Approximation Numbers of Composition Operators on Weighted Hardy Spaces

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Communicated by S. Hejazian

Abstract. In this paper we find upper and lower bounds for approximation numbers of compact composition operators on the weighted Hardy spaces $H_\sigma$ under some conditions on the weight function $\sigma$.

1. Introduction and preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$ and $H^\infty(\mathbb{D})$ the space of all bounded analytic function on $\mathbb{D}$ with the norm $||f||_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. For $z \in \mathbb{D}$, let

$$\beta_z(w) = \frac{z - w}{1 - \overline{z}w}, \quad z, w \in \mathbb{D},$$

that is, the involutive automorphism of $\mathbb{D}$ interchanging points $z$ and 0. Let $\sigma$ be a positive integrable function on $[0, 1)$. We extend $\sigma$ on $\mathbb{D}$ defining $\sigma(z) = \sigma(|z|)$ for all $z \in \mathbb{D}$ and call it a weight or a weight function. By $H_\sigma$ we denote the weighted Hardy space consisting of all $f \in H(\mathbb{D})$ such that

$$||f||^2_{H_\sigma} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \sigma(z) dA(z) < \infty,$$
where \( dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr \theta \) is the normalized area measure on \( \mathbb{D} \). A simple computation shows that a function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) belongs to \( \mathcal{H}_\sigma \) if and only if
\[
\sum_{n=0}^{\infty} |a_n|^2 \sigma_n < \infty,
\]
where \( \sigma_0 = 1 \) and
\[
\sigma_n = \sigma(n) = 2n^2 \int_0^1 r^{2n-1} w(r) dr, \quad n \in \mathbb{N}.
\]

The sequence \( (\sigma_n)_{n \in \mathbb{N}_0} \) is called the weight sequence of the weighted Hardy space \( \mathcal{H}_\sigma \). The properties of the weighted Hardy space with the weight sequence \( (\sigma_n)_{n \in \mathbb{N}_0} \), clearly depends upon \( \sigma_n \).

Let \( \mathcal{H}_\sigma \) be a weighted Hardy space with weight sequence \( \{\sigma_n\} \). Then for each \( \lambda \in \mathbb{D} \), the evaluation functional in \( \mathcal{H}_\sigma \) at \( \lambda \) is a bounded linear functional and for \( f \in \mathcal{H}_\sigma \), \( f(\lambda) = \langle f, K_\lambda \rangle \), where
\[
K_\lambda(z) = \sum_{k=0}^{\infty} \frac{(\lambda z)^k}{\sigma(k)} \quad \text{and} \quad ||K_\lambda||_{\mathcal{H}_\sigma} = \sum_{k=0}^{\infty} \frac{|\lambda|^{2k}}{\sigma(k)}.
\]

Moreover,
\[
|f(z)| \leq ||f||_{\mathcal{H}_\sigma} \left( \sum_{k=0}^{\infty} r^{2k(\sigma(k))^{-1}} \right)^{1/2}
\]
\[
|f'(z)| \leq ||f||_{\mathcal{H}_\sigma} \left( \sum_{k=0}^{\infty} k^2 r^{2(k-1)(\sigma(k))^{-1}} \right)^{1/2}
\]
for \( |z| \leq r \) where \( \sigma(k) = ||z|^2||_{\mathcal{H}_\sigma} \), see Theorem 2.10 in [2].

For more about weighted Hardy spaces and some related topics, see [2], [3] and [15].

Throughout the paper, a weight \( \sigma \) will satisfy the following properties:

\[(W_1) \ \sigma \ \text{is non-increasing}; \]

\[(W_2) \ \frac{\sigma(r)}{(1-r)^{\delta}} \ \text{is non-decreasing for some} \ \delta > 0; \]

\[(W_3) \ \lim_{r \to 1} \sigma(r) = 0. \]

We also assume that \( \sigma \) will satisfy one of the following properties:

\[(W_4) \ \sigma \ \text{is convex and} \ \lim_{r \to 1} \sigma(r) = 0; \ \text{or} \]

\[(W_5) \ \sigma \ \text{is concave}. \]

Such a weight function is called admissible (see [3]). If \( \sigma \) satisfies condition \((W_1), (W_2), (W_3)\) and \((W_4)\), then it is said that \( \sigma \) is I-admissible. If \( \sigma \) satisfies condition \((W_1), (W_2), (W_3)\) and \((W_5)\), then it is said that \( \sigma \) is II-admissible. I-admissibility corresponds to the case \( \mathcal{H}^2 \subseteq \mathcal{H}_\sigma \subsetneq \mathcal{A}_\alpha^2 \) for some \( \alpha > -1 \), whereas II-admissibility corresponds to the case \( \mathcal{D} \subseteq \mathcal{H}_\sigma \subsetneq \mathcal{H}^2 \). If we say that a weight is admissible it means that it is I-admissible or II-admissible.
Recall that for $z$ and $w$ in $\mathbb{D}$, the pseudohyperbolic distance $d$ between $z$ and $w$ is defined by

\[ d(z, w) = |\beta_z(w)|. \]

For $r \in (0, 1)$ and $z \in \mathbb{D}$, denote by $D(z, r)$, the pseudohyperbolic disk whose pseudohyperbolic center is $z$ and whose pseudohyperbolic radius is $r$, that is

\[ D(z, r) = \{ w \in \mathbb{D} : d(z, w) < r \}. \]

We need Carleson type Theorem for weighted Hardy spaces, see [11]

**Theorem 1.1.** Let $\sigma$ be an admissible weight, $r \in (0, 1)$ fixed and $\mu$ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:

1. The following quantity is bounded
   \[ C_1 := \sup_{z \in \mathbb{D}} \frac{\mu(D(z, r))}{\sigma(z)(1 - |z|^2)^2}; \]

2. There is a constant $C_2 > 0$ such that, for every $f \in H_\sigma$,
   \[ \int_{\mathbb{D}} |f'(w)|^2 d\mu(w) \leq C_2 \| f \|_{H_\sigma}^2; \]

3. The following quantity is bounded
   \[ C_3 := \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |z|^2)^{2+2\gamma}}{\sigma(z)|1 - \overline{z}w|^{4+2\gamma}} d\mu(w). \]

Moreover, the following asymptotic relationships hold

\[ C_1 \asymp C_2 \asymp C_3. \]

The generalized Nevanlinna counting function shall play a key role in our work. The generalized Nevanlinna counting function associated to a weight function $\omega$ is defined for every $z \in \mathbb{D} \setminus \{ \varphi(0) \}$ by

\[ \mathfrak{N}_{\varphi, \sigma}(z) = \sum_{\lambda, \varphi(\lambda) = z} \sigma(\lambda), \]

where $\mathfrak{N}_{\varphi, \sigma}(z) = 0$ when $z \notin \varphi(\mathbb{D})$. By convention, we define $\mathfrak{N}_{\varphi, \sigma}(z) = 0$ when $z = \varphi(0)$. When $\sigma(r) = \sigma_0(r) = \log 1/r$, $\mathfrak{N}_{\varphi, \sigma_0} = N_{\varphi}$, the usual Nevanlinna counting function associated to $\varphi$.

For more about generalized and classical Nevanlinna counting functions, see [2] and [3]. The generalized Nevanlinna counting function $\mathfrak{N}_{\varphi, \sigma}$ provides the following non-univalent change of variable formula (see [2], Theorem 2.32).

**Lemma 1.2.** If $g$ and $\sigma$ are positive measurable function on $\mathbb{D}$ and $\varphi$ a holomorphic self-map of $\mathbb{D}$, then

\[ \int_{\mathbb{D}} (g \circ \varphi)(z)|\varphi'(z)|^2 \sigma(z) dA(z) = \int_{\mathbb{D}} g(z)\mathfrak{N}_{\varphi, \sigma}(z) dA(z). \]

Recall that the essential norm $\| T \|_e$ of a bounded linear operator on a Banach space $X$ is given by

\[ \| T \|_e = \inf\{ \| T - K \| : K \text{ is compact on } X \}. \]
It provides a measure of non-compactness of $T$. Clearly, $T$ is compact if and only if $||T||_e = 0$. Let $\varphi$ be a non-constant analytic self-map (a so called Schur function) of $\mathbb{D}$ and let $C_\varphi : \mathcal{H}_\omega \to H(\mathbb{D})$ the associated composition operator:

$$C_\varphi f = f \circ \varphi.$$ 

For more about composition operators on weighted Hardy spaces, see [3], [11] and [15].

The next theorem can be found in [15].

**Theorem 1.3.** Let $\sigma_1$ and $\sigma_2$ be two admissible weights ($(I)$-admissible or $(II)$-admissible) and $\varphi$ be a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi : \mathcal{H}_{\sigma_1} \to \mathcal{H}_{\sigma_2}$ is bounded if and only if

$$\sup_{|z| < 1} \frac{\mathfrak{M}_{\varphi, \sigma_2}(z)}{\sigma_1(z)} < \infty.$$ 

Moreover, if $C_\varphi : \mathcal{H}_{\sigma_1} \to \mathcal{H}_{\sigma_2}$ is bounded, then

$$||C_\varphi||_{\mathcal{H}_{\sigma_1} \to \mathcal{H}_{\sigma_2}} \leq \sup_{|z| < 1} \frac{\mathfrak{M}_{\varphi, \sigma_2}(z)}{\sigma_1(z)}.$$ 

As in [5], we first introduce the following notations. If

$$\varphi^*(z) = \lim_{w \to z} \frac{\rho(\varphi(w), \varphi(z))}{\rho(w, z)} = \frac{||\varphi^*(z)||}{1 - |\varphi(z)|^2}$$

is the pseudo-hyperbolic derivative of $\varphi$, we set:

$$[\varphi] = \sup_{z \in \mathbb{D}} \varphi^*(z) = ||\varphi^*||_{\infty}.$$ 

Also recall that the approximation (or singular) numbers $a_n(T)$ of an operator $T \in \mathcal{L}(H_1, H_2)$, between two Hilbert spaces $H_1$ and $H_2$ are defined by:

$$a_n(T) = \inf \{ ||T - R||; \rank(R) < n \}, \quad n = 1, 2, \ldots.$$ 

We have

$$a_n(T) = c_n(T) = d_n(T),$$

where the numbers $c_n$ (resp. $d_n$) are the Gelfand (resp. Kolmogorov) numbers of $T$ ([1], page 59 and page 51 respectively). In the sequel we shall need the following quantity:

$$\tau(T) = \liminf_{n \to \infty} [a_n(T)]^{1/n}.$$ 

These approximation numbers form a non-increasing sequence such that

$$a_1(T) = ||T||, \quad a_n(T) = \sqrt{a_n(T^*T)}$$

are verify the so-called “ideal” and “subadditivity” properties ([4], see page 57 and page 68):

$$a_n(ATB) \leq ||A||a_n(T)||B||; \quad a_{n+m-1}(S + T) \leq a_n(S) + a_m(T).$$
Moreover, the sequence \((a_n(T))\) tends to 0 if and only if \(T\) is compact. If for some \(p, 1 \leq p < \infty\), \((a_n(T)) \in l_p\), where
\[
l_p = \left\{ a = \{a_n\}_{n=1}^\infty : \|a\|_p = \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} < \infty \right\},
\]
then we say that \(T\) belongs to the Schatten class \(S_p\).

The upper and lower bounds for approximation numbers of composition operators on the Hardy space were computed by Li, Queffelec and Rodriguez-Piazza in [5]. In this paper, we generalized some of the results concerning upper and lower bounds for approximation numbers of composition operators to weighted Hardy spaces \(H_\sigma\) under some conditions on the weight function \(\sigma\).

Throughout the paper constants are denoted by \(C\), they are positive and not necessarily the same at each occurrence. The notation \(A \lesssim B\) means that there is a positive constant \(C\) such that \(A \leq CB\). When \(A \lesssim B\) and \(B \lesssim A\), we write \(A \asymp B\).

2. Lower Bound

We first show that, each Möbius transformations \(\beta_z\) always induce a bounded composition operator on \(H_\sigma\). This property ensures that, we may consider the operator \(C_\varphi\) under the assumption \(\varphi(0) = 0\).

Proposition 2.1. Let \(\sigma\) be an admissible weight. Then for each \(z \in \mathbb{D}\), \(C_\beta_z\) is bounded on \(H_\sigma\).

Proof. By the change of variable formula, we have
\[
\|C_\beta_z f\|^2_{H_\sigma} = |f(\beta_z(0))|^2 + \int_D |f'(\beta_z(w))|^2 |\beta'_z(w)|^2 \sigma(w) dm(w)
\]
\[
= |f(z)|^2 + \int_D |f'(w)|^2 |\beta'_z(\xi_z(w))|^2 \sigma(\beta_z(w)) |\beta'_z(w)|^2 dm(w)
\]
\[
= |f(z)|^2 + \int_D |f'(w)|^2 |(\beta_z \circ \beta_z)'(w)|^2 \sigma(\beta_z(w)) dm(w)
\]
\[
= |f(z)|^2 + \int_D |f'(w)|^2 \sigma(\beta_z(w)) dm(z) \tag{3}
\]
By Lemma 2.1 of [3], we have
\[
\sigma(\beta_z(w)) \asymp \sigma(w) \tag{4}
\]
From (3) and (4), we have
\[
\|C_\beta_z f\|^2_{H_\sigma} \lesssim |f(z)|^2 + \|f\|^2_{H_\sigma}
\]
for each \(f \in H_\sigma\). This implies that \(C_\beta_z(H_\sigma) \subset H_\sigma\). Thus by closed graph theorem, \(C_\beta_z\) is bounded on \(H_\sigma\).

Proposition 2.2. For each \(z \in \mathbb{D}\), \(C_\beta_z\) is invertible.
Proof. By Proposition 1, $C_{\beta_z}$ is bounded. Now the proof is an easy consequence of Theorem 1.6 in [2].

In the following result, we show that if $\sigma$ is II-admissible, or $\sigma$ is I-admissible and $C_{\varphi}$ is compact on $H_{\sigma}$, then the approximation numbers of $C_{\varphi}$ on $H_{\sigma}$ cannot supersede a geometric speed. \hfill \Box

**Theorem 2.3.** Let $\sigma$ be an admissible weight and $\varphi$ be a Schur function such that for $C_{\varphi} : H_{\sigma} \to H_{\sigma}$ is bounded. Suppose that $C_{\varphi}$ is compact on $H_{\sigma}$, whenever $\omega$ is I-admissible. Then there exist positive constant $C > 0$ and $0 < r < 1$ such that

$$a_n(C_{\varphi}) \geq C r^n, \quad n = 1, 2, \ldots.$$ 

More precisely, one has $\beta(C_{\varphi}) \geq [\varphi]^2$ and hence for each $k < [\varphi]$ there exist a constant $C_k > 0$ such that

$$a_n(C_{\varphi}) \geq C_k k^{2n}.$$ 

For the proof we need the following lemma (see [5]).

**Lemma 2.4.** Let $T : H \to H$ be a compact operator. Suppose that $(\lambda_n)_{n \geq 1}$ the sequence of eigenvalues of $T$ rearranged in non-increasing order satisfies for some $\delta > 0$ and $r \in (0, 1)$

$$|\lambda_n| \geq \delta r^n, \quad n = 1, 2, \ldots.$$ 

Then there exist $\delta_1 > 0$ such that

$$a_n(T) \geq \delta_1 r^{2n}, \quad n = 1, 2, \ldots.$$ 

In particular $\beta(T) \geq r^2$.

**Proposition 2.5.** Let $\omega$ be an admissible weight and $\varphi$ be a Schur function such that for $C_{\varphi} : H_{\omega} \to H_{\omega}$ is compact. Then $\tau(C_{\varphi}) \geq [\varphi]^2$.

**Proof.** The proof follows on same lines as the proof of Proposition 3.3 in [5]. We include it for completeness. For every $z \in \mathbb{D}$, let $\beta_z$ be the involutive automorphism of $\mathbb{D}$. Then we have

$$\beta_z(z) = 0, \quad \beta_z(0) = z, \quad \beta'_z(z) = \frac{1}{|z|^2 - 1}, \quad \beta'_z(0) = |a|^2 - 1.$$ 

Let $\psi = \beta_{\varphi(z)} \circ \varphi \circ \beta_z$. Then 0 is a fixed point of $\psi$, whose derivative by the chain rule is

$$\psi'(0) = \beta'_{\varphi(z)}(\varphi(z))\varphi'(z)\beta'_z(0) = \frac{\varphi'(z)(1 - |z|^2)}{1 - |\varphi(z)|^2} = \varphi^z(z).$$ 

By Schwarz’s lemma

$$\frac{(1 - |z|^2)}{1 - |\varphi(z)|^2} |\varphi'(z)| = |\psi'(0)| \leq 1.$$ 

Let us first assume that, the composition operator $C_{\varphi}$ is compact on $H_{\sigma}$. Then so is $C_{\psi}$, since we have

$$C_{\psi} = C_{\beta_z} \circ C_{\varphi} \circ C_{\beta_{\varphi(z)}}.$$ 

If $\psi'(0) \neq 0$, the sequence of eigenvalues of $C_{\psi}$ the Hardy space $H^2$ is $([\psi'(0)]^n)_{n \geq 0}$ (see [2], page 96). Since II-admissibility corresponds to the case $H_{\sigma} \subset H^2$, so the
result given for $H^2$ holds for $\mathcal{H}_\sigma$ and would also holds for any space of analytic functions in $\mathbb{D}$ on which $C_\psi$ is compact. By Lemma 2.4, we have

$$\tau(C_\psi) \geq |\psi'(0)| = |\varphi'(z)|^2 \geq 0.$$  

This trivially still holds if $\psi'(0) = 0$. Now since $C_{\beta_z}$ and $C_{\beta_{\varphi(z)}}$ are invertible operators, we have that $\tau(C_{\varphi}) = \tau(C_\psi)$ and therefore, we have

$$\tau(C_\psi) = [\varphi]^2$$  

for all $z \in \mathbb{D}$. By passing to the supremum on $z \in \mathbb{D}$, we end the proof of Proposition 2.5 and that of Theorem 2.3 in the compact case. If $C_\varphi$ is not compact, the proposition trivially holds. Indeed, in this case, we have $\tau(C_\varphi) = 1 \geq [\varphi]^2$. \[\square\]

3. Upper Bound

**Theorem 3.1.** Let $\varphi$ be a holomorphic self-map of $\mathbb{D}$ such that $\varphi(0) = 0$. Let $\sigma$ be an admissible weight. Assume that $\sup \sigma(k) < \infty$ and $r \in (0, 1)$ is fixed. Then the approximation number of $C_\varphi : \mathcal{H}_{\sigma_1} \to \mathcal{H}_{\sigma_2}$ has the upper bound

$$a_n(C_\varphi) \lesssim \inf_{0 < h < 1} \left[ (1 - h)^{2n} \sum_{k=0}^{\infty} k^2 (1 - h)^{2(k-1)} \frac{\sigma_k}{\sigma_k + n} + (1 - h)^{2n-2} \sum_{k=0}^{\infty} (1 - h)^{2k} \right]$$

$$\left( \sup_{1 \leq j < \infty} \frac{\sigma_j}{\sigma_{j+n}} + \sup_{z \in \mathbb{D}} \frac{\mu_{\sigma, \varphi, h}(D(z, r))}{\sigma(z)(1 - |z|^2)^2} \right). \quad (3.1)$$

To prove the theorem, we need the following lemma.

**Lemma 3.2.** Let $f(z) = \sum_{k=n}^{\infty} a_k z^k$ and $g(z) = z^n f(z)$. Then

$$\|g\|_{\mathcal{H}_\sigma}^2 \leq \sup_{1 \leq j < \infty} \frac{\sigma_j}{\sigma_{j+n}} \|f\|_{\mathcal{H}_\sigma}^2.$$  

Proof. \[\square\]

Proof. We denote by $P_n$ the projection operator defined by

$$P_n f = \sum_{k=0}^{n-1} \hat{f}(k) z^k$$

and we take $R = C_\varphi \circ P_n$, that is, if we have $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k \in \mathcal{H}_\sigma$ then

$$R(f) = \sum_{k=0}^{n-1} \hat{f}(k) \varphi^k$$

so that $(C_\varphi - R)f = C_\varphi(r)$. Then, we have

$$r(z) = \sum_{k=n}^{\infty} \hat{f}(k) z^k = z^n s(z),$$
where

$$||s||_{L^2}^2 \leq C \sup_{|r|} \frac{\sigma(j)}{\sigma(j+k)} ||r||_{L^2}^2, \text{ and } ||r||_{L^2} \leq ||f||_{L^2}. \quad (3.2)$$

Assume that $||f||_{L^2} \leq 1$ and $dm_{\varphi,\sigma} = \mathcal{M}_{\varphi,\sigma}(z) dm(z)$. Fix $0 < h < 1$. Let

\[ \mu_{\varphi,\sigma}(z) = (m_{\varphi,\sigma} \circ \varphi^{-1})(z) \]

and $\mu_{\varphi,\sigma,h}$ be the restriction of the measure $\mu_{\varphi,\sigma}(z)$ to the annulus $1 - h < |z| \leq 1$. Then we have

$$|| (C_{\varphi} - R) f ||_{L^2}^2 = ||C_{\varphi}(r)||_{L^2}^2 \quad (3.3)$$

Let $(z_n)_{n \in \mathbb{N}}$ be a sequence with a positive separation constant such that

$$\cup_{n=1}^{\infty} D(z_n, r) = \mathbb{D}$$

and every point in $\mathbb{D}$ belongs to at most $M$ sets in the family $\{D(z_n, 2r)\}_{n \in \mathbb{N}}$. Since $\sigma$ is an almost standard weight we have that for $0 < r_1 < r_2 < 1$

$$\left( \frac{1-r_2}{1-r_1} \right)^{t+1} w(r_1) \leq w(r_2) \leq w(r_1).$$

From this and since $1 - |z| \asymp 1 - |z_n|$, for $z \in D(z_n, 2r)$, we obtain

$$\sigma(z) \asymp \sigma(z_n), \quad z \in D(z_n, 2r).$$

Using these facts we obtain

\[
I_1 = \int_{|z| \leq 1-h} |(z^n s'(z) + nz^{n-1} s(z))| \mathcal{M}_{\varphi,\sigma}(z) dm(z) \\
\leq \int_{|z| \leq 1-h} |z^n s'(z)| \mathcal{M}_{\varphi,\sigma}(z) dm(z) + n^2 \int_{|z| \leq 1-h} |z^{n-1} s(z)| \mathcal{M}_{\varphi,\sigma}(z) dm(z) \\
\leq (1-h)^{2n} \int_{|z| \leq 1-h} |s'(z)| \mathcal{M}_{\varphi,\sigma}(z) dm(z) \\
+ n^2 (1-h)^{2n-2} \int_{|z| \leq 1-h} |s(z)| \mathcal{M}_{\varphi,\sigma}(z) dm(z). \quad (3.4)
\]
Thus by Lemma 3.2, (2) and (6), we have

\[(1 - h)^{2n} \int_{|z| \leq 1 - h} |s'(z)|^2 \mathcal{N}_{\varphi, \sigma}(z) dm(z) \]
\[\leq (1 - h)^{2n} \|s\|_{H_\sigma}^2 \left( \sum_{k=0}^{\infty} k^2 (1 - h)^{2(k-1)} \sigma_k^{-1} \right) \int \mathcal{N}_{\varphi, \sigma}(z) dm(z) \]
\[\lesssim (1 - h)^{2n} \sup_{1 \leq j < \infty} \frac{\sigma_j}{\sigma_{j+n}} \|r\|_{H_\sigma}^2 \sum_{k=0}^{\infty} k^2 (1 - h)^{2(k-1)} \sigma_k^{-1} \|\varphi\|_{H_\sigma}^2 \]
\[\lesssim (1 - h)^{2n} \sup_{1 \leq j < \infty} \frac{\sigma_j}{\sigma_{j+n}} \|r\|_{H_\sigma}^2 \sum_{k=0}^{\infty} k^2 (1 - h)^{2(k-1)} \sigma_k^{-1} \|\varphi\|_{H_\sigma}^2 \]
\[\lesssim (1 - h)^{2n} \left( \sup_{1 \leq j < \infty} \frac{\sigma_j}{\sigma_{j+n}} \right) \sum_{k=0}^{\infty} k^2 (1 - h)^{2(k-1)} \sigma_k^{-1}. \quad (3.5) \]

Again by Lemma 3.2, (1) and (6), we have

\[(1 - h)^{2n-2} \int_{|z| \leq 1 - h} |s(z)|^2 \mathcal{N}_{\varphi, \sigma}(z) dm(z) \]
\[\lesssim (1 - h)^{2n-2} \|s\|_{H_\sigma}^2 \sum_{k=0}^{\infty} \frac{(1 - h)^{2k}}{\sigma_k} \int \mathcal{N}_{\varphi, \sigma}(z) dm(z) \]
\[\lesssim (1 - h)^{2n-2} \left( \sup_{1 \leq j < \infty} \frac{\sigma_j}{\sigma_{j+n}} \right) \|r\|_{H_\sigma}^2 \sum_{k=0}^{\infty} \frac{(1 - h)^{2k}}{\sigma_k} \|\varphi\|_{H_\sigma}^2 \]
\[\lesssim (1 - h)^{2n-2} \left( \sup_{1 \leq j < \infty} \frac{\sigma_j}{\sigma_{j+n}} \right) \|r\|_{H_\sigma}^2 \sum_{k=0}^{\infty} \frac{(1 - h)^{2k}}{\sigma_k} \|\varphi\|_{H_\sigma}^2 \]
\[\lesssim (1 - h)^{2n-2} \left( \sup_{1 \leq j < \infty} \frac{\sigma_j}{\sigma_{j+n}} \right) \sum_{k=0}^{\infty} \frac{(1 - h)^{2k}}{\sigma_k}. \quad (3.6) \]

Combining (8), (9) and (10), we have

\[I_1 \lesssim (1 - h)^{2n} \left( \sup_{1 \leq j < \infty} \frac{\sigma_j}{\sigma_{j+n}} \right) \sum_{k=0}^{\infty} \frac{k^2 (1 - h)^{2(k-1)}}{\sigma_k} \]
\[+ (1 - h)^{2n-2} \left( \sup_{1 \leq j < \infty} \frac{\sigma_j}{\sigma_{j+n}} \right) \sum_{k=0}^{\infty} \frac{(1 - h)^{2k}}{\sigma_k}. \quad (3.7) \]
Again
\[ I_2 = \int_{1-h<|z|<1} |r'(z)|^2 \mathcal{M}_{\varphi,\varphi}(z) \, dm(z) \]
\[ = \int_{\mathbb{D}} |r'(z)|^2 d\mu_{\sigma,\varphi,\varphi}(z) \]
\[ \leq \sum_{n=1}^{\infty} \int_{D(z_n,r)} |r'(z)|^2 d\mu_{\sigma,\varphi,\varphi}(z) \]
\[ \leq \sum_{n=1}^{\infty} \mu_{\sigma,\varphi,\varphi}(D(z_n,r)) \sup_{\sigma \in D(z_n,r)} |r'(\sigma)|^2 \]
\[ \leq \sum_{n=1}^{\infty} \frac{\mu(D(z_n,r))}{\sigma(z_n)(1-|z_n|^2)^2} \int_{D(z_n,2r)} |r'(z)|^2 \sigma(z) \, dm(z) \]
\[ \leq \sup_{z \in \mathbb{D}} \frac{\mu_{\sigma,\varphi,\varphi}(D(z,r))}{\sigma(z)(1-|z|^2)^2} \int_{\mathbb{D}} |r'(z)|^2 \sigma(z) \, dm(z) \]
\[ \leq \sup_{z \in \mathbb{D}} \frac{\mu_{\sigma,\varphi,\varphi}(D(z,r))}{\sigma(z)(1-|z|^2)^2} \left\| r' \right\|_{H_\sigma}^2 \]
\[ \leq \sup_{z \in \mathbb{D}} \frac{\mu_{\sigma,\varphi,\varphi}(D(z,r))}{\sigma(z)(1-|z|^2)^2} \]  \hspace{1cm} (3.8)

Combining (7), (11) and (12), we get the desired upper bound given in (5). \qed

Acknowledgments. This work is a part of the research project sponsored by National Board of Higher Mathematics (NBHM)/DAE, India (Grant No. 48/4/2009/R&D-II/426).

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