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GENERALIZATIONS OF STEFFENSEN'S INEQUALITY BY ABEL-GONTSCHAROFF POLYNOMIAL

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Abstract. In this paper generalizations of Steffensen's inequality using Abel-Gontscharoff interpolating polynomial are obtained. Moreover, in a special case generalizations by Abel-Gontscharoff polynomial reduce to known weaker conditions for Steffensen's inequality. Furthermore, Ostrowski type inequalities related to obtained generalizations are given.

1. Introduction

Let $-\infty < a < b < \infty$ and let $a \leq a_1 < a_2 < \ldots < a_n \leq b$ be the given points. For $f \in C^n[a,b]$ *Abel-Gontscharoff interpolating polynomial* P_{AG} of degree $(n-1)$ satisfying Abel-Gontscharoff conditions

$$P_{AG}^{(i)}(a_{i+1}) = f^{(i)}(a_{i+1}), \quad 0 \leq i \leq n-1$$

exists uniquely ([7], [12]).
This conditions in particular include two-point right focal conditions

$$\begin{aligned} &P_{AG2}^{(i)}(a_1) = f^{(i)}(a_1), \;\; 0 \leq i \leq \alpha, \\ &P_{AG2}^{(i)}(a_2) = f^{(i)}(a_2), \;\; \alpha + 1 \leq i \leq n-1, \;\; a \leq a_1 < a_2 \leq b. \end{aligned}$$

First, we give representations of Abel-Gontscharoff interpolating polynomial. For details and proofs see [1].

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Theorem 1.1. *Abel-Gontscharoff interpolating polynomial P_{AG} of the function f can be expressed as*

$$P_{AG}(t)=\sum_{i=0}^{n-1}T_i(t)f^{(i)}(a_{i+1}),$$

where $T_0(t)=1$ and T_i, $1\le i\le n-1$ is the unique polynomial of degree i satisfying

$$\begin{aligned}T_i^{(k)}(a_{k+1})&=0,\ 0\le k\le i-1\\ T_i^{(i)}(a_{i+1})&=1\end{aligned}$$

and it can be written as

$$\begin{aligned}T_i(t)&=\frac{1}{1!2!\cdots i!}\begin{vmatrix}1 & a_1 & a_1^2 & \dots & a_1^{i-1} & a_1^i\\ 0 & 1 & 2a_2 & \dots & (i-1)a_2^{i-2} & ia_2^{i-1}\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & (i-1)! & i!a_i\\ 1 & t & t^2 & \dots & t^{i-1} & t^i\end{vmatrix}\\ &=\int_{a_1}^{t}\int_{a_2}^{t_1}\cdots\int_{a_i}^{t_{i-1}}dt_idt_{i-1}\cdots dt_1,\ (t_0=t).\end{aligned}\tag{1}$$

In particular, we have

$$\begin{aligned}T_0(t)&=1\\ T_1(t)&=t-a_1\\ T_2(t)&=\frac{1}{2}\left[t^2-2a_2t+a_1(2a_2-a_1)\right].\end{aligned}$$

Corollary 1.2. *The two-point right focal interpolating polynomial P_{AG2} of the function f can be written as*

$$\begin{aligned}P_{AG2}(t)=&\sum_{i=0}^{\alpha}\frac{(t-a_1)^i}{i!}f^{(i)}(a_1)\\ &+\sum_{j=0}^{n-\alpha-2}\left[\sum_{i=0}^{j}\frac{(t-a_1)^{\alpha+1+i}(a_1-a_2)^{j-i}}{(\alpha+1+i)!(j-i)!}\right]f^{(\alpha+1+j)}(a_2).\end{aligned}$$

The associated error $e_{AG}(t)=f(t)-P_{AG}(t)$ can be represented in terms of the Green's function $g_{AG}(t,s)$ of the boundary value problem

$$z^{(n)}=0,\ z^{(i)}(a_{i+1})=0,\ 0\le i\le n-1$$

and appears as (see [1]):

$$g_{AG}(t,s)=\begin{cases}\sum\limits_{i=0}^{k-1}\frac{T_i(t)}{(n-i-1)!}(a_{i+1}-s)^{n-i-1}, & a_k\le s\le t;\\ -\sum\limits_{i=k}^{n-1}\frac{T_i(t)}{(n-i-1)!}(a_{i+1}-s)^{n-i-1}, & t\le s\le a_{k+1}\\ & k=0,1,\dots,n\ (a_0=a,a_{n+1}=b)\end{cases}\tag{2}$$

Corresponding to the two-point right focal conditions Green's function $g_{AG2}(t,s)$ of the boundary value problem

$$z^{(n)} = 0,\ z^{(i)}(a_1) = 0,\ 0 \le i \le \alpha,\ z^{(i)}(a_2) = 0, \alpha+1 \le i \le n-1$$

is given by (see [1]):

$$g_{AG2}(t,s) = \frac{1}{(n-1)!} \begin{cases} \sum\limits_{i=0}^{\alpha} \binom{n-1}{i}(t-a_1)^i(a_1-s)^{n-i-1}, & a \le s \le t; \\ -\sum\limits_{i=\alpha+1}^{n-1} \binom{n-1}{i}(t-a_1)^i(a_1-s)^{n-i-1}, & t \le s \le b. \end{cases} \tag{3}$$

Further, for $a_1 \le s,\ t \le a_2$ the following inequalities hold

$$(-1)^{n-\alpha-1}\frac{\partial^i g_{AG2}(t,s)}{\partial t^i} \ge 0, \quad 0 \le i \le \alpha$$

$$(-1)^{n-i}\frac{\partial^i g_{AG2}(t,s)}{\partial t^i} \ge 0, \quad \alpha+1 \le i \le n-1.$$

Theorem 1.3. *Let $f \in C^n[a,b]$, and let P_{AG} be its Abel-Gontscharoff interpolating polynomial. Then*

$$\begin{aligned} f(t) &= P_{AG}(t) + e_{AG}(t) \\ &= \sum_{i=0}^{n-1} T_i(t) f^{(i)}(a_{i+1}) + \int_a^b g_{AG}(t,s) f^{(n)}(s)ds \end{aligned} \tag{4}$$

where T_i is defined by (1) and $g_{AG}(t,s)$ is defined by (2).

Theorem 1.4. *Let $f \in C^n[a,b]$, and let P_{AG2} be its two-point right focal Abel-Gontscharoff interpolating polynomial. Then*

$$\begin{aligned} f(t) &= P_{AG2}(t) + e_{AG2}(t) \\ &= \sum_{i=0}^{\alpha} \frac{(t-a_1)^i}{i!} f^{(i)}(a_1) + \sum_{j=0}^{n-\alpha-2}\left[\sum_{i=0}^{j} \frac{(t-a_1)^{\alpha+1+i}(a_1-a_2)^{j-i}}{(\alpha+1+i)!(j-i)!}\right] f^{(\alpha+1+j)}(a_2) \\ &\quad + \int_a^b g_{AG2}(t,s) f^{(n)}(s)ds \end{aligned} \tag{5}$$

where $g_{AG2}(t,s)$ is defined by (3).

Finally, we recall the well-known Steffensen inequality which reads, [18]:

Theorem 1.5. *Suppose that f is decreasing and g is integrable on $[a,b]$ with $0 \le g \le 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int_{b-\lambda}^{b} f(t)dt \le \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt. \tag{6}$$

The inequalities are reversed for f increasing.

Since its appearance in 1918 Steffensen's inequality has been the subject of investigation by many mathematicians. Various papers have been devoted to generalizations and refinements of Steffensen's inequality and its connection to other important inequalities. In the following theorem we recall weaker conditions on the function g obtained by Milovanović and Pečarić in [14].

Theorem 1.6. *Let f and g be integrable functions on $[a,b]$ such that f is decreasing and let $\lambda = \int_a^b g(t)dt$.*

a) *If*

$$\int_a^x g(t)dt \leq x-a \quad \text{and} \quad \int_x^b g(t)dt \geq 0 \quad \text{for every } x \in [a,b], \tag{7}$$

then the second inequality in (6) holds.

b) *If*

$$\int_x^b g(t)dt \leq b-x \quad \text{and} \quad \int_a^x g(t)dt \geq 0 \quad \text{for every } x \in [a,b], \tag{8}$$

then the first inequality in (6) holds.

Steffensen's inequality is important not only in the theory of inequalities but also in many applications such as statistics, functional equations, special functions, time scales etc. Some of these applications can be found in [2], [5], [6], [9], [10], [11], and [15].

The aim of this paper is to obtain new generalizations of Steffensen's inequality using Abel-Gontscharoff interpolating polynomial. Our new generalizations involve $n-$ convex function f instead of restricting it to be a decreasing function as in Steffensen's inequality. As a special case of these generalizations (for $n=1$) we obtain weaker conditions for Steffensen's inequality given in Theorem 1.6. These new generalizations in the end enable us to construct linear functionals whose action on particularly chosen families of functions give us exponentially convex functions. However, there is lack of examples of this functions since there are no operative criteria to recognize this type of functions, so our constructed examples are valuable addition to the theory of that functions. We also get additional results about Ostrowski type inequalities.

2. Difference of integrals on two intervals

If $[a,b] \cap [c,d] \neq \emptyset$ we have four possible cases for two intervals $[a,b]$ and $[c,d]$. We observe cases $[c,d] \subset [a,b]$ and $[a,b] \cap [c,d] = [c,b]$ since other two cases are obtained by changing $a \leftrightarrow c$ and $b \leftrightarrow d$.

In this paper by $T_{w,n}^{[a,b]}$ we denote

$$T_{w,n}^{[a,b]} = \sum_{i=0}^{n-1} f^{(i)}(a_{i+1}) \int_a^b w(t) T_i(t)dt$$

where T_i is defined by (1).

Theorem 2.1. *Let $f : [a, b] \cup [c, d] \to \mathbb{R}$ be of class C^n on $[a, b] \cup [c, d]$ for some $n \geq 1$. Let $w : [a, b] \to \mathbb{R}$ and $u : [c, d] \to \mathbb{R}$. Then if $[a, b] \cap [c, d] \neq \emptyset$ we have*

$$\int_a^b w(t) f(t) \, dt - \int_c^d u(t) f(t) \, dt - T_{w,n}^{[a,b]} + T_{u,n}^{[c,d]} = \int_a^{\max\{b,d\}} K_n(s) f^{(n)}(s) \, ds, \tag{9}$$

where in case $[c, d] \subseteq [a, b]$,

$$K_n(s) = \begin{cases} \int_a^b w(t) g_{AG}(t, s) \, dt, & s \in [a, c], \\ \int_a^b w(t) g_{AG}(t, s) \, dt - \int_c^d u(t) g_{AG}(t, s) \, dt, & s \in \langle c, d], \\ \int_a^b w(t) g_{AG}(t, s) \, dt, & s \in \langle d, b], \end{cases} \tag{10}$$

and in case $[a, b] \cap [c, d] = [c, b]$,

$$K_n(s) = \begin{cases} \int_a^b w(t) g_{AG}(t, s) \, dt & s \in [a, c], \\ \int_a^b w(t) g_{AG}(t, s) \, dt - \int_c^d u(t) g_{AG}(t, s) \, dt, & s \in \langle c, b], \\ -\int_c^d u(t) g_{AG}(t, s) \, dt, & s \in \langle b, d]. \end{cases} \tag{11}$$

Proof. Multiplying identity (4) by $w(t)$, then integrating from a to b and using Fubini's theorem we obtain

$$\int_a^b w(t) f(t) dt = \sum_{i=0}^{n-1} f^{(i)}(a_{i+1}) \int_a^b w(t) T_i(t) dt + \int_a^b f^{(n)}(s) \left(\int_a^b w(t) g_{AG}(t, s) dt \right) ds. \tag{12}$$

Furthermore, multiplying identity (4) by $u(t)$, then integrating from c to d and using Fubini's theorem we obtain similar identity to identity (12). Now subtracting these two identities we obtain (9). □

Remark 2.2. Using two-point right focal Abel-Gontscharoff polynomial, i.e. using (5), inequality (9) becomes

$$\int_a^b w(t) f(t) \, dt - \int_c^d u(t) f(t) \, dt - Q_{w,n}^{[a,b]} + Q_{u,n}^{[c,d]} = \int_a^{\max\{b,d\}} K_n(s) f^{(n)}(s) \, ds,$$

where $g_{AG}(t, s)$ is replaced by $g_{AG2}(t, s)$ in definition of $K_n(s)$ and by $Q_{w,n}^{[a,b]}$ we denote

$$Q_{w,n}^{[a,b]} = \sum_{i=0}^{\alpha} \frac{f^{(i)}(a_1)}{i!} \int_a^b w(t)(t - a_1)^i dt$$
$$+ \sum_{j=0}^{n-\alpha-2} f^{(\alpha+1+j)}(a_2) \left[\sum_{i=0}^{j} \frac{(a_1 - a_2)^{j-i}}{(\alpha + 1 + j)!(j - i)!} \int_a^b w(t)(t - a_1)^{\alpha+1+i} dt \right].$$

Theorem 2.3. *Let* $f : [a, b] \cup [c, d] \to \mathbb{R}$ *be* $n-$*convex on* $[a, b] \cup [c, d]$ *and let* $w : [a, b] \to \mathbb{R}$ *and* $u : [c, d] \to \mathbb{R}$*. Then if* $[a, b] \cap [c, d] \neq \emptyset$ *and*

$$K_n(s) \geq 0, \tag{13}$$

we have

$$\int_a^b w(t) f(t)\, dt - T_{w,n}^{[a,b]} \geq \int_c^d u(t) f(t)\, dt - T_{u,n}^{[c,d]} \tag{14}$$

where in case $[c, d] \subseteq [a, b]$, $K_n(s)$ *is defined by* (10) *and in case* $[a, b] \cap [c, d] = [c, b]$, $K_n(s)$ *is defined by* (11).

Proof. Since f is n-convex, without loss of generality we can assume that f is $n-$times differentiable and $f^{(n)} \geq 0$ see [17, p. 16 and p. 293]. Now we can apply Theorem 2.1 to obtain (14). □

Remark 2.4. As in Remark 2.2, using two-point right focal Abel-Gontscharoff polynomial, inequality (14) becomes

$$\int_a^b w(t) f(t)\, dt - Q_{w,n}^{[a,b]} \geq \int_c^d u(t) f(t)\, dt - Q_{u,n}^{[c,d]}.$$

3. Generalization of Steffensen's inequality by Abel-Gontscharoff polynomial

For a special choice of weights and intervals in results from previous section we obtain generalizations of Steffensen's inequality.

Theorem 3.1. *Let* $f : [a, b] \cup [a, a+\lambda] \to \mathbb{R}$ *be* $n-$*convex on* $[a, b] \cup [a, a+\lambda]$ *for some* $n \geq 1$ *and let* $w : [a, b] \to \mathbb{R}$*. Then if*

$$K_n(s) \geq 0, \tag{15}$$

we have

$$\int_a^b w(t) f(t)\, dt - T_{w,n}^{[a,b]} \geq \int_a^{a+\lambda} f(t)\, dt - T_{1,n}^{[a,a+\lambda]} \tag{16}$$

where in case $a \leq a + \lambda \leq b$,

$$K_n(s) = \begin{cases} \int_a^b w(t) g_{AG}(t,s)\, dt - \int_a^{a+\lambda} g_{AG}(t,s)\, dt, & s \in [a, a+\lambda], \\ \int_a^b w(t) g_{AG}(t,s)\, dt, & s \in \langle a+\lambda, b], \end{cases} \tag{17}$$

and in case $a \leq b \leq a + \lambda$,

$$K_n(s) = \begin{cases} \int_a^b w(t) g_{AG}(t,s)\, dt - \int_a^{a+\lambda} g_{AG}(t,s)\, dt, & s \in [a, b], \\ -\int_a^{a+\lambda} g_{AG}(t,s)\, dt, & s \in \langle b, a+\lambda]. \end{cases} \tag{18}$$

Proof. We take $c = a$, $d = a + \lambda$ and $u(t) = 1$ in Theorem 2.3. □

Remark 3.2. For $n = 1$ and $\lambda \leq b - a$, $K_1(s)$ becomes

$$K_1(s) = \begin{cases} -\int_a^s w(t) dt + s - a, & s \in [a, a+\lambda], \\ \int_s^b w(t) dt, & s \in \langle a+\lambda, b]. \end{cases}$$

So, if

$$\int_a^s w(t)dt \leq s-a \qquad \text{for } a \leq s \leq a+\lambda \tag{19}$$

and

$$\int_s^b w(t)dt \geq 0 \qquad \text{for } a+\lambda < s \leq b \tag{20}$$

and f is increasing, from Theorem 3.1 we have

$$\int_a^b w(t)f(t)dt - f(a+\lambda)\int_a^b w(t)dt \geq \int_a^{a+\lambda} f(t)dt - \lambda f(a+\lambda).$$

Furthermore, for $\lambda = \int_a^b w(t)dt$ we obtain the right-hand side of Steffensen's inequality for an increasing function f. In [14] Milovanović and Pečarić showed that conditions (19) and (20) are equivalent to condition (7), so for $n = 1$ Theorem 3.1 reduces to Theorem 1.6 a).

Theorem 3.3. *Let $f : [a,b] \cup [b-\lambda, b] \to \mathbb{R}$ be $n-$convex on $[a,b] \cup [b-\lambda, b]$ for some $n \geq 1$ and let $w : [a,b] \to \mathbb{R}$. Then if*

$$K_n(s) \geq 0, \tag{21}$$

we have

$$\int_{b-\lambda}^b f(t)\,dt - T_{1,n}^{[b-\lambda,b]} \geq \int_a^b w(t)\,f(t)\,dt - T_{w,n}^{[a,b]} \tag{22}$$

where in case $a \leq b-\lambda \leq b$,

$$K_n(s) = \begin{cases} -\int_a^b w(t)g_{AG}(t,s)\,dt, & s \in [a, b-\lambda], \\ \int_{b-\lambda}^b g_{AG}(t,s)\,dt - \int_a^b w(t)g_{AG}(t,s)\,dt, & s \in \langle b-\lambda, b], \end{cases} \tag{23}$$

and in case $b-\lambda \leq a \leq b$,

$$K_n(s) = \begin{cases} \int_{b-\lambda}^b g_{AG}(t,s)\,dt, & s \in [b-\lambda, a], \\ \int_{b-\lambda}^b g_{AG}(t,s)\,dt - \int_a^b w(t)g_{AG}(t,s)\,dt, & s \in \langle a, b]. \end{cases} \tag{24}$$

Proof. First we change $a \leftrightarrow c$, $b \leftrightarrow d$ and $w \leftrightarrow u$ in Theorem 2.3 and consider cases $[a,b] \subseteq [c,d]$ and $[a,b] \cap [c,d] = [c,b]$. Then we take $c = b-\lambda$, $d = b$ and $u(t) = 1$ to finish the proof. □

Remark 3.4. For $n = 1$ and $\lambda \leq b-a$, $K_1(s)$ becomes

$$K_1(s) = \begin{cases} \int_a^s w(t)dt, & s \in [a, b-\lambda], \\ b-s-\int_s^b w(t)dt, & s \in \langle b-\lambda, b]. \end{cases}$$

So, if

$$\int_a^s w(t)dt \geq 0 \qquad \text{for } a \leq s \leq b-\lambda \tag{25}$$

and

$$\int_s^b w(t)dt \leq b-s \qquad \text{for } b-\lambda < s \leq b \tag{26}$$

and f is increasing from Theorem 3.3 we have

$$\int_{b-\lambda}^{b} f(t)dt - \lambda f(b-\lambda) \geq \int_{a}^{b} w(t)f(t)dt - f(b-\lambda)\int_{a}^{b} w(t)dt.$$

Furthermore, for $\lambda = \int_a^b w(t)dt$ we obtain the left-hand side of Steffensen's inequality for an increasing function f. Similar as in [14] we can show that conditions (25) and (26) are equivalent to condition (8). Hence, for $n = 1$ Theorem 3.3 reduces to Theorem 1.6 b).

4. Estimation of the difference

In this section we give Ostrowski type inequalities related to results from previous sections.

Theorem 4.1. *Suppose that all assumptions of Theorem 2.1 hold. Assume (p, q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. Let $\left|f^{(n)}\right|^p : [a, b] \cup [c, d] \to \mathbb{R}$ be an R-integrable function for some $n \geq 1$. Then we have*

$$\begin{aligned}
&\left| \int_{a}^{b} w(t) f(t)\, dt - \int_{c}^{d} u(t)\, f(t)\, dt - T_{w,n}^{[a,b]} + T_{u,n}^{[c,d]} \right| \\
&\leq \left\| f^{(n)} \right\|_p \left(\int_{a}^{\max\{b,d\}} |K_n(s)|^q\, ds \right)^{\frac{1}{q}}.
\end{aligned} \tag{27}$$

The constant $\left(\int_a^{\max\{b,d\}} |K_n(s)|^q\, ds\right)^{1/q}$ in the inequality (27) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Using inequality (9) and applying Hölder's inequality we obtain

$$\begin{aligned}
&\left| \int_{a}^{b} w(t) f(t)\, dt - \int_{c}^{d} u(t)\, f(t)\, dt - T_{w,n}^{[a,b]} + T_{u,n}^{[c,d]} \right| \\
&= \left| \int_{a}^{\max\{b,d\}} K_n(s) f^{(n)}(s) ds \right| \leq \left\| f^{(n)} \right\|_p \left(\int_{a}^{\max\{b,d\}} |K_n(s)|^q\, ds \right)^{\frac{1}{q}}.
\end{aligned}$$

For the proof of the sharpness of the constant $\left(\int_a^{\max\{b,d\}} |K_n(s)|^q\, ds\right)^{\frac{1}{q}}$ we will find a function f for which the equality in (27) is obtained.
For $1 < p < \infty$ take f to be such that

$$f^{(n)}(s) = \operatorname{sgn} K_n(s) \left| K_n(s) \right|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $f^{(n)}(s) = \operatorname{sgn} K_n(s)$.
For $p = 1$ we will prove that

$$\left| \int_{a}^{\max\{b,d\}} K_n(s)\, f^{(n)}(s) ds \right| \leq \max_{s \in [a, \max\{b,d\}]} |K_n(s)| \left(\int_{a}^{\max\{b,d\}} \left| f^{(n)}(s) \right| ds \right) \tag{28}$$

is the best possible inequality. Suppose that $|K_n(s)|$ attains its maximum at $s_0 \in [a, \max\{b, d\}]$. First we assume that $K_n(s_0) > 0$. For ε small enough we define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} 0, & a \leq s \leq s_0, \\ \frac{1}{\varepsilon\, n!}(s - s_0)^n, & s_0 \leq s \leq s_0 + \varepsilon, \\ \frac{1}{n!}(s - s_0)^{n-1}, & s_0 + \varepsilon \leq s \leq \max\{b, d\}. \end{cases}$$

Then for ε small enough

$$\left| \int_a^{\max\{b,d\}} K_n(s) f^{(n)}(s) ds \right| = \left| \int_{s_0}^{s_0+\varepsilon} K_n(s) \frac{1}{\varepsilon} ds \right| = \frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} K_n(s) ds.$$

Now from inequality (28) we have

$$\frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} K_n(s) ds \leq K_n(s_0) \int_{s_0}^{s_0+\varepsilon} \frac{1}{\varepsilon} ds = K_n(s_0).$$

Since,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} K_n(s) ds = K_n(s_0)$$

the statement follows. In case $K_n(s_0) < 0$ we define

$$f_\varepsilon(s) = \begin{cases} \frac{1}{n!}(s - s_0 - \varepsilon)^{n-1},, & a \leq s \leq s_0, \\ -\frac{1}{\varepsilon\, n!}(s - s_0 - \varepsilon)^n, & s_0 \leq s \leq s_0 + \varepsilon, \\ 0, & s_0 + \varepsilon \leq s \leq \max\{b, d\}, \end{cases}$$

and the rest of the proof is the same as above. □

Theorem 4.2. *Suppose that all assumptions of Theorem 2.1 for $c = a$ and $d = a + \lambda$ hold. Assume (p, q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. Let $\left|f^{(n)}\right|^p : [a, b] \cup [a, a + \lambda] \to \mathbb{R}$ be an R-integrable function for some $n \geq 1$. Let $K_n(s)$ be defined by (17) in case $a \leq a + \lambda \leq b$ and by (18) in case $a \leq b \leq a + \lambda$. Then we have*

$$\begin{aligned} &\left| \int_a^b w(t) f(t)\, dt - \int_a^{a+\lambda} f(t)\, dt - T_{w,n}^{[a,b]} + T_{1,n}^{[a,a+\lambda]} \right| \\ &\leq \left\| f^{(n)} \right\|_p \left(\int_a^{\max\{b,a+\lambda\}} |K_n(s)|^q\, ds \right)^{\frac{1}{q}}. \end{aligned} \tag{29}$$

The constant $\left(\int_a^{\max\{b,a+\lambda\}} |K_n(s)|^q\, ds \right)^{1/q}$ in the inequality (29) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. We take $c = a$, $d = a + \lambda$ and $u(t) = 1$ in Theorem 4.1. □

Theorem 4.3. *Suppose that all assumptions of Theorem 2.1 for $c = b - \lambda$ and $d = b$ hold. Assume (p, q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. Let $\left|f^{(n)}\right|^p : [a, b] \cup [b - \lambda, b] \to \mathbb{R}$ be an R-integrable function for* *some $n \geq 1$. Let $K_n(s)$ be defined by (23) in case $a \leq b-\lambda \leq b$ and by (24) in case $b-\lambda \leq a \leq b$. Then we have*

$$\begin{aligned} &\left| \int_{b-\lambda}^{b} f(t)\, dt - \int_{a}^{b} w(t) f(t)\, dt - T_{1,n}^{[b-\lambda,b]} + T_{w,n}^{[a,b]} \right| \\ &\leq \left\| f^{(n)} \right\|_p \left(\int_{\min\{a,b-\lambda\}}^{b} |K_n(s)|^q\, ds \right)^{\frac{1}{q}}. \end{aligned} \tag{30}$$

The constant $\left(\int_{\min\{a,b-\lambda\}}^{b} |K_n(s)|^q\, ds\right)^{1/q}$ in the inequality (30) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. First we change $a \leftrightarrow c$, $b \leftrightarrow d$ and $w \leftrightarrow u$ in Theorem 2.1 and then we take $c = b-\lambda$, $d = b$ and $u(t) = 1$. The rest of the proof is similar to the proof of Theorem 4.1. ☐

5. $n-$ EXPONETIAL CONVEXITY AND EXPONENTIAL CONVEXITY

We begin this section by giving some definitions and notions which are used frequently in the results. For more details see e.g. [4], [13] and [16].

Definition 5.1. A function $\psi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j\, \psi\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

hold for all choices $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and all choices $x_1, \ldots, x_n \in I$. A function $\psi : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 5.2. It is clear from the definition that 1-exponentially convex function in the Jensen sense is in fact a nonnegative function. Also, n-exponentially convex function in the Jensen sense is k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

Definition 5.3. A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 5.4. It is known that $\psi : I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2 \psi(y) \geq 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

Proposition 5.5. *If f is a convex function on I and if $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(y_2)-f(y_1)}{y_2-y_1}.$$

If the function f is concave, the inequality is reversed.

Definition 5.6. Let f be a real-valued function defined on the segment $[a,b]$. The *$n-$th order divided difference* of the function f at distinct points $x_0,...,x_n \in [a,b]$, is defined recursively (see [3], [17]) by

$$f[x_i] = f(x_i),\ (i = 0,\ldots,n)$$

and

$$f[x_0,\ldots,x_n] = \frac{f[x_1,\ldots,x_n]-f[x_0,\ldots,x_{n-1}]}{x_n-x_0}.$$

The value $f[x_0,\ldots,x_n]$ is independent of the order of the points $x_0,\ldots,x_n$. Previous definition may be extended to include the case in which some or all of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x,\ldots,x}_{j-times}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \tag{31}$$

Motivated by inequalities (14),(16) and (22), under assumptions of Theorems 2.3, 3.1 and 3.3 we define following linear functionals:

$$L_1(f) = \int_a^b w(t)\, f(t)\, dt - \int_c^d u(t)f(t)\, dt - T_{w,n}^{[a,b]} + T_{u,n}^{[c,d]} \tag{32}$$

$$L_2(f) = \int_a^b w(t)\, f(t)\, dt - \int_a^{a+\lambda} f(t)\, dt - T_{w,n}^{[a,b]} + T_{1,n}^{[a,a+\lambda]} \tag{33}$$

$$L_3(f) = \int_{b-\lambda}^b f(t)\, dt - \int_a^b w(t)\, f(t)\, dt - T_{1,n}^{[b-\lambda,b]} + T_{w,n}^{[a,b]} \tag{34}$$

Also, we define $I_1 = [a,b]\cup[c,d]$, $I_2 = [a,b]\cup[a,a+\lambda]$ and $I_3 = [a,b]\cup[b-\lambda,b]$.

Remark 5.7. Under the assumptions of Theorems 2.3, 3.1 and 3.3 respectively it holds $L_i(f) \geq 0$, $i = 1,2,3$ for all $n-$ convex functions f.

Now we will show how to generate means for our list of linear functionals.

Theorem 5.8. *Let $f : I_i \to \mathbb{R}$ $(i = 1,2,3)$ be such that $f \in C^n(I_i)$. If inequalities in (13) $(i = 1)$, (15) $(i = 2)$ and (21) $(i = 3)$ hold, then there exist $\xi_i \in I_i$ such that*

$$L_i(f) = f^{(n)}(\xi_i)L_i(\varphi), \quad i = 1,2,3 \tag{35}$$

where $\varphi(x) = \frac{x^n}{n!}$.

Proof. Let us denote $m = \min f^{(n)}(x)$ and $M = \max f^{(n)}(x)$. For a given function $f \in C^n(I_i)$ we define functions $F_1, F_2 : I_i \to \mathbb{R}$ with

$$F_1(x) = \frac{Mx^n}{n!} - f(x) \quad \text{and} \quad F_2(x) = f(x) - \frac{mx^n}{n!}.$$

Now $F_1^{(n)}(x) = M - f^{(n)}(x) \geq 0, x \in I_i$, so we conclude $L_i(F_1) \geq 0$ and then $L_i(f) \leq M \cdot L_i(\varphi)$. Similarly, from $F_2^{(n)}(x) = f^{(n)}(x) - m \geq 0$ we conclude $m \cdot L_i(\varphi) \leq L_i(f)$.
If $L_i(\varphi) = 0$, (35) holds for all $\xi_i \in I_i$. Otherwise, $m \leq \frac{L_i(f)}{L_i(\varphi)} \leq M$. Since $f^{(n)}(x)$ is continuous on I_i there exist $\xi_i \in I_i$ such that (35) holds and the proof is complete. □

Theorem 5.9. *Let $f, g : I_i \to \mathbb{R}$ $(i = 1, 2, 3)$ be such that $f, g \in C^n(I_i)$ and $g^{(n)}(x) \neq 0$ for every $x \in I_i$. If inequalities in (13) $(i = 1)$, (15) $(i = 2)$ and (21) $(i = 3)$ hold, then there exist $\xi_i \in I_i$ such that*

$$\frac{L_i(f)}{L_i(g)} = \frac{f^{(n)}(\xi_i)}{g^{(n)}(\xi_i)}, \quad i = 1, 2, 3. \tag{36}$$

Proof. We define functions $\phi_i(x) = f(x)L_i(g) - g(x)L_i(f),\ i = 1, 2, 3$. According to Theorem 5.8 there exists $\xi_i \in I_i$ such that $L_i(\phi_i) = \phi_i^{(n)}(\xi_i)L_i(\varphi)$. Since $L_i(\phi_i) = 0$ it follows $f^{(n)}(\xi_i)L_i(g) - g^{(n)}(\xi_i)L_i(f) = 0$ and (36) is proved. □

We will use previously defined functionals to construct exponentially convex functions, a special type of convex functions that are invented by S. N. Bernstein over eighty years ago in [4]. An elegant method of constructing $n-$ exponentially convex and exponentially convex functions is given in [13]. We use this method to prove the $n-$exponential convexity for above defined functionals. In the sequel the notion log denotes the natural logarithm function.

Theorem 5.10. *Let $\Omega = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $I_i,\ i = 1, 2, 3$ in $\mathbb{R}$ such that the function $p \mapsto f_p[x_0, \ldots, x_m]$ is $n-$exponentially convex in the Jensen sense on J for every $(m + 1)$ mutually different points $x_0, \ldots, x_m \in I_i,\ i = 1, 2, 3$. Let $L_i,\ i = 1, 2, 3$ be linear functionals defined by (32) – (34). Then $p \mapsto L_i(f_p)$ is $n-$exponentially convex function in the Jensen sense on J.*
If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is $n-$exponentially convex on J.

Proof. For $\xi_j \in \mathbb{R},\ j = 1, \ldots, n$ and $p_j \in J,\ j = 1, \ldots, n$, we define the function

$$g(x) = \sum_{j,k=1}^{n} \xi_j \xi_k f_{\frac{p_j + p_k}{2}}(x).$$

Using the assumption that the function $p \mapsto f_p[x_0, \ldots, x_m]$ is n-exponentially convex in the Jensen sense, we have

$$g[x_0, \ldots, x_m] = \sum_{j,k=1}^{n} \xi_j \xi_k f_{\frac{p_j + p_k}{2}}[x_0, \ldots, x_m] \geq 0,$$

which in turn implies that g is a m-convex function on J, so $L_i(g) \geq 0,\ i = 1, 2, 3$. Hence

$$\sum_{j,k=1}^{n} \xi_j \xi_k L_i \left(f_{\frac{p_j + p_k}{2}} \right) \geq 0.$$

We conclude that the function $p \mapsto L_i(f_p)$ is n-exponentially convex on J in the Jensen sense.

If the function $p \mapsto L_i(f_p)$ is also continuous on J, then $p \mapsto L_i(f_p)$ is n-exponentially convex by definition. □

The following corollaries are immediate consequences of the above theorem:

Corollary 5.11. *Let $\Omega = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1, 2, 3$ in $\mathbb{R}$, such that the function $p \mapsto f_p[x_0, \ldots, x_m]$ is exponentially convex in the Jensen sense on J for every $(m + 1)$ mutually different points $x_0, \ldots, x_m \in I_i$. Let L_i, $i = 1, 2, 3$, be linear functionals defined by (32)-(34). Then $p \mapsto L_i(f_p)$ is an exponentially convex function in the Jensen sense on J. If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is exponentially convex on J.*

Corollary 5.12. *Let $\Omega = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1, 2, 3$ in $\mathbb{R}$, such that the function $p \mapsto f_p[x_0, \ldots, x_m]$ is 2-exponentially convex in the Jensen sense on J for every $(m + 1)$ mutually different points $x_0, \ldots, x_m \in I_i$. Let L_i, $i = 1, 2, 3$ be linear functionals defined by (32)-(34). Then the following statements hold:*

(i) *If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is 2-exponentially convex function on J. If $p \mapsto L_i(f_p)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:*

$$[L_i(f_s)]^{t-r} \leq [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r}$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) *If the function $p \mapsto L_i(f_p)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$, such that $p \leq u$ and $q \leq v$, we have*

$$\mu_{p,q}(L_i, \Omega) \leq \mu_{u,v}(L_i, \Omega), \tag{37}$$

where

$$\mu_{p,q}(L_i, \Omega) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)} \right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp} L_i(f_p)}{L_i(f_p)} \right), & p = q, \end{cases} \tag{38}$$

for $f_p, f_q \in \Omega$.

Proof. (i) This is an immediate consequence of Theorem 5.10 and Remark 5.4.

(ii) Since $p \mapsto L_i(f_p)$ is positive and continuous, by (i) we have that $p \mapsto L_i(f_p)$ is log-convex on J, that is, the function $p \mapsto \log L_i(f_p)$ is convex on J.

Applying Proposition 5.5 we get

$$\frac{\log L_i(f_p) - \log L_i(f_q)}{p-q} \leq \frac{\log L_i(f_u) - \log L_i(f_v)}{u-v}, \tag{39}$$

for $p \leq u, q \leq v, p \neq q, u \neq v$. Hence, we conclude that

$$\mu_{p,q}(L_i, \Omega) \leq \mu_{u,v}(L_i, \Omega).$$

Cases $p = q$ and $u = v$ follow from (39) as limit cases.

□

Remark 5.13. Note that the results from the above theorem and corollaries still hold when two of the points $x_0, \ldots, x_m \in I_i,\ i = 1, 2, 3$ coincide, say $x_1 = x_0$, for a family of differentiable functions f_p such that the function $p \mapsto f_p[x_0, \ldots, x_m]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $m+1$ points coincide for a family of m differentiable functions with the same property. The proofs use (31) and suitable characterization of convexity.

6. Applications to Stolarsky type means

In this section, we present several families of functions which fullfill the conditions of Theorem 5.10, Corollary 5.11, Corollary 5.12 and Remark 5.13. This enables us to construct a large families of functions which are exponentially convex. For a discussion related to this problem see [8].

Example 6.1. Consider a family of functions

$$\Omega_1 = \{f_p : \mathbb{R} \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$f_p(x) = \begin{cases} \frac{e^{px}}{p^n}, & p \neq 0, \\ \frac{x^n}{n!}, & p = 0. \end{cases}$$

Here, $\frac{d^n f_p}{dx^n}(x) = e^{px} > 0$ which shows that f_p is n-convex on $\mathbb{R}$ for every $p \in \mathbb{R}$ and $p \mapsto \frac{d^n f_p}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 5.10 we also have that $p \mapsto f_p[x_0, \ldots, x_m]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 5.11 we conclude that $p \mapsto L_i(f_p), i = 1, 2, 3$, are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping $p \mapsto f_p$ is not continuous for $p = 0$), so it is exponentially convex. For this family of functions, $\mu_{p,q}(L_i, \Omega_1),\ i = 1, 2, 3$, from (38), becomes

$$\mu_{p,q}(L_i, \Omega_1) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{L_i(id \cdot f_p)}{L_i(f_p)} - \frac{n}{p}\right), & p = q \neq 0, \\ \exp\left(\frac{1}{n+1}\frac{L_i(id \cdot f_0)}{L_i(f_0)}\right), & p = q = 0, \end{cases}$$

where id is the identity function. Also, by Corollary 5.12 it is monotonic function in parameters p and q.

We observe here that $\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{p-q}}(\log x) = x$ so using Theorem 5.9 it follows that:

$$M_{p,q}(L_i, \Omega_1) = \log \mu_{p,q}(L_i, \Omega_1),\ i = 1, 2, 3$$

satisfies

$$\min\{a, c, b-\lambda\} \le M_{p,q}(L_i, \Omega_1) \le \max\{b, d, a+\lambda\},\ i = 1, 2, 3.$$

So, $M_{p,q}(L_i, \Omega_1)$ is a monotonic mean.

Example 6.2. Consider a family of functions

$$\Omega_2 = \{g_p : (0, \infty) \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$g_p(x) = \begin{cases} \frac{x^p}{p(p-1)\cdots(p-n+1)}, & p \notin \{0, 1, \ldots, n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & p = j \in \{0, 1, \ldots, n-1\}. \end{cases}$$

Here, $\frac{d^n g_p}{dx^n}(x) = x^{p-n} > 0$ which shows that g_p is $n-$convex for $x > 0$ and $p \mapsto \frac{d^n g_p}{dx^n}(x)$ is exponentially convex by definition. Arguing as in Example 6.1 we get that the mappings $p \mapsto L_i(g_p), i = 1, 2, 3$ are exponentially convex. For this family of functions $\mu_{p,q}(L_i, \Omega_2),\ i = 1, 2, 3$, from (38), is now equal to

$$\mu_{p,q}(L_i, \Omega_2) = \begin{cases} \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}}, & p \ne q, \\ \exp\left((-1)^{n-1}(n-1)!\frac{L_i(g_0 g_p)}{L_i(g_p)} + \sum_{k=0}^{n-1} \frac{1}{k-p}\right), & p = q \notin \{0, 1, \ldots, n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\frac{L_i(g_0 g_p)}{2L_i(g_p)} + \sum_{\substack{k=0 \\ k \ne p}}^{n-1} \frac{1}{k-p}\right), & p = q \in \{0, 1, \ldots, n-1\}. \end{cases}$$

Again, using Theorem 5.9 we conclude that

$$\min\{a, b-\lambda, c\} \le \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}} \le \max\{a+\lambda, b, d\},\ i = 1, 2, 3.$$

So, $\mu_{p,q}(L_i, \Omega_2), i = 1, 2, 3$ is a mean.

Example 6.3. Consider a family of functions

$$\Omega_3 = \{\phi_p : (0, \infty) \to \mathbb{R} : p \in (0, \infty)\}$$

defined by

$$\phi_p(x) = \begin{cases} \frac{p^{-x}}{(-\log p)^n}, & p \ne 1 \\ \frac{x^n}{n!}, & p = 1. \end{cases}$$

Since $\frac{d^n \phi_p}{dx^n}(x) = p^{-x}$ is the Laplace transform of a non-negative function (see [19]) it is exponentially convex. Obviously ϕ_p are n-convex functions for every $p > 0$.

For this family of functions, $\mu_{p,q}(L_i, \Omega_3), i = 1, 2, 3$ from (38) is equal to

$$\mu_{p,q}(L_i, \Omega_3) = \begin{cases} \left(\frac{L_i(\phi_p)}{L_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(-\frac{L_i(id \cdot \phi_p)}{p\ L_i(\phi_p)} - \frac{n}{p \log p}\right), & p = q \neq 1, \\ \exp\left(-\frac{1}{n+1}\frac{L_i(id \cdot \phi_1)}{L_i(\phi_1)}\right), & p = q = 1, \end{cases}$$

where id is the identity function. This is a monotone function in parameters p and q by (37). Using Theorem 5.9 it follows that

$$M_{p,q}(L_i, \Omega_3) = -L(p, q) \log \mu_{p,q}(L_i, \Omega_3),\ \ i = 1, 2, 3$$

satisfies

$$\min\{a, b - \lambda, c\} \leq M_{p,q}(L_i, \Omega_3) \leq \max\{a + \lambda, b, d\}.$$

So $M_{p,q}(L_i, \Omega_3)$ is a monotonic mean. $L(p, q)$ is a logarithmic mean defined by

$$L(p, q) = \begin{cases} \frac{p-q}{\log p - \log q}, & p \neq q \\ p, & p = q. \end{cases}$$

Example 6.4. Consider a family of functions

$$\Omega_4 = \{\psi_p : (0, \infty) \to \mathbb{R} : p \in (0, \infty)\}$$

defined by

$$\psi_p(x) = \frac{e^{-x\sqrt{p}}}{(-\sqrt{p})^n}.$$

Since $\frac{d^n \psi_p}{dx^n}(x) = e^{-x\sqrt{p}}$ is the Laplace transform of a non-negative function (see [19]) it is exponentially convex. Obviously ψ_p are n-convex functions for every $p > 0$. For this family of functions, $\mu_{p,q}(L_i, \Omega_4), i = 1, 2, 3$ from (38) is equal to

$$\mu_{p,q}(L_i, \Omega_4) = \begin{cases} \left(\frac{L_i(\psi_p)}{L_i(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(-\frac{L_i(id \cdot \psi_p)}{2\sqrt{p} L_i(\psi_p)} - \frac{n}{2p}\right), & p = q, \end{cases}$$

where id is the identity function. This is monotone function in parameters p and q by (37). Using Theorem 5.9 it follows that

$$M_{p,q}(L_i, \Omega_4) = -(\sqrt{p} + \sqrt{q}) \log \mu_{p,q}(L_i, \Omega_4),\ \ i = 1, 2, 3$$

satisfies $\min\{a, b - \lambda, c\} \leq M_{p,q}(L_i, \Omega_4) \leq \max\{a + \lambda, b, d\}$, so $M_{p,q}(L_i, \Omega_4)$ is a monotonic mean.

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