HERMITE-HADAMARD TYPE INEQUALITIES FOR MAPPINGS WHOSE DERIVATIVES ARE $s$–CONVEX IN THE SECOND SENSE VIA FRACTIONAL INTEGRALS

ERHAN SET$^1$, M. EMİN ÖZDEMİR$^2$, M. ZEKİ SARİKAYA$^3$ AND FİLİZ KARAKOÇ$^4$

Communicated by S. Hejazian

ABSTRACT. In this paper we establish Hermite-Hadamard type inequalities for mappings whose derivatives are $s$–convex in the second sense and concave.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2} \tag{1.1}$$

is known that the Hermite-Hadamard inequality for convex function. Both inequalities hold in the reserved direction if $f$ is concave. We note that Hadamard’s inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality. Hadamard’s inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found; see, for example see ( [1]- [21]).

Definition 1.1. ( [18]) A function $f : [0, \infty) \to \mathbb{R}$ is said to be $s$–convex in the second sense if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$
for all $x, y \in [0, \infty), \lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$. This class of $s$-convex functions is usually denoted by $K_2^s$.

In ([15]) Dragomir and Fitzpatrick proved a variant of Hadamard’s inequality which holds for $s$-convex functions in the second sense:

**Theorem 1.2.** Suppose that $f : [0, \infty) \to [0, \infty)$ is an $s$-convex function in the second sense, where $s \in (0, 1)$ and let $a, b \in [0, \infty), a < b$. If $f' \in L^1([a, b])$, then the following inequalities hold:

$$2^{s-1} f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)\,dx \leq \frac{f(a) + f(b)}{s + 1} \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2)

The following results are proved by M.I.Bhatti et al. (see [8]).

**Theorem 1.3.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I^\circ$ such that $|f''|$ is convex function on $I$. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha}} \left[ J_a^\alpha f(b) + J_a^\alpha f(a) \right] \right| \leq \frac{\alpha(b - a)^2}{2(\alpha + 1)(\alpha + 2)} \left[ \frac{|f''(a)| + |f''(b)|}{2} \right] \tag{1.3}$$

where $\beta$ is Euler Beta function.

**Theorem 1.4.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I^\circ$. Assume that $p \in \mathbb{R}, p > 1$ such that $|f''|^{\frac{1}{p-1}}$ is convex function on $I$. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha}} \left[ J_a^\alpha f(b) + J_a^\alpha f(a) \right] \right| \leq \frac{(b - a)^2}{\alpha + 1} \beta^\frac{1}{p} (p + 1, \alpha p + 1) \left( \frac{|f''(a)|^q + |f''(b)|^q}{2} \right)^\frac{1}{q} \tag{1.4}$$

where $\beta$ is Euler Beta function.

**Theorem 1.5.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I^\circ$. Assume that $q \geq 1$ such that $|f''|^q$ is convex function on $I$. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha}} \left[ J_a^\alpha f(b) + J_a^\alpha f(a) \right] \right| \leq \frac{\alpha(b - a)^2}{4(\alpha + 1)(\alpha + 2)} \left[ \left( \frac{2\alpha + 4}{3\alpha + 9} |f''(a)|^q + \frac{2\alpha + 5}{3\alpha + 9} |f''(b)|^q \right)^\frac{1}{q} \right]. \tag{1.5}$$
Theorem 1.6. Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a twice differentiable function on \( I^\circ \). Assume that \( p \in \mathbb{R}, p > 1 \) with \( q = \frac{p}{p-1} \) such that \( |f''|^q \) is concave function on \( I \). Suppose that \( a, b \in I^\circ \) with \( a < b \) and \( f'' \in L[a, b] \), then the following inequality for fractional integrals holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma (\alpha + 1)}{2(b - a)^\alpha} \left[ J_α^a f(b) + J_α^b f(a) \right] \right| \leq \frac{(b - a)^2}{2(\alpha + 1)} \beta \frac{1}{p} (p + 1, \alpha p + 1) \left| f'' \left( \frac{a + b}{2} \right) \right|
\]

where \( \beta \) is Euler Beta function.

We will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further this paper.

Definition 1.7. Let \( f \in L[a, b] \). The Reimann-Liouville integrals \( J_α^a f(x) \) and \( J_α^b f(x) \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J_α^a f(x) = \frac{1}{\Gamma (\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a
\]

and

\[
J_α^b f(x) = \frac{1}{\Gamma (\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b
\]

respectively, where \( \Gamma (\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du \) is the Gamma function and \( J_α^a f(x) = J_α^0 f(x) = f(x) \).

In the case of \( \alpha = 1 \) the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities, see [3]-[25].

In this paper, we establish fractional integral inequalities of Hermite-Hadamard type for mappings whose derivatives are \( s \)-convex and concave.

2. Main results

In order to prove our main theorems we need the following lemma (see [8]).

Lemma 2.1. Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a twice differentiable function on \( I^\circ \), the interior of \( I \). Assume that \( a, b \in I^\circ \) with \( a < b \) and \( f'' \in L[a, b] \), then the following identity for fractional integral with \( \alpha > 0 \) holds:

\[
\frac{f(a) + f(b)}{2} - \frac{\Gamma (\alpha + 1)}{2(b - a)^\alpha} \left[ J_α^a f(b) + J_α^b f(a) \right] = \frac{(b - a)^2}{2(\alpha + 1)} \int_0^1 t (1 - t^\alpha) \left[ f'' (ta + (1 - t) b) + f'' ((1 - t) a + tb) \right] dt
\]

where \( \Gamma (\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du \).
Theorem 2.2. Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a twice differentiable function on \( I^\circ \) and let \( a,b \in I^\circ \) with \( a < b \) and \( f'' \in L[a,b] \). If \( |f''| \) is \( s \)--convex in the second sense on \( I \) for some fixed \( s \in (0,1) \), then the following inequality for fractional integrals with \( \alpha > 0 \) holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma (\alpha + 1)}{2 (b - a)^\alpha} [J_a^\alpha f(b) + J_b^\alpha f(a)] \right| 
\leq \frac{(b - a)^2}{2 (\alpha + 1)} \left[ \frac{\alpha}{(s + 2)(\alpha + s + 2)} + \beta(2, s + 1) - \beta(\alpha + 2, s + 1) \right] \times \left| |f''(a)| + |f''(b)| \right|
\]

where \( \beta \) is Euler Beta function.

Proof. From Lemma 2.1 since \( |f''| \) is \( s \)--convex in the second sense on \( I \), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma (\alpha + 1)}{2 (b - a)^\alpha} [J_a^\alpha f(b) + J_b^\alpha f(a)] \right| 
\leq \frac{(b - a)^2}{2 (\alpha + 1)} \int_0^1 t (1 - t^\alpha) |t^s |f''(ta + (1 - t)b)| + |f''((1 - t)a + tb)| dt 
\]

\[
\leq \frac{(b - a)^2}{2 (\alpha + 1)} \left[ \int_0^1 t (1 - t^\alpha) t^s |f''(a)| + (1 - t)^s |f''(b)| dt + \int_0^1 t (1 - t^\alpha) t^s |f''(a)| + t^s |f''(b)| dt \right] 
\]

\[
= \frac{(b - a)^2}{2 (\alpha + 1)} \left[ \frac{\alpha}{(s + 2)(\alpha + s + 2)} + \beta(2, s + 1) - \beta(\alpha + 2, s + 1) \right] \times \left| |f''(a)| + |f''(b)| \right|
\]

where we used the fact that

\[
\int_0^1 t^{s+1} (1 - t^\alpha) dt = \frac{\alpha}{(s + 2)(\alpha + s + 2)}
\]

and

\[
\int_0^1 t (1 - t^\alpha) (1 - t)^s dt = \beta(2, s + 1) - \beta(\alpha + 2, s + 1)
\]

which completes the proof. \( \square \)

Remark 2.3. In Theorem 2.2 if we choose \( s = 1 \) then (2.2) reduces the inequality (1.3) of Theorem 1.3.

Theorem 2.4. Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a twice differentiable function on \( I^\circ \). Suppose that \( a,b \in I^\circ \) with \( a < b \) and \( f'' \in L[a,b] \). If \( |f''|^q \) is \( s \)--convex in the second sense on \( I \) for some fixed \( s \in (0,1) \), \( p,q > 1 \), then the following inequality for fractional integrals with \( \alpha \in (0,1] \) holds:
where we used the fact that

$$\int f(a) + f(b) \frac{2}{2} - \frac{\Gamma (\alpha + 1)}{2(b-a)^\alpha} [J_{a}^{\alpha} f(b) + J_{b}^{\alpha} f(a)]$$

(2.3)

Proof. From Lemma 2.1, using the well known Hölder inequality and \(|f''|^q\) is s–convex in the second sense on I, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma (\alpha + 1)}{2(b-a)^\alpha} [J_{a}^{\alpha} f(b) + J_{b}^{\alpha} f(a)] \right|$$

\[
\leq \frac{(b-a)^2}{\alpha} \left( \int_0^1 t(1-t)^{\alpha} \right) \left[ \frac{|f''(ta + (1-t)b)|}{s+1} + \frac{|f''((1-t)a + tb)|}{s+1} \right] \]

\[
\leq \frac{(b-a)^2}{\alpha} \left( \int_0^1 t^p (1-t)^{\alpha_p} dt \right) \frac{1}{p} \]

\[
\times \left[ \left( \int_0^1 |f''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} + \left( \int_0^1 |f''((1-t)a + tb)|^q dt \right)^{\frac{1}{q}} \right]
\]

\[
= \frac{(b-a)^2}{\alpha} \left( \int_0^1 t^p (1-t)^{\alpha_p} dt \right) \frac{1}{p} \left[ \left( \int_0^1 |f''(ta + (1-t)b)|^q + \frac{|f''((1-t)a + tb)|^q}{s+1} \right)^{\frac{1}{q}} \right]
\]

\[
\leq \frac{(b-a)^2}{\alpha + 1} \beta^{\frac{1}{p}} (p + 1, \alpha p + 1) \left[ \frac{|f''(a)|^q + |f''(b)|^q}{s+1} \right]^{\frac{1}{q}}
\]

where we used the fact that

\[
\int_0^1 t^s dt = \int_0^1 (1-t)^s dt = \frac{1}{s+1}
\]

and

\[
\int_0^1 t^p (1-t)^{\alpha_p} dt \leq \int_0^1 t^p (1-t)^{\alpha p} dt = \beta (p + 1, \alpha p + 1)
\]

which completes the proof. □

Remark 2.5. In Theorem 2.4 if we choose \(s = 1\) then (2.3) reduces the inequality (1.4) of Theorem 1.4.
Theorem 2.6. Let \( f : I \subseteq [0, \infty) \rightarrow \mathbb{R} \) be a twice differentiable function on \( I^c \). Suppose that \( a, b \in I^c \) with \( a < b \) and \( f'' \in L[a, b] \). If \( |f''|^q \) is \( s \)-convex in the second sense on \( I \) for some fixed \( s \in (0, 1] \) and \( q \geq 1 \) then the following inequality for fractional integrals with \( \alpha > 0 \) holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J^\alpha_a f(b) + J^\alpha_b f(a)] \right| \\
\leq \frac{\alpha (b-a)^2}{4(\alpha+1)(\alpha+2)} \\
\times \left[ \left| f''(a) \right|^q \frac{2\alpha + 4}{(s+2)(\alpha + s + 2)} + \left| f''(b) \right|^q \frac{\beta(2,s+1) - \beta(\alpha+2,s+1)(2\alpha+4)}{\alpha} \right]^{\frac{1}{q}} \\
+ \left| f''(a) \right|^q \frac{\beta(2,s+1) - \beta(\alpha+2,s+1)(2\alpha+4)}{\alpha} + \left| f''(b) \right|^q \frac{2\alpha + 4}{(s+2)(\alpha + s + 2)} \right]^{\frac{1}{q}}.
\]

Proof. From Lemma 2.1, using power mean inequality and \( |f''|^q \) is \( s \)-convex in the second sense on \( I \) we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J^\alpha_a f(b) + J^\alpha_b f(a)] \right| \\
\leq \frac{(b-a)^2}{2(\alpha + 1)} \int_0^1 |t(1-t^\alpha)| \left[ |f''(ta + (1-t)b)| + |f''((1-t)a + tb)| \right] dt \\
\leq \frac{(b-a)^2}{2(\alpha + 1)} \left( \int_0^1 t(1-t^\alpha) dt \right)^{1 - \frac{1}{q}} \left[ \left( \int_0^1 t(1-t^\alpha) \left| f''(ta + (1-t)b) \right|^q dt \right)^\frac{1}{q} \\
+ \left( \int_0^1 t(1-t^\alpha) \left| f''((1-t)a + tb) \right|^q dt \right)^\frac{1}{q} \right] \\
\leq \frac{(b-a)^2}{2(\alpha + 1)} \left( \int_0^1 t(1-t^\alpha) dt \right)^{1 - \frac{1}{q}} \\
\times \left[ \left( \int_0^1 t^{s+1}(1-t^\alpha) \left| f''(a) \right|^q + t(1-t^\alpha)(1-t)^s \left| f''(b) \right|^q \right) dt \right]^{\frac{1}{q}} \\
+ \left( \int_0^1 t(1-t^\alpha)(1-t)^s \left| f''(a) \right|^q + t^{s+1}(1-t^\alpha) \left| f''(b) \right|^q \right)^{\frac{1}{q}} \right] \\
= \frac{(b-a)^2}{2(\alpha + 1)} \left( \frac{\alpha}{2(\alpha + 2)} \right)^{1 - \frac{1}{q}} \\
\times \left[ \left( |f''(a)|^q \frac{\alpha}{(s+2)(\alpha + s + 2)} + |f''(b)|^q \frac{\beta(2,s+1) - \beta(\alpha+2,s+1)}{\alpha} \right) \right]^{\frac{1}{q}} \\
+ \left( |f''(a)|^q \frac{\beta(2,s+1) - \beta(\alpha+2,s+1)}{\alpha (s+2)(\alpha + s + 2)} \right)^{\frac{1}{q}} \\
+ \left( |f''(b)|^q \frac{\beta(2,s+1) - \beta(\alpha+2,s+1)}{\alpha (s+2)(\alpha + s + 2)} \right)^{\frac{1}{q}}.\]
Proof. From Lemma 2.1 and using the Hölder inequality we have
\[
\frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)} \times \left[ (|f''(a)|^q \frac{(2\alpha+4)}{(s+2)(\alpha+s+2)} + |f''(b)|^q \frac{\beta(2,s+1) - \beta(\alpha,2,s+1)(2\alpha+4)}{\alpha} \right]^{\frac{1}{q}}
\]
+ \left[ (|f''(a)|^q \frac{\beta(2,s+1) - \beta(\alpha+2,s+1)(2\alpha+4)}{\alpha} + |f''(b)|^q \frac{2\alpha+4}{(s+2)(\alpha+s+2)} \right]^{\frac{1}{q}}
\]
where we used the fact that
\[
\int_0^1 t^{s+1} (1 - t^\alpha) dt = \frac{\alpha}{(s+2)(\alpha+s+2)}
\]
and
\[
\int_0^1 t (1 - t^\alpha) (1 - t)^s dt = \beta(2, s+1) - \beta(\alpha+2, s+1)
\]
which completes the proof. \(\square\)

Remark 2.7. In Theorem 2.6 if we choose \(s = 1\) then (2.4) reduces the inequality (1.5) of Theorem 1.5.

The following result holds for \(s\)–concavity.

Theorem 2.8. Let \(f : I \subseteq [0, \infty) \to \mathbb{R}\) be a twice differentiable function on \(I^\circ\). Suppose that \(a, b \in I^\circ\) with \(a < b\) and \(f'' \in L[a, b]\). If \(|f''|^q\) is \(s\)–concave in the second sense on \(I\) for some fixed \(s \in (0, 1]\) and \(p, q > 1\), then the following inequality for fractional integrals with \(\alpha \in (0, 1]\) holds:
\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \leq \frac{(b-a)^2}{\alpha + 1} \beta \frac{1}{\alpha} (p+1, \alpha p+1) 2^{\frac{\alpha+1}{p}} \left| f'' \left( \frac{a+b}{2} \right) \right|
\]
where \(\frac{1}{p} + \frac{1}{q} = 1\) and \(\beta\) is Euler Beta function.

Proof. From Lemma 2.1 and using the Hölder inequality we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \leq \left( \int_0^1 |f''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}}
\]
\[
\leq \frac{(b-a)^2}{2(\alpha+1)} \left( \int_0^1 t^p (1 - t^\alpha)^p dt \right)^{\frac{1}{p}}
\]
\[
\times \left[ \left( \int_0^1 |f''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} + \left( \int_0^1 |f''((1-t)a + tb)|^q dt \right)^{\frac{1}{q}} \right]
\]
Since \(|f''|^q\) is \(s\)–concave using inequality (1.2) we get (see [2])
\[
\int_0^1 |f''(ta + (1-t)b)|^q dt \leq 2^{s-1} \left| f'' \left( \frac{a+b}{2} \right) \right|^q
\]
(2.7)
and
\[ \int_0^1 |f''((1-t)a + tb)|^q \, dt \leq 2^{s-1} \left| f'' \left( \frac{b + a}{2} \right) \right|^q \] (2.8)

Using (2.7) and (2.8) in (2.6), we have
\[ \left| f(a) + f(b) - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_\alpha^a f(b) + J_\alpha^b f(a) \right] \right| \leq \frac{(b-a)^2}{\alpha + 1} \beta^{\frac{p}{q}} (p + 1, \alpha p + 1) 2^{\frac{s-1}{q}} \left| f'' \left( \frac{a + b}{2} \right) \right| \]
which completes the proof. □

**Remark 2.9.** In Theorem 2.8 if we choose \( s = 1 \) then (2.5) reduces inequality (1.6) of Theorem 1.6.

**References**


1 Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu, Turkey
E-mail address: erhanset@yahoo.com

2 Atatürk University, K.K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey
E-mail address: emos@atauni.edu.tr

3 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
E-mail address: sarikayamz@gmail.com

4 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
E-mail address: filinz_41@hotmail.com