IN Variant MEANS ON CHART GROUPS

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Abstract. The purpose of this paper is to give a stream-lined proof of the existence and uniqueness of a right-invariant mean on a CHART group. A CHART group is a slight generalisation of a compact topological group. The existence of an invariant mean on a CHART group can be used to prove Furstenberg’s fixed point theorem.

1. Introduction and preliminaries

Given a nonempty set $X$ and a linear subspace $S$ of $\mathbb{R}^X$ that contains all the constant functions we say that a linear functional $m : S \to \mathbb{R}$ is a mean on $S$ if:

(i) $m(f) \geq 0$ for all $f \in S$ that satisfy $f(x) \geq 0$ for all $x \in X$;
(ii) $m(1) = 1$, where $1$ is the function that is identically equal to $1$.

If all the functions in $S$ are bounded on $X$ then this definition is equivalent to the following:

$$1 = m(1) = \|m\|$$

where, $\|m\| := \sup\{m(f) : f \in S \text{ and } \|f\|_{\infty} \leq 1\}$.

If $(X, \cdot)$ is a semigroup then we can define, for each $g \in X$, $L_g : \mathbb{R}^X \to \mathbb{R}^X$ and $R_g : \mathbb{R}^X \to \mathbb{R}^X$ by,

$$L_g(f)(x) := f(gx) \text{ for all } x \in X \quad \text{and} \quad R_g(f)(x) := f(xg) \text{ for all } x \in X.$$  

Note that for all $g, h \in X$, $L_g \circ L_h = L_{hg}$, $R_g \circ R_h = R_{gh}$ and $L_g \circ R_h = R_h \circ L_g$.

If $S$ is a subspace of $\mathbb{R}^X$ that contains all the constant functions and $L_g(S) \subseteq S$
invariant group (left topological group). We shall call a triple $$(G, \cdot, \tau)$$ a right topological group and a left topological group if $$(G, \cdot)$$ is both a right topological group and a left topological group then we call it a semitopological group.

Proposition 1.1. Let $$(G, \cdot, \tau)$$ and $$(H, \cdot, \tau')$$ be compact Hausdorff right topological groups and let $$\pi : G \to H$$ be a continuous epimorphism (i.e., a surjective homomorphism). If $$m$$ is a right-invariant mean on $$C(H)$$ then $$m^* : \pi^#(C(H)) \to \mathbb{R}$$ defined by, $$m^*(f) := m((\pi^#)^{-1}(f))$$ for all $$f \in \pi^#(C(H))$$ is a right-invariant mean on $$\pi^#(C(H))$$. If $$C(H)$$ has a unique right-invariant mean then $$\pi^#(C(H))$$ has a unique right-invariant mean.

We can now state and prove our main theorem for this section.

Theorem 1.2. Let $$(G, \cdot, \tau)$$ and $$(H, \cdot, \tau')$$ be compact Hausdorff right topological groups and let $$\pi : G \to H$$ be a continuous epimorphism. If the mapping

$$m : G \times \ker(\pi) \to G$$

defined by, $$m(x, y) := x \cdot y$$ for all $$(x, y) \in G \times \ker(\pi)$$

is continuous and $$C(H)$$ has a right-invariant mean then $$C(G)$$ has a right-invariant mean. Furthermore, if $$C(H)$$ has a unique right-invariant mean then so does $$C(G)$$.

Proof. Let $$L := \ker(\pi)$$. Then from the hypotheses and [1, Theorem 2] $$(L, \cdot, \tau_L)$$ (here $$\tau_L$$ is the relative $$\tau$$-topology on $$L$$) is a compact topological group. Thus $$(L, \cdot, \tau_L)$$ admits a unique Borel probability measure $$\lambda$$ (called the Haar measure on $$L$$)

such that

$$\int_{L} L_g(f)(t) \, d\lambda(t) = \int_{L} R_g(f)(t) \, d\lambda(t) = \int_{L} f(t) \, d\lambda(t)$$

for all $$g \in L$$ and $$f \in C(L)$$. 

[Note: The author has not included a complete proof here, but the statement is correct as per the proposition and theorem restated.]
Let $P : C(G) \to \pi^\#(C(H))$ be defined by,

$$P(f)(g) := \int_L f(g \cdot t) \, d\lambda(t) \text{ i.e., } P(f)(g) \text{ is the "average" of } f \text{ over the coset } gL.$$ 

Firstly, since $m$ is continuous on $G \times L$ (and $L$ is compact) $P(f) \in C(G)$ for each $f \in C(G)$. Secondly, since $\lambda$ is invariant on $L$ it is routine to check that $P(f)$ is constant on the fibers of $\pi$. Hence, $P(f) \in \pi^\#(C(H))$. We now show that for each $g \in G$ and $f \in C(G)$,

$$\int_L L_g(f)(t) \, d\lambda(t) = \int_L f(g \cdot t) \, d\lambda(t) = \int_L f(t \cdot g) \, d\lambda(t) = \int_L R_g(f)(t) \, d\lambda(t). \quad (*)$$

To this end, fixed $g \in G$ and define $G : C(L) \to C(L)$ by, $G(f)(t) := f(g^{-1} \cdot t \cdot g)$. Since $m$ is continuous, $t \mapsto (g^{-1} \cdot t) \cdot g$ is continuous and so $G$ is well-defined, i.e., $G(f) \in C(L)$ for each $f \in C(L)$. We claim that

$$f \mapsto \int_L G(f)(t) \, d\lambda(t)$$

is a right-invariant mean on $C(L)$. Clearly, this mapping is a mean so it remains to show that it is right-invariant. To see this, let $l \in L$. Then $g \cdot l \cdot g^{-1} \in L$ and

$$\int_L G(R_l(f))(t) \, d\lambda(t) = \int_L R_l(f)(g^{-1} \cdot t \cdot g) \, d\lambda(t)$$

$$= \int_L f(g^{-1} \cdot t \cdot g \cdot l) \, d\lambda(t)$$

$$= \int_L f(g^{-1} \cdot [t \cdot (g \cdot l \cdot g^{-1})] \cdot g) \, d\lambda(t)$$

$$= \int_L G(f)(t \cdot (g \cdot l \cdot g^{-1})) \, d\lambda(t)$$

$$= \int_L R_{g^{-1} \cdot g^{-1}}(G(f))(t) \, d\lambda(t)$$

$$= \int_L G(f)(t) \, d\lambda(t) \text{ since } \lambda \text{ is right-invariant.}$$

Now, since there is only one right-invariant mean on $C(L)$ we must have that

$$\int_L G(f)(t) \, d\lambda(t) = \int_L f(g^{-1} \cdot t \cdot g) \, d\lambda(t) = \int_L f(t) \, d\lambda(t) \text{ for all } f \in C(L).$$

It now follows that equation $(*)$ holds. Next, we show that $R_g(P(f)) = P(R_g(f))$ for all $g \in G$ and $f \in C(G)$. To this end, let $g \in G$ and $f \in C(G)$. Then for any $x \in G$,

$$R_g(P(f))(x) = P(f)(x \cdot g) = \int_L f(x \cdot g \cdot t) \, d\lambda(t) = \int_L f(x \cdot t \cdot g) \, d\lambda(t) \text{ by (*)}$$

$$= \int_L R_g(f)(x \cdot t) \, d\lambda(t) = P(R_g(f))(x).$$
Let $\mu$ be the unique right-invariant mean on $\pi^\#(C(H))$, given to us by Proposition 1.1. Let $\mu^* : C(G) \to \mathbb{R}$ be defined by, $\mu^*(f) := \mu(P(f))$. It is now easy to verify that $\mu^*$ is a right-invariant mean on $C(G)$.

So it remains to prove uniqueness. Suppose that $\mu^*$ and $\nu^*$ are right-invariant means on $C(G)$. Since, by Proposition 1.1, we know that $\mu^*|_{\pi^\#(C(H))} = \nu^*|_{\pi^\#(C(H))}$ it will be sufficient to show that $\mu^*(f) = \mu^*(P(f))$ and $\nu^*(f) = \nu^*(P(f))$ for each $f \in C(G)$. We shall apply Riesz’s representation theorem along with Fubini’s theorem. Let $\mu$ be the probability measure on $G$ that represents $\mu^*$ and let $f \in C(G)$. Then

$$\mu^*(f) = \int_G f(s) \, d\mu(s) = \int_L \int_G f(s \cdot t) \, d\mu(s) \, d\lambda(t)$$
$$= \int_G \int_L f(s \cdot t) \, d\lambda(t) \, d\mu(s)$$
$$= \int_G P(f)(s) \, d\mu(s) = \mu^*(P(f)).$$

A similar argument show that $\nu^*(f) = \nu^*(P(f))$. This completes the proof. □

This paper is the culmination of work done many people, starting with the work of H. Furstenberg in [4] on the existence of invariant measures on distal flows. This work was later simplified and phrased in terms of CHART groups by I. Namioka in [8]. The results of Namioka were further generalised by R. Ellis, [3]. In 1992, P. Milnes and J. Pym, [5] showed that every CHART group (that satisfies some countability condition) admits a unique right-invariant mean (unique right-invariant measure) called the Haar mean (Haar measure). Later, in [6], Milne and Pym managed to remove the countability condition from the proof contained in [5] by appealing to a result from [3]. Finally, in [7], a direct proof of the existence and uniqueness of a right-invariant mean on a CHART group was given, however, this proof still relied upon the results from [5].

In the present paper we give a streamlined proof (that does not require knowledge from topological dynamics) of the existence and uniqueness of a right-invariant mean on a CHART group.

2. Groups

Let $(G, \cdot, \tau)$ be a right topological group and let $H$ be a subgroup of $G$. We shall denote by $(H, \tau_H)$ the set $H$ equipped with the relative $\tau$-topology. It is easy to see that $(H, \cdot, \tau_H)$ is also a right topological group.

Now let $G/H$ be the set $\{xH : x \in G\}$ of all left cosets of $H$ in $G$ and give $G/H$ the quotient topology $q(\tau)$ induced from $(G, \tau)$ by the map $\pi : G \to G/H$ defined by $\pi(x) := xH$.

Note that $\pi$ is an open mapping because, if $U$ is an open subset of $G$ then

$$\pi^{-1}(\pi(U)) = UH = \bigcup\{Ux : x \in H\}$$

and this last set is open since right multiplication is a homeomorphism on $G$. 


If $H$ is a normal subgroup of a right(left) semi topological group $(G, \cdot, \tau)$ then one can check that $(G/H, \cdot, q(\tau))$ is also a right(left) semi topological group.

In order to continue our investigations further we need to introduce a new topology.

2.1. The $\sigma$-topology. Let $(G, \cdot, \tau)$ be a right topological group and let $\varphi : G \times G \to G$ be the map defined by

$$\varphi(x, y) := x^{-1} \cdot y.$$ 

Then the quotient topology on $G$ induced from $(G \times G, \tau \times \tau)$ by the map $\varphi$ is called the $\sigma(G, \tau)$-topology or $\sigma$-topology.

The proof of the next result can be found in [8, Theorem 1.1, Theorem 1.3] or [9, Lemma 4.3].

**Lemma 2.1.** Let $(G, \cdot, \tau)$ be a right topological group. Then,

(i) $(G, \sigma)$ is a semitopological group.

(ii) $\sigma \subseteq \tau$.

(iii) $(G/H, q(\tau))$ is Hausdorff provided the subgroup $H$ is closed with respect to the $\sigma$-topology on $G$.

2.2. Admissibility and CHART groups. Let $(G, \cdot, \tau)$ be a right topological group and let $\Lambda(G, \tau)$ be the set of all $x \in G$ such that the map $y \mapsto x \cdot y$ is $\tau$ continuous. If $\Lambda(G, \tau)$ is $\tau$-dense in $G$ then $(G, \tau)$ is said to be admissible.

The proof for the following proposition may be found in [8, Theorem 1.2, Corollary 1.1] or [9, Proposition 4.4, Proposition 4.5].

**Proposition 2.2.** Let $(G, \cdot, \tau)$ be an admissible right topological group.

(i) If $U$ is the family of all $\tau$-open neighborhoods of $e$ in $G$ then $\{U^{-1}U : U \in U\}$ is a base of open neighborhoods of $e$ in $(G, \sigma)$.

(ii) If $N(G, \tau) := \bigcap\{U^{-1}U : U \in U\}$ then $N(G, \tau) = \{e\}^\sigma$.

A compact Hausdorff admissible right topological group $(G, \cdot, \tau)$ is called a CHART group.

The proof for the following result may be found [8, Proposition 2.1] or [9, Proposition 4.6].

**Proposition 2.3.** Let $(G, \cdot, \tau)$ be a CHART group. Then the following hold:

(i) If $L$ is a $\sigma$-closed normal subgroup of $G$, then so is $N(L, \sigma_L)$.

(ii) If $m : (G/N(L, \sigma_L), q(\tau)) \times (L/N(L, \sigma_L), q(\tau)) \to (G/N(L, \sigma_L), q(\tau))$ is defined by

$$m(xN(L, \sigma_L), yN(L, \sigma_L)) := x \cdot yN(L, \sigma_L) \quad \text{for all} \quad (x, y) \in G \times L$$

then $m$ is well-defined and continuous.

**Remark 2.4.** By considering the mapping $\pi : G/N(L, \sigma_L) \to G/L$, Theorem 1.2 and Proposition 2.3 we see that if $(G/L, q(\tau))$ admits a unique right-invariant mean then so does $(G/N(L, \sigma_L), q(\tau))$. Hence if $N(L, \sigma_L)$ is a proper subset of $L$ then we have made some progress towards showing that $G \cong G/\{e\}$ admits a unique right-invariant mean.
3. $N(L, \sigma_L) \neq L$

In this section we will show that if $L$ is a nontrivial $\sigma$-closed normal subgroup of a CHART group $(G, \cdot, \tau)$ then $N(L, \sigma_L)$ is a proper subset of $L$.

**Lemma 3.1.** Let $(H, \cdot)$ be a group and $X$ be a nonempty set. Then for any $f : H \to X$, $S := \{s \in H : f(hs) = f(h) \text{ for all } h \in H\}$ is a subgroup of $H$.

**Proof.** Clearly, $e \in S$. Now suppose that, $s_1, s_2 \in S$. Let $h$ be any element of $H$ then

$$f(h(s_1s_2)) = f((hs_1)s_2) = f(hs_1) = f(h).$$

Therefore, $s_1s_2 \in S$. Next, let $s$ be any element of $S$ and $h$ be any element of $H$ then

$$f(h) = f(hs^{-1}s) = f((hs^{-1})s) = f(hs^{-1}).$$

Therefore, $s^{-1} \in S$. \hfill $\square$

**Lemma 3.2.** Let $(G, \cdot, \tau)$ be a compact right topological group and let $\sigma$ be a topology on $G$ weaker than $\tau$ such that $(G, \cdot, \sigma)$ is also a right topological group. If $U$ is a dense open subset of $(G, \sigma)$ then $U$ is also a dense subset of $(G, \tau)$.

**Proof.** Let $C := G \setminus U$. Then $C$ is a $\sigma$-closed (hence $\tau$-closed) nowhere-dense subset of $G$. If $U$ is not $\tau$-dense in $G$ then $C$ contains a nonempty $\tau$-open subset. By the compactness of $(G, \tau)$ there exists a finite subset $F$ of $G$ such that $G = \bigcup \{Cg : g \in F\}$. Now each $Cg$ is nowhere dense in $(G, \sigma)$ since each right multiplication is a homeomorphism. This forms a contradiction since a nonempty topological space can never be the union of a finite number of nowhere dense subsets. \hfill $\square$

**Lemma 3.3.** Let $(G, \cdot, \tau)$ be a CHART group and let $\Lambda = \Lambda(G, \tau)$. If $A$ and $B$ are nonempty open subsets of $(G, \tau)$, then $A^{-1}B = (A \cap \Lambda)^{-1}B$.

**Proof.** Let $x \in A^{-1}B$. Then for some $a \in A, ax \in B$. Since $B$ is open and $A \cap \Lambda$ is dense in $A$ there is a $c \in A \cap \Lambda$ such that $cx \in B$. Hence $x \in c^{-1}B \subseteq (A \cap \Lambda)^{-1}B$. Thus, $A^{-1}B \subseteq (A \cap \Lambda)^{-1}B$. The reverse inclusion is obvious. \hfill $\square$

**Lemma 3.4.** Let $(G, \cdot, \tau)$ be a compact Hausdorff right topological group. If $S$ is a nonempty subsemigroup of $\Lambda(G, \tau)$ then $\overline{S}$ is a subgroup of $G$.

**Proof.** In this proof we shall repeatedly use the following fact, [2, Lemma 1] “Every nonempty compact right topological semigroup admits an idempotent element (i.e., an element $u$ such that $u \cdot u = u$). Firstly, it is easy to see that $\overline{S}$ is a subsemigroup of $G$. Hence, $(\overline{S}, \cdot)$ is a nonempty compact right topological semigroup and so has an idempotent element $u$. However, since $G$ is a group it has only one idempotent element, namely $e$. Therefore, $e = u \in \overline{S}$. Next, let $s$ be any element of $\overline{S}$. Then $\overline{S} \cdot s$ is a nonempty compact right topological semigroup of $\overline{S}$. Therefore, there exists an element $s' \in \overline{S}$ such that $(s' \cdot s) \cdot (s' \cdot s) = (s' \cdot s)$. Again, since $G$ is a group, $s' \cdot s = e$. By multiplying both sides of this equation by $s^{-1}$ we see that $s^{-1} = s' \in \overline{S}$. \hfill $\square$

The following lemma is a simplified form of the structure theorem found in [7].
Lemma 3.5. Let \((G, \cdot, \tau)\) be a CHART group and let \(\sigma\) denote its \(\sigma\)-topology. Suppose \(L\) is a nontrivial \(\sigma\)-closed subgroup of \(G\). Then \(N(L, \sigma_L)\) is a proper subset of \(L\).

Proof. Let \(U\) denote the family of all open neighbourhoods of \(e\) in \((G, \tau)\). Then it follows from Proposition 2.2 that \(\mathcal{V} = \{U^{-1}U : U \in U\}\) is a base for the system of open neighbourhoods of \(e\) in \((G, \sigma)\). Then \(\{V \cap L : V \in \mathcal{V}\}\) is a basis for the system of neighbourhoods of \(e\) in \((L, \sigma_L)\). From the definition of \(N(L, \sigma_L)\) (see Proposition 2.2 part (ii)) it follows that

\[
N(L, \sigma_L) = \bigcap \{(V \cap L)^{-1}(V \cap L) : V \in \mathcal{V}\}.
\]

The proof is by contradiction. So assume that \(N(L, \sigma_L) = L\). Then

\[
L = \bigcap \{(V \cap L)^{-1}(V \cap L) : V \in \mathcal{V}\}.
\]

Hence, for each \(V \in \mathcal{V}\), \((V \cap L)^{-1}(V \cap L) = L\), or equivalently, for each \(V \in \mathcal{V}\), \((V \cap L)\) is dense in \((L, \sigma_L)\). That is, for each \(U \in \mathcal{U}\), \((U^{-1}U \cap L)\) is open and dense in \((L, \sigma_L)\) and hence, by Lemma 3.2, dense in \((L, \tau_L)\).

Since \(L \neq \{e\}\), there exists a point \(a \in L\) such that \(a \neq e\). Note that since \((G, \tau)\) is compact and Hausdorff there is a continuous function \(f\) on \((G, \tau)\) such that \(f(e) = 0\) and \(f \equiv 1\) on a \(\tau\)-neighbourhood of \(a\).

For the rest of the proof, the topology always refers to \(\tau\) and we shall denote \(\Lambda(G, \tau)\) by \(\Lambda\). By induction on \(n\), we construct a sequence \(\{U_n : n \in \mathbb{N}\}\) in \(\mathcal{U}\), a sequence \(\{V_n : n \in \mathbb{N}\}\) of nonempty open subsets of \(G\), each of which intersects \(L\) and sequences \(\{u_n : n \in \mathbb{N}\}\) and \(\{v_n : n \in \mathbb{N}\}\) in \(G\) which satisfy the following conditions:

(i) \(v_n \in U_{n-1}^{-1}U_{n-1} \cap (V_{n-1} \cap \Lambda) = (U_{n-1} \cap \Lambda)^{-1}U_{n-1} \cap (V_{n-1} \cap \Lambda)\); by Lemma 3.3.

(ii) \(u_n \in U_{n-1} \cap \Lambda\);

(iii) \(V_n \subseteq V_n \cap V_{n-1} \subseteq f^{-1}(1)\) and \(V_n \cap L \neq \emptyset\);

(iv) \(u_n V_n \subseteq U_{n-1}\);

(v) if \(H_n\) denotes the semigroup generated by \(\{u_1, v_1, u_2, v_2, \ldots, u_n, v_n\}\); which we enumerate as: \(H_n := \{h^n_j : j \in \mathbb{N}\}\) and

\[
U_n := \{t \in G : |f(h^n_i t) - f(h^n_j)| < 1/n \quad \text{for} \quad 1 \leq i, j \leq n\}
\]

then \(H_n \subseteq \Lambda\) and \(e \in U_n \subseteq U_n \subseteq U_{n-1}\).

Construction. We let \(U_0 := G\) and let \(V_0\) be the interior of \(f^{-1}(1)\) and \(u_0, v_0\) are not defined. Assume that \(n \in \mathbb{N}\) and that \(U_k, V_k\) are defined for \(0 \leq k < n\) and \(v_k, u_k\) are defined for \(0 < k < n\). By our assumption there exists an \(x \subseteq (U_{n-1} \cap \Lambda)^{-1}U_{n-1} \cap (V_{n-1} \cap \Lambda)\). So there is a \(u_n \in U_{n-1} \cap \Lambda\) such that \(u_n x \in U_{n-1}\). Since \(u_n \in \Lambda\), \(x \in V_n, V_n \subseteq U_{n-1}\) is open, there is an open neighbourhood \(V_n\) of \(x\) such that \(x \in V_n \subseteq V_n \subseteq V_{n-1}\) and \(u_n V_n \subseteq U_{n-1}\). Then \(V_n \cap L \neq \emptyset\) since \(x \in V_n \cap L\). Thus (ii)-(iv) are satisfied. Let \(v_n\) be any element of \(V_n \cap \Lambda\), then by (iv) and (ii), (i) is satisfied and \(H_n \subseteq \Lambda\) is defined. Finally, since the map \(t \mapsto |f(gt) - f(g)|\) is continuous for \(g \in \Lambda\), the set \(U_n\) is an open neighbourhood of \(e\) and so condition (v) is satisfied. This completes the construction.

We let

\[
U_\infty = \bigcap \{U_n : n \in \mathbb{N}\} \quad \text{and} \quad H = \bigcup \{H_n : n \in \mathbb{N}\}
\]
and let $u_\infty$, $v_\infty$ be cluster points of the sequences $\{u_n : n \in \mathbb{N}\}$, $\{v_n : n \in \mathbb{N}\}$ respectively. Clearly $u_\infty \in U_\infty$, $v_\infty \in V_0$ and $H$ is a subgroup of $G$, by Lemma 3.4. Moreover, by the construction, $f(ht) = f(h)$ for each $h \in H$ and each $t \in \bigcap\{U_n : n \in \mathbb{N}\}$. Therefore, if we let

\[
S = \{s \in H : f(hs) = f(h) \text{ for each } h \in H\}
\]

then $\bigcap\{U_n : n \in \mathbb{N}\} \cap H \subset S$ and $S$ is a subgroup of $G$ by Lemma 3.1. Furthermore, by (ii), $u_\infty \in U_\infty \cap H \subset S$ and by (iv) $u_nv_\infty \in \overline{U_{n-1}} \cap H$ for each $n \in \mathbb{N}$. Hence

\[
u_\infty v_\infty \in \bigcap_{n \in \mathbb{N}} U_{n-1} \cap H \subset S.
\]

Therefore, $v_\infty = u_\infty^{-1}(u_\infty v_\infty) \in S^{-1}S \subset S$. Now, $f(s) = 0$ for all $s \in S$ since

\[
f(es) = f(e) = 0 \quad \text{for all } s \in S.
\]

Therefore, $f(v_\infty) = 0$. On the other hand, since $v_\infty \in V_0 \subset f^{-1}(1)$, $f(v_\infty) = 1$. This contradiction completes the proof. \qed

4. INVARIANT MEANS ON CHART GROUPS

In this section we will show that every CHART group admits a unique right-invariant mean.

**Theorem 4.1**. Every CHART group $(G, \cdot, \tau)$ possesses a unique right-invariant mean $m$ on $C(G)$.

**Proof.** Let $\mathcal{L}$ be the family of all $\sigma$-closed normal subgroups $L$ of $G$ for which $C(G/L)$ has a unique right-invariant mean. Clearly, $\mathcal{L} \neq \emptyset$ as $G \in \mathcal{L}$. Now, $(\mathcal{L}, \subset)$ is a partially ordered set. We claim that $(\mathcal{L}, \subset)$ possesses a minimal element. To prove this, it is sufficient to show that every totally ordered subfamily $\mathcal{M}$ of $\mathcal{L}$ has a lower bound (in $\mathcal{L}$). To this end, let $\mathcal{M} := \{M_\alpha : \alpha \in A\}$ be a nonempty totally ordered subfamily of $\mathcal{L}$. Let

\[
M_0 := \bigcap\{M_\alpha : \alpha \in A\}.
\]

Then $M$ is a $\sigma$-closed normal subgroup of $G$ and $M_0 \subseteq M_\alpha$ for every $\alpha \in A$. Thus, to complete the proof of the claim we must show that $M_0 \in \mathcal{L}$, i.e., show that $C(G/M_0)$ admits a unique right-invariant mean. For each $\alpha \in A$, let $\pi_\alpha : G/M_0 \to G/M_\alpha$ be defined by $\pi_\alpha(gM_0) := gM_\alpha$. Then $\pi_\alpha$ is a continuous, open and onto map and its dual map $\pi_\alpha^\# : C(G/M_\alpha) \to C(G/M_0)$ is an isometric algebra isomorphism of $C(G/M_\alpha)$ into $C(G/M_0)$. By Proposition 1.1, for each $\alpha \in A$, there exists a unique right-invariant mean $m_\alpha$ on $\pi_\alpha^\#(C(G/M_\alpha))$. From the Hahn-Banach extension theorem it follows that each mean $m_\alpha$ has an extension to a mean $m_\alpha^*$ on $C(G/M_0)$. Let $\mathcal{A} := \bigcup\{\pi_\alpha^*(C(G/M_\alpha)) : \alpha \in A\}$. Then $\mathcal{A}$ is a subalgebra of $C(G/M_0)$, that contains all the constant functions and separates the point of $G/M_0$ since $M_0 := \bigcap\{M_\alpha : \alpha \in A\}$. Therefore, by the Stone-Weierstrass theorem, $\mathcal{A}$ is dense in $C(G/M_0)$. Let $m$ be a weak* cluster-point of the net $(m_\alpha^* : \alpha \in A)$ in $B_{C(G/M_0)^*}$. Clearly, $m$ is a mean on $C(G/M_0)$. Furthermore, it is routine to show that (i) $m|_{\mathcal{A}}$ is a right-invariant mean on $\mathcal{A}$ and (ii) $m|_{\mathcal{A}}$ is the
only (unique) right-invariant mean on $\mathcal{A}$. It now follows from continuity that $m$ is the one and only right-invariant mean on $C(G/M_0)$, i.e., $M_0 \in \mathcal{L}$.

Let $L_0$ be a minimal element of $\mathcal{L}$. Then by Remark 2.4, $N(L_0, \sigma_{L_0}) \in \mathcal{L}$. However, since $N(L, \sigma_L) \subseteq L_0$ and $L_0$ is a minimal element of $\mathcal{L}$ we must have that $N(L, \sigma_L) = L_0$. Thus, by Lemma 3.5, it must be the case that $L_0 = \{e\}$. This completes the proof. □

Let us now note that the unique right-invariant mean given above is also partially left invariant in the sense that for each $g \in \Lambda(G, \tau)$, $m(L_g(f)) = m(f)$ for all $f \in C(G)$. To see why this is true, consider the mean $m^*$ on $C(G)$ defined by, $m^*(f) := m(L_g(f))$ for each $f \in C(G)$ and some $g \in \Lambda(G, \tau)$. Then for any $h \in G$,

$$m^*(R_h(f)) = m(L_g(R_h(f))) = m(R_h(L_g(f))) = m(L_g(f)) = m^*(f).$$

Therefore, $m^*$ is a right-invariant mean on $C(G)$. Thus, $m^* = m$ and so

$$m(L_g(f)) = m^*(f) = m(f)$$

for all $f \in C(G)$ and all $g \in \Lambda(G, \tau)$.

REFERENCES


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