results obtained by the author related to Pompeiu's mean value theorem and inequality. Natural applications to Ostrowski type inequalities that play an important role in Numerical Analysis, Approximation Theory, Probability Theory & Statistics, Information Theory and other fields, are given as well.

1. Introduction

In 1946, Pompeiu [11] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [13, p. 83]).

Theorem 1.1 (Pompeiu, 1946 [11])**.** *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{1.1}$$

Following [13, p. 84 – 85], we will mention here a geometrical interpretation of Pompeiu's theorem.

The equation of the secant line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$y = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

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This line intersects the $y-$axis at the point $(0,y)$, where y is

$$\begin{aligned} y &= f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(0-x_1) \\ &= \frac{x_1 f(x_2) - x_2 f(x_1)}{x_1-x_2}. \end{aligned}$$

The equation of the tangent line at the point $(\xi, f(\xi))$ is

$$y = (x-\xi) f'(\xi) + f(\xi).$$

The tangent line intersects the $y-$axis at the point $(0,y)$, where

$$y = -\xi f'(\xi) + f(\xi).$$

Hence, the geometric meaning of Pompeiu's mean value theorem is that *the tangent of the point $(\xi, f(\xi))$ intersects on the $y-$axis at the same point as the secant line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.*

The following inequality is a simple consequence of *Pompeiu's mean value theorem.*

Corollary 1.2 (Pompeiu's Inequality)**.** *With the assumptions of Theorem 1.1 and if $\|f-\ell f'\|_\infty = \sup_{t\in(a,b)} |f(t) - tf'(t)| < \infty$ where $\ell(t) = t$, $t \in [a,b]$, then*

$$|tf(x) - xf(t)| \le \|f-\ell f'\|_\infty |x-t| \tag{1.2}$$

for any $t, x \in [a,b]$.

The inequality (1.2) was obtained by the author in [3], see also [4].

In 1938, A. Ostrowski [9] proved the following result in the estimating the integral mean:

Theorem 1.3 (Ostrowski, 1938 [9])**.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $|f'(t)| \le M < \infty$ for all $t\in(a,b)$. Then for any $x\in[a,b]$, we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right] M(b-a). \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

In order to provide another approximation of the integral mean, by making use of the Pompeiu's mean value theorem, the author proved the following result:

Theorem 1.4 (Dragomir, 2005 [3])**.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then for any $x\in[a,b]$, we have the inequality*

$$\begin{aligned} \left| \frac{a+b}{2}\cdot\frac{f(x)}{x} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| &\le \frac{b-a}{|x|}\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right] \\ &\quad \times \|f-\ell f'\|_\infty, \end{aligned} \tag{1.4}$$

where $\ell(t) = t,\ t \in [a,b]$.

The constant $\frac{1}{4}$ *is sharp in the sense that it cannot be replaced by a smaller constant.*

In [12], E. C. Popa using a mean value theorem obtained a generalization of (1.4) as follows:

Theorem 1.5 (Popa, 2007 [12])**.** *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b). *Assume that* $\alpha \notin [a,b]$. *Then for any* $x \in [a,b]$, *we have the inequality*

$$\left| \left(\frac{a+b}{2} - \alpha \right) f(x) + \frac{\alpha - x}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \left\| f - \ell_\alpha f' \right\|_\infty , \tag{1.5}$$

where $\ell_\alpha(t) = t - \alpha,\ t \in [a,b]$.

In [10], J. Pečarić and S. Ungar have proved a general estimate with the p-norm, $1 \leq p \leq \infty$ which for $p = \infty$ give Dragomir's result.

Theorem 1.6 (Pečarić & Ungar, 2006 [10])**.** *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $0 < a < b$. *Then for* $1 \leq p, q \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *we have the inequality*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq PU(x,p) \left\| f - \ell f' \right\|_p , \tag{1.6}$$

for $x \in [a,b]$, *where*

$$PU(x,p) := (b-a)^{\frac{1}{p}-1} \left[\left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q} x^{1-2q}}{(1-2q)(1+q)} \right)^{1/q} + \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q} x^{1-2q}}{(1-2q)(1+q)} \right)^{1/q} \right].$$

In the cases $(p,q) = (1,\infty), (\infty,1)$ *and* $(2,2)$ *the quantity* $PU(x,p)$ *has to be taken as the limit as* $p \to 1, \infty$ *and* 2, *respectively.*

For other inequalities in terms of the p-norm of the quantity $f - \ell_\alpha f'$, where $\ell_\alpha(t) = t - \alpha$, $t \in [a,b]$ and $\alpha \notin [a,b]$ see [2] and [1].

In this paper we survey some recent results obtained by the author related to Pompeiu's inequality presented above. Natural applications to Ostrowski type inequalities that play an important role in Numerical Analysis, Approximation Theory, Probability Theory & Statistics, Information Theory and other fields are given as well.

2. Ostrowski Via a Generalized Pompeiu's Inequality

2.1. **Pompeiu's Inequality for p-Norms.** We can generalize the above inequality (1.2) for the larger class of functions that are absolutely continuous and complex valued as well as for other norms of the difference $f - \ell f'$ as follows:

Lemma 2.1 (Dragomir, 2013 [5]). *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ with $b > a > 0$. Then for any $t, x \in [a, b]$ we have*

$$|tf(x) - xf(t)| \tag{2.1}$$
$$\leq \begin{cases} \|f - \ell f'\|_\infty |x - t| & \text{if } f - \ell f' \in L_\infty [a, b] \\ \frac{1}{2q-1} \|f - \ell f'\|_p \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} & \begin{array}{c} \text{if } f - \ell f' \in L_p [a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1 \end{array} \\ \|f - \ell f'\|_1 \frac{\max\{t,x\}}{\min\{t,x\}} & \end{cases}$$

or, equivalently

$$\left| \frac{f(x)}{x} - \frac{f(t)}{t} \right| \tag{2.2}$$
$$\leq \begin{cases} \|f - \ell f'\|_\infty \left| \frac{1}{t} - \frac{1}{x} \right| & \text{if } f - \ell f' \in L_\infty [a, b] \\ \frac{1}{2q-1} \|f - \ell f'\|_p \left| \frac{1}{t^{2q-1}} - \frac{1}{x^{2q-1}} \right|^{1/q} & \begin{array}{c} \text{if } f - \ell f' \in L_p [a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1 \end{array} \\ \|f - \ell f'\|_1 \frac{1}{\min\{t^2,x^2\}} & \end{cases}$$

Proof. If f is absolutely continuous, then f/ℓ is absolutely continuous on the interval $[a, b]$ that does not containing 0 and

$$\int_t^x \left(\frac{f(s)}{s} \right)' ds = \frac{f(x)}{x} - \frac{f(t)}{t}$$

for any $t, x \in [a, b]$ with $x \neq t$.

Since

$$\int_t^x \left(\frac{f(s)}{s} \right)' ds = \int_t^x \frac{f'(s) s - f(s)}{s^2} ds$$

then we get the following identity

$$tf(x) - xf(t) = xt \int_t^x \frac{f'(s) s - f(s)}{s^2} ds \tag{2.3}$$

for any $t, x \in [a, b]$.

We notice that the equality (2.3) was proved for the smaller class of differentiable function and in a different manner in [10].

Taking the modulus in (2.3) we have

$$
\begin{aligned}
|tf(x) - xf(t)| &= \left| xt \int_t^x \frac{f'(s)\,s - f(s)}{s^2} ds \right| \\
&\leq xt \left| \int_t^x \left| \frac{f'(s)\,s - f(s)}{s^2} \right| ds \right| := I
\end{aligned} \tag{2.4}
$$

and utilizing Hölder's integral inequality we deduce

$$
\begin{aligned}
I &\leq xt \begin{cases} \sup_{s\in[t,x]([x,t])} |f'(s)\,s - f(s)| \left|\int_t^x \frac{1}{s^2} ds\right| \\ \\ \left|\int_t^x |f'(s)\,s - f(s)|^p ds\right|^{1/p} \left|\int_t^x \frac{1}{s^{2q}} ds\right|^{1/q} & \begin{array}{c} p>1, \\ \frac{1}{p}+\frac{1}{q}=1 \end{array} \\ \\ \left|\int_t^x |f'(s)\,s - f(s)|\, ds\right| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{s^2}\right\} \end{cases} \\
&\leq \begin{cases} \|f-\ell f'\|_\infty |x-t| \\ \\ \frac{1}{2q-1} \|f-\ell f'\|_p \left|\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}}\right|^{1/q} & \begin{array}{c} p>1, \\ \frac{1}{p}+\frac{1}{q}=1 \end{array} \\ \\ \|f-\ell f'\|_1 \frac{\max\{t,x\}}{\min\{t,x\}} \end{cases}
\end{aligned} \tag{2.5}
$$

and the inequality (2.2) is proved. □

Remark 2.2. The first inequality in (2.1) also holds in the same form for $0 > b > a$.

Remark 2.3. If we take in (2.1) $x = A = A(a,b) := \frac{a+b}{2}$ (the arithmetic mean) and $t = G = G(a,b) := \sqrt{ab}$ (the geometric mean) then we get the simple inequality for functions of means:

$$
\begin{aligned}
&|Gf(A) - Af(G)| \\
&\leq \begin{cases} \|f-\ell f'\|_\infty (A-G) & \text{if } f-\ell f' \in L_\infty[a,b] \\ \\ \frac{1}{2q-1} \|f-\ell f'\|_p \frac{\left(A^{2q-1}-G^{2q-1}\right)^{1/q}}{A^{1/p}G^{1/p}} & \begin{array}{c} \text{if } f-\ell f' \in L_p[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1 \end{array} \\ \\ \|f-\ell f'\|_1 \frac{A}{G} \end{cases}
\end{aligned} \tag{2.6}
$$

2.2. **Evaluating the Integral Mean.** The following new result holds.

Theorem 2.4 (Dragomir, 2013 [5]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b>a>0$. Then for any $x\in[a,b]$ we have*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.7}$$
$$\leq\begin{cases}\frac{b-a}{x}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f-\ell f'\|_\infty & \text{if } f-\ell f'\in L_\infty[a,b]\\ \\ \frac{1}{(2q-1)x(b-a)^{1/q}}\|f-\ell f'\|_p\left[B_q(a,b;x)\right]^{1/q} & \begin{array}{c}\text{if } f-\ell f'\in L_p[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1\end{array}\\ \\ \frac{1}{b-a}\|f-\ell f'\|_1\left(\ln\frac{x}{a}+\frac{b^2-x^2}{2x^2}\right),\end{cases},$$

where

$$B_q(a,b;x)=\begin{cases}\frac{x^q}{2-q}\left(2x^{q-2}-a^{q-2}-b^{q-2}\right)\\ +\frac{1}{x^{q-1}(q+1)}\left(b^{q+1}+a^{q+1}-2x^{q+1}\right), & q\neq 2\\ \\ x^2\ln\frac{x^2}{ab}+\frac{b^3+a^3-2x^3}{3x}, & q=2\end{cases} \tag{2.8}$$

Proof. The first inequality can be proved in an identical way to the case of differentiable functions from [3] by utilizing the first inequality in (2.1).

Utilising the second inequality in (2.1) we have

$$\begin{aligned}&\left|\frac{a+b}{2}\cdot f(x)-\frac{x}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq\frac{1}{b-a}\int_a^b|tf(x)-xf(t)|\,dt\\ &\leq\frac{1}{(2q-1)(b-a)}\|f-\ell f'\|_p\int_a^b\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q}dt\end{aligned} \tag{2.9}$$

Utilising Hölder's integral inequality we have

$$\int_a^b\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q}dt\leq(b-a)^{1/p}\left(\int_a^b\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|dt\right)^{1/q}. \tag{2.10}$$

For $q\neq 2$ we have

$$\begin{aligned}&\int_a^b\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|dt\\ &=\frac{x^q}{2-q}\left(2x^{q-2}-a^{q-2}-b^{q-2}\right)+\frac{1}{x^{q-1}(q+1)}\left(b^{q+1}+a^{q+1}-2x^{q+1}\right)\\ &=B_q(a,b;x).\end{aligned}$$

For $q=2$ we have

$$\int_a^b \left| \frac{x^2}{t} - \frac{t^2}{x} \right| dt = x^2 \ln \frac{x^2}{ab} + \frac{1}{x} \frac{b^3 + a^3 - 2x^3}{3} = B_2(a, b; x).$$

Utilizing (2.9) and (2.10) we get the second inequality in (2.7).

Utilising the third inequality in (2.1) we have

$$\begin{aligned} \left| \frac{a+b}{2} \cdot f(x) - \frac{x}{b-a} \int_a^b f(t)\, dt \right| &\le \frac{1}{b-a} \int_a^b |t f(x) - x f(t)|\, dt \qquad (2.11) \\ &\le \frac{1}{b-a} \|f - \ell f'\|_1 \int_a^b \frac{\max\{t, x\}}{\min\{t, x\}} dt. \end{aligned}$$

Since

$$\int_a^b \frac{\max\{t, x\}}{\min\{t, x\}} dt = \int_a^x \frac{x}{t} dt + \int_x^b \frac{t}{x} dt = x \ln \frac{x}{a} + \frac{1}{x} \frac{b^2 - x^2}{2},$$

then by (2.11) we have

$$\begin{aligned} \left| \frac{a+b}{2} \cdot f(x) - \frac{x}{b-a} \int_a^b f(t)\, dt \right| &\le \frac{1}{b-a} \int_a^b |t f(x) - x f(t)|\, dt \\ &\le \frac{1}{b-a} \|f - \ell f'\|_1 \left[x \ln \frac{x}{a} + \frac{1}{x} \frac{b^2 - x^2}{2} \right], \end{aligned}$$

and the last part of (2.7) is thus proved. □

Remark 2.5. If we take in (2.7) $x = A = A(a, b) := \frac{a+b}{2}$, then we get

$$\left| f(A) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \qquad (2.12)$$

$$\le \begin{cases} \frac{b-a}{4A} \|f - \ell f'\|_\infty & \text{if } f - \ell f' \in L_\infty[a, b] \\ \frac{1}{(2q-1) A (b-a)^{1/q}} \|f - \ell f'\|_p [B_q(a, b; A)]^{1/q} & \begin{array}{l} \text{if } f - \ell f' \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1 \end{array} \\ \frac{1}{b-a} \|f - \ell f'\|_1 \left(\ln \frac{A}{a} + \frac{1}{2}(b-a) \frac{a+3b}{4} A \right), \end{cases},$$

where

$$B_q(a, b; A) = \begin{cases} \frac{2A^q}{2-q} \left(A^{q-2} - A\left(a^{q-2}, b^{q-2}\right) \right) & \\ + \frac{2}{(q+1)A^{q-1}} \left(A\left(b^{q+1}, a^{q+1}\right) - A^{q+1} \right), & q \ne 2 \\ 2A^2 \ln \frac{A}{G} + \frac{1}{2}(b-a)^2, & q = 2 \end{cases}$$

2.3. **A Related Result.** The following new result also holds.

Theorem 2.6 (Dragomir, 2013 [5]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b>a>0$. Then for any $x\in[a,b]$ we have*

$$\left|\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\right| \tag{2.13}$$
$$\leq\begin{cases}\frac{2}{b-a}\left\|f-\ell f'\right\|_\infty\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right) & \text{if } f-\ell f'\in L_\infty[a,b]\\ \frac{1}{(2q-1)(b-a)^{1/q}}\left\|f-\ell f'\right\|_p\left(C_q(a,b;x)\right)^{1/q} & \text{if } f-\ell f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \frac{1}{b-a}\left\|f-\ell f'\right\|_1\frac{x^2+ab-2ax}{x^2a}, & \end{cases},$$

where

$$C_q(a,b;x)=\frac{1}{x^{2q-1}}(b+a-2x)+\frac{a^{2-2q}+b^{2-2q}-2x^{2-2q}}{2(q-1)},q>1. \tag{2.14}$$

Proof. From the first inequality in (2.8) we have

$$\left|\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\right|\leq\frac{1}{b-a}\int_a^b\left|\frac{f(x)}{x}-\frac{f(t)}{t}\right|dt \tag{2.15}$$
$$\leq\left\|f-\ell f'\right\|_\infty\frac{1}{b-a}\int_a^b\left|\frac{1}{t}-\frac{1}{x}\right|dt.$$

Since

$$\int_a^b\left|\frac{1}{x}-\frac{1}{t}\right|dt=\left[\int_a^x\left(\frac{1}{t}-\frac{1}{x}\right)dt+\int_x^b\left(\frac{1}{x}-\frac{1}{t}\right)dt\right]$$
$$=\left(\ln\frac{x^2}{ab}+\frac{a+b-2x}{x}\right)$$

for any $x\in[a,b]$, then we deduce from (2.15) the first inequality in (2.13).

From the second inequality in (2.8) we have

$$\left|\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\right| \tag{2.16}$$
$$\leq\frac{1}{b-a}\int_a^b\left|\frac{f(x)}{x}-\frac{f(t)}{t}\right|dt$$
$$\leq\frac{1}{(2q-1)(b-a)}\left\|f-\ell f'\right\|_p\int_a^b\left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q}dt.$$

Utilising Hölder's integral inequality we have

$$\int_a^b\left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q}dt\leq(b-a)^{1/p}\left(\int_a^b\left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|dt\right)^{1/q}. \tag{2.17}$$

Since

$$\int_a^b \left| \frac{1}{t^{2q-1}} - \frac{1}{x^{2q-1}} \right| dt = \frac{1}{x^{2q-1}} (b + a - 2x) + \frac{a^{2-2q} + b^{2-2q} - 2x^{2-2q}}{2(q-1)} = C_q(a, b; x)$$

then by (2.16) and (2.17) we get

$$\left| \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b \frac{f(t)}{t} dt \right| \leq \frac{1}{(2q-1)(b-a)} \|f - \ell f'\|_p (b-a)^{1/p} (C_q(a, b; x))^{1/q}$$

and the second inequality in (2.13) is proved.

From the third inequality in (2.8) we have

$$\left| \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b \frac{f(t)}{t} dt \right| \leq \frac{1}{b-a} \int_a^b \left| \frac{f(x)}{x} - \frac{f(t)}{t} \right| dt \leq \frac{1}{b-a} \|f - \ell f'\|_1 \int_a^b \frac{1}{\min\{t^2, x^2\}} dt. \tag{2.18}$$

Since

$$\int_a^b \frac{1}{\min\{t^2, x^2\}} dt = \int_a^x \frac{dt}{t^2} + \int_x^b \frac{dt}{x^2} = \frac{x^2 + ab - 2ax}{x^2 a},$$

then by (2.18) we deduce the last part of (2.13). □

Remark 2.7. If we take in (2.13) $x = A = A(a, b) := \frac{a+b}{2}$, then we get

$$\left| \frac{f(A)}{A} - \frac{1}{b-a} \int_a^b \frac{f(t)}{t} dt \right| \leq \begin{cases} \frac{2}{b-a} \|f - \ell f'\|_\infty \ln\left(\frac{A}{G}\right) & \text{if } f - \ell f' \in L_\infty[a, b] \\ \frac{1}{(2q-1)(b-a)^{1/q}} \|f - \ell f'\|_p (C_q(a, b; A))^{1/q} & \text{if } f - \ell f' \in L_p[a, b],\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{1}{2} \|f - \ell f'\|_1 \frac{A+a}{A^2 a}, \end{cases} \tag{2.19}$$

where

$$C_q(a, b; A) = \frac{A(a^{2-2q}, b^{2-2q}) - A^{2-2q}}{q-1}, q > 1.$$

3. Ostrowski Via Power Pompeiu's Inequality

3.1. **Power Pompeiu's Inequality.** We can generalize the above (1.2) inequality for the power function as follows.

Lemma 3.1 (Dragomir, 2013 [6]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b>a>0$. If $r\in\mathbb{R}$, $r\neq 0$, then for any $t,x\in[a,b]$ we have*

$$\begin{aligned}
&\left|t^{r}f(x)-x^{r}f(t)\right| \qquad (3.1)\\
&\leq\begin{cases}
\frac{1}{|r|}\left\|f'\ell-rf\right\|_{\infty}\left|t^{r}-x^{r}\right|, \text{ if } f'\ell-rf\in L_{\infty}[a,b]\\[2ex]
\left\|f'\ell-rf\right\|_{p}\begin{cases}
\frac{1}{|1-q(r+1)|}\left|\frac{t^{r}}{x^{1-q(r+1)-r}}-\frac{x^{r}}{t^{1-q(r+1)-r}}\right|,\\
\text{ for } r\neq-\frac{1}{p}\\[2ex]
t^{r}x^{r}\left|\ln x-\ln t\right|, \text{ for } r=-\frac{1}{p}
\end{cases}\\
\text{if } f'\ell-rf\in L_{p}[a,b]\\[2ex]
\left\|f'\ell-rf\right\|_{1}\frac{t^{r}x^{r}}{\min\{x^{r+1},t^{r+1}\}}
\end{cases}
\end{aligned}$$

or, equivalently

$$\begin{aligned}
&\left|\frac{f(x)}{x^{r}}-\frac{f(t)}{t^{r}}\right| \qquad (3.2)\\
&\leq\begin{cases}
\frac{1}{|r|}\left\|f'\ell-rf\right\|_{\infty}\left|\frac{1}{x^{r}}-\frac{1}{t^{r}}\right|, \text{ if } f'\ell-rf\in L_{\infty}[a,b]\\[2ex]
\left\|f'\ell-rf\right\|_{p}\begin{cases}
\frac{1}{|1-q(r+1)|}\left|\frac{1}{x^{1-q(r+1)}}-\frac{1}{t^{1-q(r+1)}}\right|, \text{ for } r\neq-\frac{1}{p}\\[2ex]
\left|\ln x-\ln t\right|, \text{ for } r=-\frac{1}{p}
\end{cases}\\
\text{if } f'\ell-rf\in L_{p}[a,b]\\[2ex]
\left\|f'\ell-rf\right\|_{1}\frac{1}{\min\{x^{r+1},t^{r+1}\}}
\end{cases}
\end{aligned}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. If f is absolutely continuous, then $f/(\cdot)^{r}$ is absolutely continuous on the interval $[a,b]$ and

$$\int_{t}^{x}\left(\frac{f(s)}{s^{r}}\right)'ds=\frac{f(x)}{x^{r}}-\frac{f(t)}{t^{r}}$$

for any $t,x\in[a,b]$ with $x\neq t$.

Since

$$\int_{t}^{x}\left(\frac{f(s)}{s^{r}}\right)'ds=\int_{t}^{x}\frac{f'(s)s^{r}-rs^{r-1}f(s)}{s^{2r}}ds=\int_{t}^{x}\frac{f'(s)s-rf(s)}{s^{r+1}}ds$$

then we get the following identity

$$t^{r}f(x)-x^{r}f(t)=x^{r}t^{r}\int_{t}^{x}\frac{f'(s)s-rf(s)}{s^{r+1}}ds \qquad (3.3)$$

for any $t,x\in[a,b]$.

Taking the modulus in (3.3) we have

$$\begin{aligned} \left|t^{r}f(x)-x^{r}f(t)\right| &= x^{r}t^{r}\left|\int_{t}^{x}\frac{f'(s)\,s-rf(s)}{s^{r+1}}ds\right| \\ &\leq x^{r}t^{r}\left|\int_{t}^{x}\frac{|f'(s)\,s-rf(s)|}{s^{r+1}}ds\right| := I \end{aligned} \tag{3.4}$$

and utilizing Hölder's integral inequality we deduce

$$I \leq x^{r}t^{r}\begin{cases} \sup_{s\in[t,x]([x,t])}|f'(s)\,s-rf(s)|\left|\int_{t}^{x}\frac{1}{s^{r+1}}ds\right| \\ \\ \left|\int_{t}^{x}|f'(s)\,s-rf(s)|^{p}ds\right|^{1/p}\left|\int_{t}^{x}\frac{1}{s^{q(r+1)}}ds\right|^{1/q} \\ \\ \left|\int_{t}^{x}|f'(s)\,s-rf(s)|\,ds\right|\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{s^{r+1}}\right\} \end{cases} \tag{3.5}$$

$$\leq x^{r}t^{r}\begin{cases} \frac{1}{|r|}\left\|f'\ell-rf\right\|_{\infty}\left|\frac{1}{x^{r}}-\frac{1}{t^{r}}\right| \\ \\ \left\|f'\ell-rf\right\|_{p}\begin{cases}\frac{1}{|1-q(r+1)|}\left|\frac{1}{x^{1-q(r+1)}}-\frac{1}{t^{1-q(r+1)}}\right|, \\ r\neq-\frac{1}{p} \\ \\ |\ln x-\ln t|\,,\ r=-\frac{1}{p}\end{cases} \\ \\ \left\|f'\ell-rf\right\|_{1}\frac{1}{\min\{x^{r+1},t^{r+1}\}}. \end{cases}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$, and the inequality (3.1) is proved. □

3.2. **Some Ostrowski Type Results.** The following new result also holds.

Theorem 3.2 (Dragomir, 2013 [6])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b>a>0$. If $r\in\mathbb{R}$, $r\neq0$, and $f'\ell-rf\in L_{\infty}[a,b]$, then for any $x\in[a,b]$ we have*

$$\begin{aligned} &\left|\frac{b^{r+1}-a^{r+1}}{r+1}f(x)-x^{r}\int_{a}^{b}f(t)\,dt\right| \\ &\leq\frac{1}{|r|}\left\|f'\ell-rf\right\|_{\infty} \\ &\times\begin{cases}\frac{2rx^{r+1}-x^{r}(a+b)(r+1)+b^{r+1}+a^{r+1}}{r+1}, & \text{if } r>0 \\ \\ \frac{x^{r}(a+b)(r+1)-2rx^{r+1}-b^{r+1}-a^{r+1}}{r+1}, & \text{if } r\in(-\infty,0)\setminus\{-1\}.\end{cases} \end{aligned} \tag{3.6}$$

Also, for $r=-1$, we have

$$\left|f(x)\ln\frac{b}{a}-\frac{1}{x}\int_{a}^{b}f(t)\,dt\right|\leq2\left\|f'\ell+f\right\|_{\infty}\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right) \tag{3.7}$$

for any $x\in[a,b]$, provided $f'\ell+f\in L_{\infty}[a,b]$

The constant 2 in (3.7) is best possible.

Proof. Utilising the first inequality in (3.1) for $r \neq -1$ we have

$$\left|\frac{b^{r+1}-a^{r+1}}{r+1}f(x)-x^{r}\int_{a}^{b}f(t)\,dt\right| \leq \int_{a}^{b}\left|t^{r}f(x)-x^{r}f(t)\right|dt \leq \frac{1}{|r|}\left\|f'\ell-rf\right\|_{\infty}\int_{a}^{b}\left|t^{r}-x^{r}\right|dt. \tag{3.8}$$

Observe that

$$\int_{a}^{b}\left|t^{r}-x^{r}\right|dt=\begin{cases}\int_{a}^{x}\left(x^{r}-t^{r}\right)dt+\int_{x}^{b}\left(t^{r}-x^{r}\right)dt, & \text{if } r>0\\ \\ \int_{a}^{x}\left(t^{r}-x^{r}\right)dt+\int_{x}^{b}\left(x^{r}-t^{r}\right)dt, & \text{if } r\in(-\infty,0)\setminus\{-1\}.\end{cases}$$

Then for $r>0$ we have

$$\int_{a}^{x}\left(x^{r}-t^{r}\right)dt+\int_{x}^{b}\left(t^{r}-x^{r}\right)dt=\frac{2rx^{r+1}-x^{r}(a+b)(r+1)+b^{r+1}+a^{r+1}}{r+1}$$

and for $r\in(-\infty,0)\setminus\{-1\}$ we have

$$\int_{a}^{x}\left(t^{r}-x^{r}\right)dt+\int_{x}^{b}\left(x^{r}-t^{r}\right)dt=-\frac{2rx^{r+1}-x^{r}(a+b)(r+1)+b^{r+1}+a^{r+1}}{r+1}.$$

Making use of (3.8) we get (3.6).

Utilizing the inequality (3.1) for $r=-1$ we have

$$\left|t^{-1}f(x)-x^{-1}f(t)\right|\leq\left\|f'\ell+f\right\|_{\infty}\left|t^{-1}-x^{-1}\right|$$

if $f'\ell+f\in L_{\infty}[a,b]$.

Integrating this inequality, we have

$$\left|f(x)\ln\frac{b}{a}-x^{-1}\int_{a}^{b}f(t)\,dt\right|\leq\int_{a}^{b}\left|t^{-1}f(x)-x^{-1}f(t)\right|dt \leq\left\|f'\ell+f\right\|_{\infty}\int_{a}^{b}\left|t^{-1}-x^{-1}\right|dt. \tag{3.9}$$

Since

$$\int_{a}^{b}\left|\frac{1}{x}-\frac{1}{t}\right|dt=2\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right),$$

then by (3.9) we get the desired inequality (3.7).

Now, assume that (3.7) holds with a constant $C>0$, i.e.

$$\left|f(x)\ln\frac{b}{a}-x^{-1}\int_{a}^{b}f(t)\,dt\right|\leq C\left\|f'\ell+f\right\|_{\infty}\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right) \tag{3.10}$$

for any $x\in[a,b]$.

If we take in (3.10) $f(t)=1, t\in[a,b]$, then we get

$$\left|\ln\frac{b}{a}-\frac{b-a}{x}\right|\leq C\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right) \tag{3.11}$$

for any for any $x\in[a,b]$.

Making $x = a$ in (3.10) produces the inequality

$$\left|\ln\frac{b}{a} - \frac{b-a}{a}\right| \leq C\left(\frac{b-a}{2a} - \frac{1}{2}\ln\frac{b}{a}\right)$$

which implies that $C \geq 2$.

This proves the sharpness of the constant 2 in (3.7). □

Remark 3.3. Consider the r-Logarithmic mean

$$L_r = L_r(a,b) := \left[\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right]^{1/r}$$

defined for $r \in \mathbb{R} \setminus \{0, -1\}$ and the Logarithmic mean, defined as

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

If $A = A(a,b) := \frac{a+b}{2}$, then from (3.6) we get for $x = A$ the inequality

$$\left|L_r^r (b-a) f(A) - A^r \int_a^b f(t)\,dt\right| \tag{3.12}$$

$$\leq \frac{2}{|r|}\left\|f'\ell - rf\right\|_\infty \begin{cases} \frac{A\left(b^{r+1},a^{r+1}\right) - A^{r+1}}{r+1}, & \text{if } r > 0 \\ \\ \frac{A^{r+1} - A\left(b^{r+1},a^{r+1}\right)}{r+1}, & \text{if } r \in (-\infty, 0) \setminus \{-1\}, \end{cases}$$

while from (3.7) we get

$$\left|L^{-1}(b-a) f(A) - A^{-1}\int_a^b f(t)\,dt\right| \leq 2\left\|f'\ell + f\right\|_\infty \ln\frac{A}{G} \tag{3.13}$$

The following related result holds.

Theorem 3.4 (Dragomir, 2013 [6])**.** *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b > a > 0$. If $r \in \mathbb{R}$, $r \neq 0$, then for any $x \in [a,b]$ we have*

$$\left|\frac{f(x)}{x^r}(b-a) - \int_a^b \frac{f(t)}{t^r}dt\right| \tag{3.14}$$

$$\leq \frac{1}{|r|}\left\|f'\ell - rf\right\|_\infty$$

$$\times \begin{cases} \frac{2x^{1-r} - a^{1-r} - b^{1-r}}{1-r} + \frac{1}{x^r}(b+a-2x), & r \in (0,\infty) \setminus \{1\} \\ \\ \frac{a^{1-r} + b^{1-r} - 2x^{1-r}}{1-r} + \frac{1}{x^r}(2x-a-b), & \text{if } r < 0. \end{cases},$$

Also, for $r = 1$, we have

$$\left|\frac{f(x)}{x}(b-a) - \int_a^b \frac{f(t)}{t}dt\right| \leq 2\left\|f'\ell - f\right\|_\infty \left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x}\right) \tag{3.15}$$

for any $x \in [a,b]$, provided $f'\ell - f \in L_\infty[a,b]$.

The constant 2 is best possible in (3.15).

Proof. From the first inequality in (3.2) we have

$$\left|\frac{f(x)}{x^r}-\frac{f(t)}{t^r}\right|\leq\frac{1}{|r|}\|f'\ell-rf\|_\infty\left|\frac{1}{x^r}-\frac{1}{t^r}\right|, \tag{3.16}$$

for any $t,x\in[a,b]$, provided $f'\ell-rf\in L_\infty[a,b]$.

Integrating over $t\in[a,b]$ we get

$$\begin{aligned}\left|\frac{f(x)}{x^r}(b-a)-\int_a^b\frac{f(t)}{t^r}dt\right|&\leq\int_a^b\left|\frac{f(x)}{x^r}-\frac{f(t)}{t^r}\right|dt \\ &\leq\frac{1}{|r|}\|f'\ell-rf\|_\infty\int_a^b\left|\frac{1}{x^r}-\frac{1}{t^r}\right|dt,\end{aligned} \tag{3.17}$$

for $r\in\mathbb{R}$, $r\neq0$.

For $r\in(0,\infty)\setminus\{1\}$ we have

$$\int_a^b\left|\frac{1}{x^r}-\frac{1}{t^r}\right|dt=\frac{2x^{1-r}-a^{1-r}-b^{1-r}}{1-r}+\frac{1}{x^r}(b+a-2x)$$

for any $x\in[a,b]$.

For $r<0$, we also have

$$\int_a^b\left|\frac{1}{x^r}-\frac{1}{t^r}\right|dt=\frac{a^{1-r}+b^{1-r}-2x^{1-r}}{1-r}+\frac{1}{x^r}(2x-a-b)$$

for any $x\in[a,b]$.

For $r=1$ we have

$$\int_a^b\left|\frac{1}{x}-\frac{1}{t}\right|dt=2\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right)$$

for any $x\in[a,b]$, and the inequality (3.15) is obtained.

The sharpness of the constant 2 follows as in the proof of Theorem 3.2 and the details are omitted. □

Remark 3.5. If we take $x=A$ in Theorem 3.4, then we we have

$$\begin{aligned}&\left|\frac{f(A)}{A^r}(b-a)-\int_a^b\frac{f(t)}{t^r}dt\right| \\ &\leq\frac{2}{|r|}\|f'\ell-rf\|_\infty\begin{cases}\frac{A^{1-r}-A\left(a^{1-r},b^{1-r}\right)}{1-r}, & r\in(0,\infty)\setminus\{1\}\\[2mm] \frac{A\left(a^{1-r},b^{1-r}\right)-A^{1-r}}{1-r}, & \text{if } r<0.\end{cases},\end{aligned} \tag{3.18}$$

Also, for $r=1$, we have

$$\left|\frac{f(A)}{A}(b-a)-\int_a^b\frac{f(t)}{t}dt\right|\leq2\|f'\ell-f\|_\infty\ln\frac{A}{G}. \tag{3.19}$$

Remark 3.6. The interested reader may obtain other similar results in terms of the p-norms $\|f'\ell-rf\|_p$ with $p\geq1$. However, since some calculations are too complicated, the details are not presented here.

4. Ostrowski Via an Exponential Pompeiu's Inequality

4.1. **An Exponential Pompeiu's Inequality.** We can provide some similar results for complex-valued functions with the exponential instead of ℓ.

Lemma 4.1 (Dragomir, 2013 [7]). *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) \neq 0$. Then for any $t, x \in [a, b]$ we have*

$$\left| \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)} \right| \leq \begin{cases} |\operatorname{Re}(\alpha)| \, \|f' - \alpha f\|_\infty \\ \times \left| \frac{1}{\exp(t\operatorname{Re}(\alpha))} - \frac{1}{\exp(x\operatorname{Re}(\alpha))} \right| & \text{if } f' - \alpha f \in L_\infty[a, b], \\ \\ q^{1/q} |\operatorname{Re}(\alpha)|^{1/q} \|f' - \alpha f\|_p \\ \times \left| \frac{1}{\exp(tq\operatorname{Re}(\alpha))} - \frac{1}{\exp(xq\operatorname{Re}(\alpha))} \right|^{1/q} & \text{if } f' - \alpha f \in L_p[a, b], \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \|f' - \alpha f\|_1 \frac{1}{\min\{\exp(t\operatorname{Re}(\alpha)), \exp(x\operatorname{Re}(\alpha))\}}, \end{cases} \tag{4.1}$$

or, equivalently

$$|\exp(\alpha t) f(x) - f(t) \exp(\alpha x)| \leq \begin{cases} |\operatorname{Re}(\alpha)| \, \|f' - \alpha f\|_\infty \\ \times |\exp(x\operatorname{Re}(\alpha)) - \exp(t\operatorname{Re}(\alpha))| & \text{if } f' - \alpha f \in L_\infty[a, b], \\ \\ q^{1/q} |\operatorname{Re}(\alpha)|^{1/q} \|f' - \alpha f\|_p \\ \times |\exp(xq\operatorname{Re}(\alpha)) - \exp(tq\operatorname{Re}(\alpha))|^{1/q} & \text{if } f' - \alpha f \in L_p[a, b], \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \|f' - \alpha f\|_1 \max\{\exp(t\operatorname{Re}(\alpha)), \exp(x\operatorname{Re}(\alpha))\}. \end{cases} \tag{4.2}$$

Proof. If f is absolutely continuous, then $f/\exp(\alpha\cdot)$ is absolutely continuous on the interval $[a, b]$ and

$$\int_t^x \left(\frac{f(s)}{\exp(\alpha s)} \right)' ds = \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)}$$

for any $t, x \in [a, b]$ with $x \neq t$.

Since

$$\int_t^x \left(\frac{f(s)}{\exp(\alpha s)}\right)' ds = \int_t^x \frac{f'(s)\exp(\alpha s) - \alpha f(s)\exp(\alpha s)}{\exp(2\alpha s)} ds = \int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)} ds,$$

then we get the following identity

$$\frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)} = \int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)} ds \tag{4.3}$$

for any $t, x \in [a, b]$ with $x \neq t$.

Taking the modulus in (4.3) we have

$$\left|\frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)}\right| = \left|\int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)} ds\right| \leq \left|\int_t^x \frac{|f'(s) - \alpha f(s)|}{|\exp(\alpha s)|} ds\right| := I \tag{4.4}$$

and utilizing Hölder's integral inequality we deduce

$$I \leq \begin{cases} \sup_{s\in[t,x]([x,t])} |f'(s) - \alpha f(s)| \left|\int_t^x \frac{1}{|\exp(\alpha s)|} ds\right|, \\ \left|\int_t^x |f'(s) - \alpha f(s)|^p ds\right|^{1/p} \left|\int_t^x \frac{1}{|\exp(\alpha s)|^q} ds\right|^{1/q}, \\ \left|\int_t^x |f'(s) - \alpha f(s)| ds\right| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|\exp(\alpha s)|}\right\}, \end{cases} \tag{4.5}$$

$$\leq \begin{cases} \|f' - \alpha f\|_\infty \left|\int_t^x \frac{1}{|\exp(\alpha s)|} ds\right|, \\ \|f' - \alpha f\|_p \left|\int_t^x \frac{1}{|\exp(\alpha s)|^q} ds\right|^{1/q}, \\ \|f' - \alpha f\|_1 \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|\exp(\alpha s)|}\right\}. \end{cases}$$

Now, since $\alpha = \mathrm{Re}(\alpha) + i\mathrm{Im}(\alpha)$ and $s \in [a, b]$, then

$$|\exp(\alpha s)| = \exp(s\mathrm{Re}(\alpha)).$$

We have

$$\int_t^x \frac{1}{|\exp(\alpha s)|} ds = \mathrm{Re}(\alpha)\left[\frac{1}{\exp(t\mathrm{Re}(\alpha))} - \frac{1}{\exp(x\mathrm{Re}(\alpha))}\right]$$

and by (4.4) and (4.5) we get

$$\left|\frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)}\right| \leq \|f' - \alpha f\|_\infty |\mathrm{Re}(\alpha)| \left|\frac{1}{\exp(t\mathrm{Re}(\alpha))} - \frac{1}{\exp(x\mathrm{Re}(\alpha))}\right|$$

and the first part of (4.1) is proved.

We have

$$\int_t^x \frac{1}{|\exp(\alpha s)|^q} ds = q\mathrm{Re}(\alpha)\left[\frac{1}{\exp(tq\mathrm{Re}(\alpha))} - \frac{1}{\exp(xq\mathrm{Re}(\alpha))}\right]$$

and by (4.4) and (4.5) we get the second part of (4.1).

We have

$$\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|\exp(\alpha s)|}\right\} = \frac{1}{\min\{\exp(t\mathrm{Re}(\alpha)),\exp(x\mathrm{Re}(\alpha))\}}$$

and by (4.4) and (4.5) we get the last part of (4.1).

The inequality (4.2) follows by (4.1) on multiplying with $|\exp(\alpha x)\exp(\alpha t)|$ and performing the required calculation. □

The following particular case is of interest.

Corollary 4.2. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$. Then for any $t,x \in [a,b]$ we have*

$$\left|\frac{f(x)}{\exp(x)} - \frac{f(t)}{\exp(t)}\right| \tag{4.6}$$
$$\leq \begin{cases} \|f'-f\|_\infty \left|\frac{1}{\exp(t)} - \frac{1}{\exp(x)}\right| & \text{if } f'-f \in L_\infty[a,b], \\ q^{1/q}\|f'-f\|_p \left|\frac{1}{\exp(tq)} - \frac{1}{\exp(xq)}\right|^{1/q} & \text{if } f'-f \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|f'-f\|_1 \frac{1}{\min\{\exp(t),\exp(x)\}}, \end{cases}$$

or, equivalently

$$|\exp(t)f(x) - f(t)\exp(x)| \tag{4.7}$$
$$\leq \begin{cases} \|f'-f\|_\infty |\exp(x)-\exp(t)| & \text{if } f'-f \in L_\infty[a,b], \\ q^{1/q}\|f'-f\|_p |\exp(xq)-\exp(tq)|^{1/q} & \text{if } f'-f \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|f'-f\|_1 \max\{\exp(t),\exp(x)\}. \end{cases}$$

Remark 4.3. If $\mathrm{Re}(\alpha)=0$ then the inequality (4.5) becomes

$$I \leq \begin{cases} \sup_{s\in[t,x]([x,t])} \left|f'(s)-i\mathrm{Im}(\alpha) f(s)\right| \left|\int_t^x \frac{1}{|\exp(i\mathrm{Im}(\alpha)s)|} ds\right|, \\ \left|\int_t^x \left|f'(s)-i\mathrm{Im}(\alpha) f(s)\right|^p ds\right|^{1/p} \left|\int_t^x \frac{1}{|\exp(i\mathrm{Im}(\alpha)s)|^q} ds\right|^{1/q}, \\ \left|\int_t^x \left|f'(s)-i\mathrm{Im}(\alpha) f(s)\right| ds\right| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|\exp(i\mathrm{Im}(\alpha)s)|}\right\}, \end{cases}$$

$$\leq \begin{cases} \|f'-i\mathrm{Im}(\alpha) f\|_\infty \left|\int_t^x ds\right|, \\ \|f'-i\mathrm{Im}(\alpha) f\|_p \left|\int_t^x ds\right|^{1/q}, \\ \|f'-i\mathrm{Im}(\alpha) f\|_1, \end{cases} = \begin{cases} \|f'-i\mathrm{Im}(\alpha) f\|_\infty |x-t|, \\ \|f'-i\mathrm{Im}(\alpha) f\|_p |x-t|^{1/q}, \\ \|f'-i\mathrm{Im}(\alpha) f\|_1. \end{cases}$$

Therefore we have

$$\left|\frac{f(x)}{\exp(i\mathrm{Im}(\alpha)x)}-\frac{f(t)}{\exp(i\mathrm{Im}(\alpha)t)}\right| \leq \begin{cases} \|f'-i\mathrm{Im}(\alpha) f\|_\infty |x-t|, \\ \|f'-i\mathrm{Im}(\alpha) f\|_p |x-t|^{1/q}, \\ \|f'-i\mathrm{Im}(\alpha) f\|_1, \end{cases} \tag{4.8}$$

or, equivalently

$$\left|\exp(i\mathrm{Im}(\alpha)t) f(x)-f(t)\exp(i\mathrm{Im}(\alpha)x)\right| \tag{4.9}$$
$$\leq \begin{cases} \|f'-i\mathrm{Im}(\alpha) f\|_\infty |x-t|, \\ \|f'-i\mathrm{Im}(\alpha) f\|_p |x-t|^{1/q}, \\ \|f'-i\mathrm{Im}(\alpha) f\|_1 \end{cases}$$

for any $t,x\in[a,b]$.

In particular, we have

$$\left|\frac{f(x)}{\exp(ix)}-\frac{f(t)}{\exp(it)}\right| \leq \begin{cases} \|f'-if\|_\infty |x-t|, \\ \|f'-if\|_p |x-t|^{1/q}, \\ \|f'-if\|_1, \end{cases} \tag{4.10}$$

or, equivalently

$$\left|\exp(it) f(x)-f(t)\exp(ix)\right| \leq \begin{cases} \|f'-if\|_\infty |x-t|, \\ \|f'-if\|_p |x-t|^{1/q}, \\ \|f'-if\|_1, \end{cases} \tag{4.11}$$

for any $t,x\in[a,b]$.

4.2. **Inequalities of Ostrowski Type.** The following result holds:

Theorem 4.4 (Dragomir, 2013 [7]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $\alpha\in\mathbb{C}$ with $\operatorname{Re}(\alpha)>0$. Then for any $x\in[a,b]$ we have*

$$\left|f(x)\frac{\exp(\alpha b)-\exp(\alpha a)}{\alpha}-\exp(\alpha x)\int_a^b f(t)\,dt\right|$$

$$\leq\begin{cases}|\operatorname{Re}(\alpha)|\,\|f'-\alpha f\|_\infty B_1(a,b,x,\alpha) & \text{if } f'-\alpha f\in L_\infty[a,b],\\[2ex] q^{1/q}|\operatorname{Re}(\alpha)|^{1/q}(b-a)^{1/p} \\ \times\|f'-\alpha f\|_p\,|B_q(a,b,x,\alpha)|^{1/q} & \text{if } f'-\alpha f\in L_p[a,b]\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \|f'-\alpha f\|_1 B_\infty(a,b,x,\alpha)\end{cases}\tag{4.12}$$

where

$$B_q(a,b,x,\alpha):=2\left[\exp(xq\operatorname{Re}(\alpha))\left(x-\frac{a+b}{2}\right)\right.$$
$$\left.+\frac{1}{q\operatorname{Re}(\alpha)}\left(\frac{\exp(bq\operatorname{Re}(\alpha))+\exp(aq\operatorname{Re}(\alpha))}{2}-\exp(xq\operatorname{Re}(\alpha))\right)\right]$$

for $q\geq1$ and

$$B_\infty(a,b,x,\alpha):=\exp(x\operatorname{Re}(\alpha))(x-a)+\frac{\exp(b\operatorname{Re}(\alpha))-\exp(x\operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)}.$$

Proof. Utilising the first inequality in (4.2) we have

$$\left|f(x)\int_a^b\exp(\alpha t)\,dt-\exp(\alpha x)\int_a^b f(t)\,dt\right|\tag{4.13}$$
$$\leq\int_a^b|\exp(\alpha t)f(x)-f(t)\exp(\alpha x)|\,dt$$
$$\leq|\operatorname{Re}(\alpha)|\,\|f'-\alpha f\|_\infty\int_a^b|\exp(x\operatorname{Re}(\alpha))-\exp(t\operatorname{Re}(\alpha))|\,dt$$

for any $x\in[a,b]$.

Observe that, since $\operatorname{Re}(\alpha)>0$, then

$$\int_a^b|\exp(x\operatorname{Re}(\alpha))-\exp(t\operatorname{Re}(\alpha))|\,dt$$
$$=2\left[\exp(x\operatorname{Re}(\alpha))\left(x-\frac{a+b}{2}\right)\right.$$
$$\left.+\frac{1}{\operatorname{Re}(\alpha)}\left(\frac{\exp(b\operatorname{Re}(\alpha))+\exp(a\operatorname{Re}(\alpha))}{2}-\exp(x\operatorname{Re}(\alpha))\right)\right]$$

for any $x \in [a,b]$.

Also

$$f(x)\int_a^b \exp(\alpha t)\,dt - \exp(\alpha x)\int_a^b f(t)\,dt = f(x)\frac{\exp(\alpha b)-\exp(\alpha a)}{\alpha} - \exp(\alpha x)\int_a^b f(t)\,dt$$

for any $x \in [a,b]$ and by (4.13) we get the first inequality in (4.12).

Using the second inequality in (4.2) we have

$$\left|f(x)\int_a^b \exp(\alpha t)\,dt - \exp(\alpha x)\int_a^b f(t)\,dt\right| \tag{4.14}$$
$$\le \int_a^b |\exp(\alpha t) f(x) - f(t)\exp(\alpha x)|\,dt$$
$$\le q^{1/q}\,|\mathrm{Re}(\alpha)|^{1/q}\,\|f'-\alpha f\|_p \int_a^b |\exp(xq\mathrm{Re}(\alpha)) - \exp(tq\mathrm{Re}(\alpha))|^{1/q}\,dt$$

for any $x \in [a,b]$.

By Hölder's integral inequality we also have

$$\int_a^b |\exp(xq\mathrm{Re}(\alpha)) - \exp(tq\mathrm{Re}(\alpha))|^{1/q}\,dt \le (b-a)^{1/p}\left[\int_a^b |\exp(xq\mathrm{Re}(\alpha)) - \exp(tq\mathrm{Re}(\alpha))|\,dt\right]^{1/q},$$

for any $x \in [a,b]$.

Observe that, as above, we have

$$\int_a^b |\exp(xq\mathrm{Re}(\alpha)) - \exp(tq\mathrm{Re}(\alpha))|\,dt$$
$$= 2\left[\exp(xq\mathrm{Re}(\alpha))\left(x-\frac{a+b}{2}\right) + \frac{1}{q\mathrm{Re}(\alpha)}\left(\frac{\exp(bq\mathrm{Re}(\alpha))+\exp(aq\mathrm{Re}(\alpha))}{2} - \exp(xq\mathrm{Re}(\alpha))\right)\right]$$
$$= B_q(a,b,x,\alpha)$$

for any $x \in [a,b]$ and by (4.14) we get the second part of (4.12).

Using the third inequality in (4.2) we have

$$\left|f(x)\int_a^b \exp(\alpha t)\,dt - \exp(\alpha x)\int_a^b f(t)\,dt\right| \tag{4.15}$$
$$\le \int_a^b |\exp(\alpha t) f(x) - f(t)\exp(\alpha x)|\,dt$$
$$\le \|f'-\alpha f\|_1 \int_a^b \max\{\exp(t\mathrm{Re}(\alpha)), \exp(x\mathrm{Re}(\alpha))\}\,dt$$

for any $x \in [a,b]$.

Observe that,

$$\begin{aligned}
&\int_a^b \max\{\exp(t\mathrm{Re}(\alpha)), \exp(x\mathrm{Re}(\alpha))\}\,dt \\
&= \int_a^x \max\{\exp(t\mathrm{Re}(\alpha)), \exp(x\mathrm{Re}(\alpha))\}\,dt \\
&+ \int_x^b \max\{\exp(t\mathrm{Re}(\alpha)), \exp(x\mathrm{Re}(\alpha))\}\,dt \\
&= \int_a^x \exp(x\mathrm{Re}(\alpha))\,dt + \int_x^b \exp(t\mathrm{Re}(\alpha))\,dt = \\
&= \exp(x\mathrm{Re}(\alpha))(x-a) + \frac{\exp(b\mathrm{Re}(\alpha)) - \exp(x\mathrm{Re}(\alpha))}{\mathrm{Re}(\alpha)}
\end{aligned}$$

and by (4.15) we get the third part of (4.12). □

Remark 4.5. If $\mathrm{Re}(\alpha) < 0$, then a similar result may be stated. However the details are left to the interested reader.

Corollary 4.6. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$. Then for any $x \in [a,b]$ we have*

$$\left| f(x)[\exp(b) - \exp(a)] - \exp(x)\int_a^b f(t)\,dt \right|$$

$$\leq \begin{cases} \|f' - f\|_\infty B_1(a,b,x) & \text{if } f' - f \in L_\infty[a,b], \\ \\ q^{1/q}(b-a)^{1/p}\|f'-f\|_p & \text{if } f'-f \in L_p[a,b] \\ \times |B_q(a,b,x)|^{1/q} & p>1, \\ & \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|f'-f\|_1 B_\infty(a,b,x) \end{cases} \tag{4.16}$$

where

$$\begin{aligned}
&B_q(a,b,x) \\
&:= 2\left[\left(x - \frac{a+b}{2}\right)\exp(xq) + \frac{1}{q}\left(\frac{\exp(bq)+\exp(aq)}{2} - \exp(xq)\right)\right]
\end{aligned}$$

for $q \geq 1$ and

$$B_\infty(a,b,x) := (x-a)\exp(x) + \exp(b) - \exp(x).$$

Remark 4.7. The midpoint case is as follows:

$$\left|f\left(\frac{a+b}{2}\right)[\exp(b)-\exp(a)]-\exp\left(\frac{a+b}{2}\right)\int_a^b f(t)\,dt\right|$$
$$\leq\begin{cases}\|f'-f\|_\infty B_1(a,b) & \text{if } f'-f\in L_\infty[a,b],\\ q^{1/q}(b-a)^{1/p}\|f'-f\|_p \times|B_q(a,b)|^{1/q} & \text{if } f'-f\in L_p[a,b]\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \|f'-f\|_1 B_\infty(a,b)\end{cases}\tag{4.17}$$

where

$$B_q(a,b,x):=\frac{2}{q}\left(\frac{\exp(bq)+\exp(aq)}{2}-\exp\left(\frac{a+b}{2}q\right)\right)$$

for $q\geq 1$ and

$$B_\infty(a,b):=\frac{b-a}{2}\exp\left(\frac{a+b}{2}\right)+\exp(b)-\exp\left(\frac{a+b}{2}\right).$$

The case $\operatorname{Re}(\alpha)=0$ is different and may be stated as follows.

Theorem 4.8 (Dragomir, 2013 [7])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $\alpha\in\mathbb{C}$ with $\operatorname{Re}(\alpha)=0$ and $\operatorname{Im}(\alpha)\neq 0$. Then for any $x\in[a,b]$ we have*

$$\left|f(x)\frac{\exp(i\operatorname{Im}(\alpha)b)-\exp(i\operatorname{Im}(\alpha)a)}{i\operatorname{Im}(\alpha)}-\exp(i\operatorname{Im}(\alpha)x)\int_a^b f(t)\,dt\right|$$
$$\leq\begin{cases}\|f'-i\operatorname{Im}(\alpha)f\|_\infty\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2 & \text{if } f'-i\operatorname{Im}(\alpha)f\in L_\infty[a,b],\\ \frac{q}{q+1}\|f'-i\operatorname{Im}(\alpha)f\|_p\times\left[\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}+\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\right](b-a)^{\frac{q+1}{q}} & \text{if } f'-i\operatorname{Im}(\alpha)f\in L_p[a,b]\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \|f'-i\operatorname{Im}(\alpha)f\|_1(b-a).\end{cases}\tag{4.18}$$

Proof. Utilizing the inequality (4.9) we have

$$\left|f(x)\frac{\exp(i\mathrm{Im}(\alpha)b)-\exp(i\mathrm{Im}(\alpha)a)}{i\mathrm{Im}(\alpha)}-\exp(i\mathrm{Im}(\alpha)x)\int_a^b f(t)\,dt\right| \tag{4.19}$$

$$\leq \int_a^b \left|\exp(i\mathrm{Im}(\alpha)t)f(x)-f(t)\exp(i\mathrm{Im}(\alpha)x)\right|dt$$

$$\leq \begin{cases} \left\|f'-i\mathrm{Im}(\alpha)f\right\|_\infty \int_a^b |x-t|\,dt, \\ \\ \left\|f'-i\mathrm{Im}(\alpha)f\right\|_p \int_a^b |x-t|^{1/q}\,dt, \\ \\ \left\|f'-i\mathrm{Im}(\alpha)f\right\|_1 \int_a^b dt. \end{cases}$$

Since

$$\int_a^b |x-t|\,dt = \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2$$

and

$$\int_a^b |x-t|^{1/q}\,dt = \frac{q}{q+1}\left[\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}+\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\right](b-a)^{\frac{q+1}{q}},$$

then we get from (4.19) the desired result (4.18). □

Corollary 4.9. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$. Then for any $x\in[a,b]$ we have*

$$\left|f(x)\frac{\exp(ib)-\exp(ia)}{i}-\exp(ix)\int_a^b f(t)\,dt\right|$$

$$\leq \begin{cases} \left\|f'-if\right\|_\infty \\ \times\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2 & \text{if } f'-if \in L_\infty[a,b], \\ \\ \frac{q}{q+1}\left\|f'-if\right\|_p \\ \times\left[\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}+\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\right](b-a)^{\frac{q+1}{q}} & \text{if } f'-if \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left\|f'-if\right\|_1(b-a). \end{cases} \tag{4.20}$$

Remark 4.10. The midpoint case is as follows

$$\left|f\left(\frac{a+b}{2}\right)\frac{\exp(ib)-\exp(ia)}{i}-\exp\left(i\frac{a+b}{2}\right)\int_a^b f(t)\,dt\right| \tag{4.21}$$

$$\leq\begin{cases}\frac{1}{4}\left\|f'-if\right\|_\infty(b-a)^2, \text{ if } f'-if\in L_\infty[a,b],\\ \\ \frac{q}{(q+1)2^{1/q}}\left\|f'-if\right\|_p(b-a)^{\frac{q+1}{q}}, \text{ if } f'-if\in L_p[a,b].\end{cases}$$

Similar inequalities may be stated if one uses (4.1) and integrates over t on $[a,b]$. The details are left to the interested reader.

5. Ostrowski Via a Two Functions Pompeiu's Inequality

5.1. A General Pompeiu's Inequality.

We start with the following generalization of Pompeiu's inequality:

Theorem 5.1 (Dragomir, 2013 [8]). *Let $f,g:[a,b]\to\mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$ with $g(t)\neq 0$ for all $t\in[a,b]$. Then for any $t,x\in[a,b]$ we have*

$$\left|\frac{f(x)}{g(x)}-\frac{f(t)}{g(t)}\right|$$

$$\leq\begin{cases}\left\|f'g-fg'\right\|_\infty\left|\int_t^x\frac{1}{|g(s)|^2}ds\right| & \text{if } f'g-fg'\in L_\infty[a,b],\\ \\ \left\|f'g-fg'\right\|_p\left|\int_t^x\frac{1}{|g(s)|^{2q}}ds\right|^{1/q} & \begin{array}{c}\text{if } f'g-fg'\in L_p[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1,\end{array}\\ \\ \left\|f'g-fg'\right\|_1\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|g(s)|^2}\right\}\end{cases} \tag{5.1}$$

or, equivalently

$$|g(t)f(x)-f(t)g(x)|$$

$$\leq\begin{cases}\left\|f'g-fg'\right\|_\infty|g(t)g(x)|\left|\int_t^x\frac{1}{|g(s)|^2}ds\right| & \text{if } f'g-fg'\in L_\infty[a,b],\\ \\ \left\|f'g-fg'\right\|_p|g(t)g(x)|\left|\int_t^x\frac{1}{|g(s)|^{2q}}ds\right|^{1/q} & \begin{array}{c}\text{if } f'g-fg'\in L_p[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1,\end{array}\\ \\ \left\|f'g-fg'\right\|_1|g(t)g(x)|\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|g(s)|^2}\right\}.\end{cases} \tag{5.2}$$

Proof. If f and g are absolutely continuous and $g(t) \neq 0$ for all $t \in [a,b]$, then f/g is absolutely continuous on the interval $[a,b]$ and

$$\int_t^x \left(\frac{f(s)}{g(s)}\right)' ds = \frac{f(x)}{g(x)} - \frac{f(t)}{g(t)}$$

for any $t, x \in [a,b]$ with $x \neq t$.

Since

$$\int_t^x \left(\frac{f(s)}{g(s)}\right)' ds = \int_t^x \frac{f'(s)\,g(s) - f(s)\,g'(s)}{g^2(s)} ds,$$

then we get the following identity

$$\frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} = \int_t^x \frac{f'(s)\,g(s) - f(s)\,g'(s)}{g^2(s)} ds \tag{5.3}$$

for any $t, x \in [a,b]$.

Taking the modulus in (5.3) we have

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{f(t)}{g(t)}\right| &= \left|\int_t^x \frac{f'(s)\,g(s) - f(s)\,g'(s)}{g^2(s)} ds\right| \\ &\leq \left|\int_t^x \frac{|f'(s)\,g(s) - f(s)\,g'(s)|}{|g(s)|^2} ds\right| := I \end{aligned} \tag{5.4}$$

and utilizing Hölder's integral inequality we deduce

$$I \leq \begin{cases} \sup_{s\in[t,x]([x,t])} |f'(s)\,g(s) - f(s)\,g'(s)| \left|\int_t^x \frac{1}{|g(s)|^2} ds\right|, \\ \\ \left|\int_t^x |f'(s)\,g(s) - f(s)\,g'(s)|^p ds\right|^{1/p} \left|\int_t^x \frac{1}{|g(s)|^{2q}} ds\right|^{1/q} & \begin{array}{c} p>1, \\ \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \\ \left|\int_t^x |f'(s)\,g(s) - f(s)\,g'(s)|\, ds\right| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|g(s)|^2}\right\}, \end{cases}$$

$$\leq \begin{cases} \|f'g - fg'\|_\infty \left|\int_t^x \frac{1}{|g(s)|^2} ds\right|, \\ \\ \|f'g - fg'\|_p \left|\int_t^x \frac{1}{|g(s)|^{2q}} ds\right|^{1/q} & \begin{array}{c} p>1, \\ \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \\ \|f'g - fg'\|_1 \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{|g(s)|^2}\right\} \end{cases}$$

and the inequality (5.1) is proved. □

The following particular case extends Pompeiu's inequality to other p-norms than $p = \infty$ obtained in (5.2).

Corollary 5.2. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b>a>0$. Then for any $t,x\in[a,b]$ we have*

$$\left|\frac{f(x)}{x}-\frac{f(t)}{t}\right| \leq \begin{cases} \|f-\ell f'\|_\infty \left|\frac{1}{t}-\frac{1}{x}\right| & \text{if } f-\ell f'\in L_\infty[a,b], \\ \frac{1}{2q-1}\|f-\ell f'\|_p \left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q} & \text{if } f-\ell f'\in L_p[a,b] \\ & p>1, \\ & \frac{1}{p}+\frac{1}{q}=1, \\ \|f-\ell f'\|_1 \frac{1}{\min\{t^2,x^2\}} \end{cases} \tag{5.5}$$

or, equivalently

$$|tf(x)-xf(t)| \leq \begin{cases} \|f-\ell f'\|_\infty |x-t| & \text{if } f-\ell f'\in L_\infty[a,b], \\ \frac{1}{2q-1}\|f-\ell f'\|_p \left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q} & \text{if } f-\ell f'\in L_p[a,b] \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|f-\ell f'\|_1 \frac{\max\{t,x\}}{\min\{t,x\}}, \end{cases} \tag{5.6}$$

where $\ell(t)=t, t\in[a,b]$.

The proof follows by (5.1) for $g(t)=\ell(t)=t$, $t\in[a,b]$.

The general case for power functions is as follows.

Corollary 5.3. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b>a>0$. If $r\in\mathbb{R}$, $r\neq 0$, then for any $t,x\in[a,b]$ we have*

$$\left|\frac{f(x)}{x^r}-\frac{f(t)}{t^r}\right| \leq \begin{cases} \frac{1}{|r|}\|f'\ell-rf\|_\infty \left|\frac{1}{x^r}-\frac{1}{t^r}\right|, \ \text{if } f'\ell-rf\in L_\infty[a,b], \\ \|f'\ell-rf\|_p \\ \times \begin{cases} \frac{1}{|1-q(r+1)|}\left|\frac{1}{x^{1-q(r+1)}}-\frac{1}{t^{1-q(r+1)}}\right|, \ \text{for } r\neq -\frac{1}{p} \\ |\ln x-\ln t|, \ \text{for } r=-\frac{1}{p} \end{cases} \\ \text{if } f'\ell-rf\in L_p[a,b], \\ \|f'\ell-rf\|_1 \frac{1}{\min\{x^{r+1},t^{r+1}\}}, \end{cases} \tag{5.7}$$

or, equivalently

$$|t^r f(x) - x^r f(t)|$$

$$\leq \begin{cases} \frac{1}{|r|}\left\|f'\ell - rf\right\|_\infty |t^r - x^r|, \text{ if } f'\ell - rf \in L_\infty[a,b], \\ \\ \left\|f'\ell - rf\right\|_p \\ \times \begin{cases} \frac{1}{|1-q(r+1)|}\left|\frac{t^r}{x^{1-q(r+1)-r}} - \frac{x^r}{t^{1-q(r+1)-r}}\right|, \text{ for } r \neq -\frac{1}{p} \\ \\ t^r x^r |\ln x - \ln t|, \text{ for } r = -\frac{1}{p} \end{cases} \\ \text{if } f'\ell - rf \in L_p[a,b], \\ \\ \left\|f'\ell - rf\right\|_1 \frac{t^r x^r}{\min\{x^{r+1}, t^{r+1}\}}, \end{cases} \tag{5.8}$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$.

The proof follows by (5.1) for $g(t) = t^r$, $t \in [a,b]$. The details for calculations are omitted.

We have the following result for exponential.

Corollary 5.4. *Let* $f : [a,b] \to \mathbb{C}$ *be an absolutely continuous function on the interval* $[a,b]$ *and* $\alpha \in \mathbb{R}$, $\alpha \neq 0$. *Then for any* $t, x \in [a,b]$ *we have*

$$\left|\frac{f(x)}{\exp(i\alpha x)} - \frac{f(t)}{\exp(i\alpha t)}\right|$$

$$\leq \begin{cases} \left\|f' - i\alpha f\right\|_\infty |x - t| & \text{if } f' - i\alpha f \in L_\infty[a,b], \\ \\ \left\|f' - i\alpha f\right\|_p |x - t|^{1/q} & \begin{array}{c}\text{if } f' - i\alpha f \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1,\end{array} \\ \\ \left\|f' - i\alpha f\right\|_1 \end{cases} \tag{5.9}$$

or, equivalently

$$|\exp(i\alpha t) f(x) - f(t)\exp(i\alpha x)|$$

$$\leq \begin{cases} \left\|f' - i\alpha f\right\|_\infty |x - t| & \text{if } f' - i\alpha f \in L_\infty[a,b], \\ \\ \left\|f' - i\alpha f\right\|_p |x - t|^{1/q} & \begin{array}{c}\text{if } f' - i\alpha f \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1,\end{array} \\ \\ \left\|f' - i\alpha f\right\|_1 . \end{cases} \tag{5.10}$$

5.2. **An Inequality Generalizing Ostrowski's.** The following result holds:

Theorem 5.5 (Dragomir, 2013 [8])**.** *Let $f, g : [a,b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$. If $0 < m \le |g(t)| \le M < \infty$ for any $t \in [a,b]$, then*

$$\left| f(x) \int_a^b g(t)\,dt - g(x) \int_a^b f(t)\,dt \right|$$

$$\le \left(\frac{M}{m}\right)^2 \begin{cases} \|f'g - fg'\|_\infty (b-a)^2 \left[\frac{1}{4} + \left(\frac{t - \frac{a+b}{2}}{b-a}\right)^2\right] & \text{if } f'g - fg' \in L_\infty[a,b], \\ \|f'g - fg'\|_p \left[\frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1+1/q}\right] & \text{if } f'g - fg' \in L_p[a,b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'g - fg'\|_1 (b-a) \end{cases} \tag{5.11}$$

for any $x \in [a,b]$.

Proof. Utilizing (5.2) we have

$$\left| f(x) \int_a^b g(t)\,dt - g(x) \int_a^b f(t)\,dt \right|$$

$$\le \int_a^b |g(t) f(x) - f(t) g(x)|\,dt$$

$$\le \begin{cases} \|f'g - fg'\|_\infty |g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt, \\ \|f'g - fg'\|_p |g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt, \\ \|f'g - fg'\|_1 |g(x)| \int_a^b \left(|g(t)| \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \end{cases} \tag{5.12}$$

for any $x \in [a,b]$, which is of interest in itself.

Since $0 < m \le |g(t)| \le M < \infty$ for any $t \in [a,b]$, then

$$|g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt \le \left(\frac{M}{m}\right)^2 \int_a^b |x - t|\,dt$$
$$= \left(\frac{M}{m}\right)^2 \left[\frac{1}{4} + \left(\frac{t - \frac{a+b}{2}}{b-a}\right)^2\right],$$

$$\left|g\left(x\right)\right|\int_{a}^{b}\left(\left|g\left(t\right)\right|\left|\int_{t}^{x}\frac{1}{\left|g\left(s\right)\right|^{2q}}ds\right|^{1/q}\right)dt$$
$$\leq\left(\frac{M}{m}\right)^{2}\int_{a}^{b}\left|x-t\right|^{1/q}dt=\left(\frac{M}{m}\right)^{2}\frac{\left(b-x\right)^{1+1/q}+\left(x-a\right)^{1+1/q}}{1+1/q}$$

and

$$\left|g\left(x\right)\right|\int_{a}^{b}\left(\left|g\left(t\right)\right|\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{\left|g\left(s\right)\right|^{2}}\right\}\right)dt\leq\left(\frac{M}{m}\right)^{2}\int_{a}^{b}dt=\left(\frac{M}{m}\right)^{2}\left(b-a\right)$$

for any $x\in[a,b]$ and by (5.12) we get the desired result (5.11). □

Remark 5.6. If we take $g\left(t\right)=1, t\in[a,b]$ in the first inequality (5.11) we recapture Ostrowski's inequality.

Corollary 5.7. *With the assumptions in Theorem 5.5 we have the midpoint inequalities*

$$\left|f\left(\frac{a+b}{2}\right)\int_{a}^{b}g\left(t\right)dt-g\left(\frac{a+b}{2}\right)\int_{a}^{b}f\left(t\right)dt\right|$$
$$\leq\left(\frac{M}{m}\right)^{2}\begin{cases}\frac{1}{4}\left(b-a\right)^{2}\left\Vert f'g-fg'\right\Vert_{\infty} & \text{if } f'g-fg'\in L_{\infty}[a,b],\\ \\ \frac{1}{2^{1/q}(1+1/q)}\left(b-a\right)^{1+1/q}\left\Vert f'g-fg'\right\Vert_{p} & \begin{array}{l}\text{if } f'g-fg'\in L_{p}[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1.\end{array}\end{cases} \tag{5.13}$$

The following result also holds:

Theorem 5.8 (Dragomir, 2013 [8])**.** *Let $f,g:[a,b]\to\mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$, $g\left(x\right)\neq0$ for $x\in[a,b]$ and $g^{-2}\in L_{\infty}[a,b]$. Then*

$$\left|\frac{f\left(x\right)}{g\left(x\right)}\int_{a}^{b}g\left(t\right)dt-\int_{a}^{b}f\left(t\right)dt\right|$$
$$\leq\left\Vert g^{-2}\right\Vert_{\infty}\times\begin{cases}\left\Vert f'g-fg'\right\Vert_{\infty}\int_{a}^{b}\left|g\left(t\right)\right|\left|x-t\right|dt, & \text{if } f'g-fg'\in L_{\infty}[a,b],\\ \\ \left\Vert f'g-fg'\right\Vert_{p}\int_{a}^{b}\left|g\left(t\right)\right|\left|x-t\right|^{1/q}dt & \begin{array}{l}\text{if } f'g-fg'\in L_{p}[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1,\end{array}\\ \\ \left\Vert f'g-fg'\right\Vert_{1}\int_{a}^{b}\left|g\left(t\right)\right|dt\end{cases} \tag{5.14}$$

for any $x \in [a, b]$.

Proof. Utilizing (5.2) we have

$$\left| \frac{f(x)}{g(x)} \int_a^b g(t)\,dt - \int_a^b f(t)\,dt \right|$$

$$\leq \begin{cases} \|f'g - fg'\|_\infty \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt, \\ \\ \|f'g - fg'\|_p \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt, \\ \\ \|f'g - fg'\|_1 \int_a^b \left(|g(t)| \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \end{cases} \tag{5.15}$$

for any $x \in [a, b]$.

Since

$$\left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \leq \left\| g^{-2} \right\|_\infty |x - t|, \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \leq \left\| g^{-2} \right\|_\infty |x - t|$$

and

$$\sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \leq \left\| g^{-2} \right\|_\infty$$

for any $x, t \in [a, b]$, then on making use of (5.15) we get the desired result (5.14). □

We have the midpoint inequalities:

Corollary 5.9. *With the assumptions of Theorem 5.8 we have*

$$\left| \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \int_a^b g(t)\,dt - \int_a^b f(t)\,dt \right|$$

$$\leq \left\| g^{-2} \right\|_\infty \times \begin{cases} \|f'g - fg'\|_\infty \int_a^b |g(t)| \left| \frac{a+b}{2} - t \right| dt, & \text{if } f'g - fg' \in L_\infty[a, b], \\ \\ \|f'g - fg'\|_p \int_a^b |g(t)| \left| \frac{a+b}{2} - t \right|^{1/q} dt & \begin{array}{c} \text{if } f'g - fg' \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1. \end{array} \end{cases} \tag{5.16}$$

We have the following exponential version of Ostrowski's inequality as well:

Theorem 5.10 (Dragomir, 2013 [8]). *Let* $f : [a, b] \to \mathbb{C}$ *be an absolutely continuous function on the interval* $[a, b]$ *and* $\alpha \in \mathbb{R}$, $\alpha \neq 0$. *Then for any* $x \in [a, b]$ *we* *have*

$$\left|\frac{\exp\left(i\alpha\left(b-x\right)\right)-\exp\left(-i\alpha\left(x-a\right)\right)}{i\alpha}f\left(x\right)-\int_a^b f\left(t\right)dt\right|$$

$$\leq\begin{cases}\left\|f'-i\alpha f\right\|_\infty\left(b-a\right)^2\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right], & \text{if } f'-i\alpha f\in L_\infty\left[a,b\right],\\ \left\|f'-i\alpha f\right\|_p\frac{\left(b-x\right)^{1+1/q}+\left(x-a\right)^{1+1/q}}{1+1/q}, & \begin{array}{c}\text{if } f'-i\alpha f\\ \in L_p\left[a,b\right]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1,\end{array}\\ \left\|f'-i\alpha f\right\|_1.\end{cases}\tag{5.17}$$

Proof. If we write the inequality (5.12) for $g\left(t\right)=\exp\left(i\alpha t\right),\ t\in\left[a,b\right],$ then we get

$$\left|f\left(x\right)\int_a^b\exp\left(i\alpha t\right)dt-\exp\left(i\alpha x\right)\int_a^b f\left(t\right)dt\right|$$

$$\leq\begin{cases}\left\|f'-i\alpha f\right\|_\infty\int_a^b\left|x-t\right|dt, & \text{if } f'-i\alpha f\in L_\infty\left[a,b\right]\\ \left\|f'-i\alpha f\right\|_p\left|g\left(x\right)\right|\int_a^b\left|x-t\right|^{1/q}dt, & \begin{array}{c}\text{if } f'-i\alpha f\\ \in L_p\left[a,b\right]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1\end{array}\\ \left\|f'-i\alpha f\right\|_1,\end{cases}$$

which, after simple calculation, is equivalent with (5.17).

The details are omitted. □

Corollary 5.11. *With the assumptions of Theorem 5.10 we have the midpoint inequalities*

$$\left|\frac{\exp\left(i\alpha\left(\frac{b-a}{2}\right)\right)-\exp\left(-i\alpha\left(\frac{b-a}{2}\right)\right)}{i\alpha}f\left(\frac{a+b}{2}\right)-\int_a^b f\left(t\right)dt\right|$$

$$\leq\begin{cases}\frac{1}{4}\left\|f'-i\alpha f\right\|_\infty\left(b-a\right)^2, & \text{if } f'-i\alpha f\in L_\infty\left[a,b\right],\\ \frac{1}{2^{1/q}\left(1+1/q\right)}\left(b-a\right)^{1+1/q}\left\|f'-i\alpha f\right\|_p, & \begin{array}{l}\text{if } f'-i\alpha f\in L_p\left[a,b\right]\\ p>1,\frac{1}{p}+\frac{1}{q}=1,\end{array}\end{cases}\tag{5.18}$$

or, equivalently

$$\left|\frac{2\sin\left(\alpha\left(\frac{b-a}{2}\right)\right)}{\alpha}f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right|$$

$$\leq\begin{cases}\frac{1}{4}\|f'-i\alpha f\|_\infty(b-a)^2, & \text{if } f'-i\alpha f\in L_\infty[a,b],\\ \\ \frac{1}{2^{1/q}(1+1/q)}(b-a)^{1+1/q}\|f'-i\alpha f\|_p, & \begin{array}{l}\text{if } f'-i\alpha f\in L_p[a,b]\\ p>1,\ \frac{1}{p}+\frac{1}{q}=1.\end{array}\end{cases}\tag{5.19}$$

5.3. **An Application for CBS-Inequality.** The following inequality is well known in the literature as the Cauchy-Bunyakovsky-Schwarz inequality, or the CBS-inequality, for short:

$$\left|\int_a^b f(t)\,g(t)\,dt\right|^2\leq\int_a^b|f(t)|^2\,dt\int_a^b|g(t)|^2\,dt,\tag{5.20}$$

provided that $f,g\in L_2[a,b]$.

We have the following result concerning some reverses of the CBS-inequality:

Theorem 5.12 (Dragomir, 2013 [8])**.** *Let $f,g:[a,b]\to\mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$ with $g(t)\neq 0$ for all $t\in[a,b]$. Then*

$$0\leq\int_a^b|g(t)|^2\,dt\int_a^b|f(t)|^2\,dt-\left|\int_a^b f(t)\,g(t)\,dt\right|^2$$

$$\leq\frac{1}{2}\times\begin{cases}\|f'\overline{g}-f\overline{g}'\|_\infty^2\left(\int_a^b|g(t)|^2\,dt\right)^2\left(\int_a^b\frac{1}{|g(t)|^2}dt\right)^2, & \text{if } \begin{array}{c}f'\overline{g}-f\overline{g}'\in L_\infty[a,b],\\ \frac{1}{|g|^2}\in L[a,b]\end{array}\\ \\ \|f'\overline{g}-f\overline{g}'\|_p^2\left(\int_a^b|g(t)|^2\,dt\right)^2\left(\int_a^b\frac{1}{|g(t)|^{2q}}dt\right)^{2/q}, & \text{if } \begin{array}{c}f'\overline{g}-f\overline{g}'\in L_p[a,b],\\ \frac{1}{|g|^{2q}}\in L[a,b]\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1,\end{array}\\ \\ \|f'\overline{g}-f\overline{g}'\|_1^2\left(\int_a^b|g(t)|^2\,dt\right)^2 ess\sup_{t\in[a,b]}\left\{\frac{1}{|g(t)|^4}\right\}, & \text{if } \frac{1}{|g|}\in L_\infty[a,b].\end{cases}\tag{5.21}$$

Proof. Utilising the inequality (5.2) we have

$$\left|\overline{g(t)}f(x)-f(t)\overline{g(x)}\right|$$

$$\leq\begin{cases}\|f'\overline{g}-f\overline{g}'\|_\infty |g(t)g(x)|\left|\int_t^x \frac{1}{|g(s)|^2}ds\right| & \text{if } f'\overline{g}-f\overline{g}' \in L_\infty[a,b],\\ \|f'\overline{g}-f\overline{g}'\|_p |g(t)g(x)|\left|\int_t^x \frac{1}{|g(s)|^{2q}}ds\right|^{1/q} & \text{if } f'\overline{g}-f\overline{g}' \in L_p[a,b] \; p>1, \; \frac{1}{p}+\frac{1}{q}=1,\\ \|f'\overline{g}-f\overline{g}'\|_1 |g(t)g(x)|\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|g(s)|^2}\right\}. \end{cases} \tag{5.22}$$

for any $t,x\in[a,b]$.

Taking the square in (5.22) and integrating over $(t,x)\in[a,b]^2$ we have

$$\int_a^b\int_a^b\left|\overline{g(t)}f(x)-f(t)\overline{g(x)}\right|^2 dtdx$$

$$\leq\begin{cases}\|f'\overline{g}-f\overline{g}'\|_\infty^2 \int_a^b\int_a^b |g(t)g(x)|^2\left|\int_t^x \frac{1}{|g(s)|^2}ds\right|^2 dtdx,\\ \|f'\overline{g}-f\overline{g}'\|_p^2 \int_a^b\int_a^b |g(t)g(x)|^2\left|\int_t^x \frac{1}{|g(s)|^{2q}}ds\right|^{2/q} dtdx,\\ \|f'\overline{g}-f\overline{g}'\|_1^2 \int_a^b\int_a^b |g(t)g(x)|^2\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{|g(s)|^4}\right\} dtdx.\end{cases} \tag{5.23}$$

Observe that

$$\begin{aligned}&\int_a^b\int_a^b\left|\overline{g(t)}f(x)-f(t)\overline{g(x)}\right|^2 dtdx\\ &=\int_a^b\int_a^b\left(|g(t)|^2|f(x)|^2-2\mathrm{Re}\left[\overline{g(t)}f(x)\overline{f(t)\overline{g(x)}}\right]+|g(x)|^2|f(t)|^2\right)dtdx\\ &=2\left[\int_a^b|g(t)|^2dt\int_a^b|f(t)|^2dt-\left|\int_a^b f(t)g(t)dt\right|^2\right],\end{aligned}$$

$$\int_a^b\int_a^b\left[|g(t)g(x)|^2\left|\int_t^x\frac{1}{|g(s)|^2}ds\right|^2\right]dtdx\leq\left(\int_a^b|g(t)|^2dt\right)^2\left(\int_a^b\frac{1}{|g(t)|^2}dt\right)^2,$$

$$\int_a^b \int_a^b \left[|g(t) g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{2/q} \right] dtdx$$
$$\leq \left(\int_a^b |g(t)|^2 dt \right)^2 \left(\int_a^b \frac{1}{|g(t)|^{2q}} dt \right)^{2/q}$$

and

$$\int_a^b \int_a^b \left[|g(t) g(x)|^2 \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^4} \right\} \right] dtdx$$
$$\leq \left(\int_a^b |g(t)|^2 dt \right)^2 ess \sup_{t \in [a,b]} \left\{ \frac{1}{|g(t)|^4} \right\},$$

then by (5.23) we get the desired result (5.21). □

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