



**STRONGLY ZERO-PRODUCT PRESERVING MAPS ON
NORMED ALGEBRAS INDUCED BY A BOUNDED LINEAR
FUNCTIONAL**

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Communicated by H.R. Ebrahimi Vishki

ABSTRACT. We introduce the notions of strongly zero-product (strongly Jordan zero-product) preserving maps on normed algebras. These notions are generalization of the concepts of zero-product and Jordan zero-product preserving maps. Also for a non-zero vector space \mathcal{V} and for a non-zero linear functional f on \mathcal{V} , we equip \mathcal{V} with a multiplication, converting \mathcal{V} into an associative algebra, denoted by \mathcal{V}_f . We characterize the zero-product (Jordan zero-product) preserving maps on \mathcal{V}_f . Also we characterize the strongly zero-product (strongly Jordan zero-product) preserving maps on \mathcal{V}_f in the case where \mathcal{V} is a normed vector space and f is a continuous linear functional on \mathcal{V} . Finally, for polynomials in one variable x over \mathcal{V}_f , we shall show that each polynomial of precise degree $n \geq 0$, with non-zero constant term has precisely n -zeros (counted with multiplicity) in \mathcal{V}_f . While, polynomials of precise degree $n \geq 2$, with zero constant term have infinitely many zeros when $\dim(\mathcal{V}) \geq 2$. This shows that the algebraic fundamental theorem for polynomial equations over an arbitrary algebra, is not valid in general.

1. INTRODUCTION AND PRELIMINARIES

Let \mathcal{A} and \mathcal{B} be algebras over \mathbb{C} . A linear map $\theta : \mathcal{A} \rightarrow \mathcal{B}$ is called a zero-product preserving map if $\theta(a)\theta(b) = 0$ whenever $ab = 0$. Also a linear map $\theta : \mathcal{A} \rightarrow \mathcal{B}$ is called a Jordan zero-product preserving map if $\theta(a) \circ \theta(b) = 0$ whenever $a \circ b = 0$, where “ \circ ” denotes the Jordan product $a \circ b = ab + ba$. The canonical form of a zero-product preserving map, $\theta = c\varphi$, arises from an element c in the center of

Date: Received: 18 October 2014; Revised: 5 December 2014; Accepted: 13 December 2014.
2010 *Mathematics Subject Classification.* Primary 46B99; Secondary 46H99, 47B37, 65H04.

Key words and phrases. (Jordan) zero-product preserving map, strongly (Jordan) zero-product preserving map, Arens product, polynomial equation.

\mathcal{B} and an algebra homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$. But it is not the case in general. For some good references in the field of zero-product (Jordan zero-product) preserving maps we refer the reader to [1] and [2].

Let \mathcal{A} and \mathcal{B} be two normed algebras over \mathbb{C} . We shall say that a linear map $\theta : \mathcal{A} \rightarrow \mathcal{B}$ is a strongly zero-product preserving map if, for any two sequences $\{a_n\}_n, \{c_n\}_n$ in \mathcal{A} , $\theta(a_n)\theta(c_n) \rightarrow 0$ whenever $a_n c_n \rightarrow 0$.

Also we shall say that θ is a strongly Jordan zero-product preserving map if, for any two sequences $\{a_n\}_n, \{c_n\}_n$ in \mathcal{A} , $\theta(a_n) \circ \theta(c_n) \rightarrow 0$ whenever $a_n \circ c_n \rightarrow 0$. Let \mathcal{V} be a non-zero vector space and let $f \in \mathcal{V}^*$ be a non-zero linear functional. For each $a, b \in \mathcal{V}$ define $a \cdot b = f(a)b$. One can simply verify that “ \cdot ” converts \mathcal{V} into an associative algebra. We denote (\mathcal{V}, \cdot) by \mathcal{V}_f , that is an algebra. Note that \mathcal{V}_f is not unital in general. Indeed \mathcal{V}_f is unital if and only if $\dim \mathcal{V} = 1$. Also if $\dim \mathcal{V} > 1$ then $Z(\mathcal{V}_f) = \{0\}$, where $Z(\mathcal{V}_f)$ is the algebra center of \mathcal{V}_f . It is obvious that \mathcal{V}_f is not a commutative algebra. Indeed \mathcal{V}_f is commutative if and only if $\dim \mathcal{V} = 1$. Many basic properties of \mathcal{V}_f such as Arens regularity, n -weak amenability, minimal idempotents and ideal structure are investigated in [6] in the case where \mathcal{V} is a Banach space. Also the endomorphisms and automorphisms of \mathcal{V}_f are characterized in [5] when \mathcal{V} is a vector space.

For an algebra \mathcal{A} let \mathcal{A}^{**} be the second dual of \mathcal{A} . We introduce the Arens products Δ and \odot on the second dual \mathcal{A}^{**} . Let $a, b \in \mathcal{A}$, $f \in \mathcal{A}^*$ and $m, n \in \mathcal{A}^{**}$.

$\langle f \cdot a, b \rangle = \langle f, ab \rangle$, $\langle n \cdot f, a \rangle = \langle n, f \cdot a \rangle$ and $\langle m \Delta n, f \rangle = \langle m, n \cdot f \rangle$. Similarly $\langle b, a \cdot f \rangle = \langle ba, f \rangle$, $\langle a, f \cdot n \rangle = \langle a \cdot f, n \rangle$ and $\langle f, m \odot n \rangle = \langle f \cdot m, n \rangle$. One can simply verify that $(\mathcal{A}^{**}, \Delta)$ and $(\mathcal{A}^{**}, \odot)$ are associative algebras.

An algebra \mathcal{A} is called Arens regular if the two Arens products coincide. One can simply verify that \mathcal{V}_f is Arens regular [6, proposition 2.1].

A polynomial P in one variable x over a ring R is defined to be a formal expression of the form $P = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$, where $n \in \mathbb{N} \cup \{0\}$ and coefficients a_0, a_1, \dots, a_n are elements of R . A polynomial equation is an equation such as, $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0 = 0$. In case of a polynomial equation, the variable is considered an unknown and one seeks to find the possible values that satisfies the equation. The solutions to the equation are called the roots of the polynomial. By one of the most fundamental results in algebra, it is well known that any polynomial of precise degree n , with coefficients in a given field F , has precisely n zeros (when counted with multiplicity) in the algebraic closure of F . This result fails for polynomials with coefficients in a finite dimensional F -algebra [7]. In [4] Herstein proved that a polynomial of degree n , with coefficients in the center of an associative division algebra \mathcal{A} , has either infinitely many or at most n zeros in \mathcal{A} . In [3] Gordon and Motzkin showed that a polynomial of degree n , with coefficients in an associative division algebra \mathcal{A} which is d -dimensional over its center, has either infinitely many or at most n^d zeros in \mathcal{A} .

In this paper our purpose is to characterize zero-product (Jordan zero-product) preserving maps on \mathcal{V}_f . Also we characterize zero-product (Jordan zero-product) preserving maps on $(\mathcal{V}_f)^{(2n)}$ for all $n \geq 0$, where $(\mathcal{V}_f)^0 = \mathcal{V}_f$ and $(\mathcal{V}_f)^{(2n)}$ is the $2n$ -th dual of \mathcal{V}_f that is an algebra equipped with one of the Arens products.

Also after introducing the notions of strongly zero-product (strongly Jordan zero-product) preserving maps, we characterize them on \mathcal{V}_f in the case where \mathcal{V} is a normed vector space and f is a continuous linear functional on \mathcal{V} . It is worthwhile mentioning that the study of the zero-product (Jordan zero-product) preserving maps on \mathcal{V}_f is very interesting. Finally, we prove some interesting results concerning the number of roots of polynomial equations with coefficients in \mathcal{V}_f . The following examples are some different zero-product preserving maps on \mathcal{V}_f that are worthy of consideration.

- (1) $\theta : \mathcal{V}_f \longrightarrow \mathcal{V}_f$, $\theta(a) = g(a)c$, where g is a linear functional on \mathcal{V} and c is a constant element of $\ker f$.
- (2) $\theta : \mathcal{V}_f \longrightarrow \mathcal{V}_f$, $\theta(a) = f(a)b$, where b is a constant element of \mathcal{V}_f . (Note that f is an algebra homomorphism from \mathcal{V}_f onto \mathbb{C}).

2. ZERO-PRODUCT PRESERVING MAPS ON \mathcal{V}_f

In this section we characterize the zero-product (Jordan zero-product) preserving maps on \mathcal{V}_f . Also we emphasize that the zero-product preserving maps are not homomorphisms multiplied by central elements, in general.

Theorem 2.1. *Let \mathcal{V} be a non-zero vector space and let $0 \neq f \in \mathcal{V}^*$. Then a linear map $\theta : \mathcal{V}_f \longrightarrow \mathcal{V}_f$ is a Jordan zero-product preserving map if and only if $\theta(\ker f) \subseteq \ker f$.*

Proof. In the case where $\theta \equiv 0$ the result is obvious. Let θ be a non-zero Jordan zero-product preserving map. Since for each $a \in \ker f$, $2a^2 = a^2 + a^2 = 0$, so $2\theta(a)^2 = \theta(a)\theta(a) + \theta(a)\theta(a) = 0$. It follows that $f(\theta(a)^2) = f(\theta(a))^2 = 0$, which is equivalent to $\theta(a) \in \ker f$. So $\theta(\ker f) \subseteq \ker f$. For the converse let $\theta(\ker f) \subseteq \ker f$ and $ab + ba = 0$. So $f(ab + ba) = f(a)f(b) + f(b)f(a) = 0$. It follows that $2f(a)f(b) = 0$, which is equivalent to $a \in \ker f$ or $b \in \ker f$. Without loss of generality let $a \in \ker f$. The equality $0 = ab + ba = f(a)b + f(b)a$ implies that $f(b)a = 0$ and so $f(b)\theta(a) = 0$. It follows that $b \in \ker f$ or $\theta(a) = 0$. Since $a \in \ker f$ and $\theta(\ker f) \subseteq \ker f$ so, $\theta(a)\theta(b) + \theta(b)\theta(a) = f(\theta(a))\theta(b) + f(\theta(b))\theta(a) = 0 + 0 = 0$. This shows that θ is a Jordan zero-product preserving map. \square

Theorem 2.2. *Let \mathcal{V} be a non-zero vector space and let $0 \neq f \in \mathcal{V}^*$. Then a linear map $\theta : \mathcal{V}_f \longrightarrow \mathcal{V}_f$ is a zero-product preserving map if and only if $\theta(\ker f) \subseteq \ker f$.*

Proof. In the case where $\theta \equiv 0$ the result is obvious. Let θ be a non-zero zero-product preserving map. Let $a \in \ker f$ and $b \in \mathcal{V}_f$ such that $\theta(b) \neq 0$. As $ab = 0$ so $f(\theta(a))\theta(b) = \theta(a)\theta(b) = 0$. It follows that $f(\theta(a)) = 0$, which is equivalent to $\theta(\ker f) \subseteq \ker f$. For the converse let $\theta(\ker f) \subseteq \ker f$ and $ab = 0$. So $f(a)b = 0$, which is equivalent to $a \in \ker f$ or $b = 0$. Since $\theta(\ker f) \subseteq \ker f$ it follows that $f(\theta(a)) = 0$ or $b = 0$. So $\theta(a)\theta(b) = f(\theta(a))\theta(b) = 0$. This shows that θ is a zero-product preserving map. \square

Corollary 2.3. *Let \mathcal{V} be a non-zero vector space and let $0 \neq f \in \mathcal{V}^*$. Then a linear map $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$ is a zero-product preserving map if and only if it is a Jordan zero-product preserving map.*

Corollary 2.4. *Let \mathcal{V} be a non-zero vector space and let $0 \neq f \in \mathcal{V}^*$. Also let the bijective linear map $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$ be a zero-product (Jordan zero-product) preserving map. Then θ^{-1} is a zero-product (Jordan zero-product) preserving map if and only if $\theta(\ker f) = \ker f$.*

Remark 2.5. Note that in the case where $\dim \mathcal{V} > 1$ the linear map $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$, $\theta(a) = f(a)c$, where c is a non-zero element of $\ker f$ is a zero-product (Jordan zero-product) preserving map but it is not a homomorphism (Jordan homomorphism) multiplied by a central element.

Remark 2.6. Let \mathcal{V} be a non-zero vector space and $f \in \mathcal{V}^*$ be a non-zero linear functional. Also let \hat{f} be the canonical image of f in $\mathcal{V}^{(2n+1)}$, $n \geq 1$. As by [6, Proposition 2.1] $(\mathcal{V}_f)^{(2n)} = (\mathcal{V}^{(2n)})_{\hat{f}}$ so by Theorems 2.1 and 2.2 a linear map $\varphi : (\mathcal{V}_f)^{(2n)} \rightarrow (\mathcal{V}_f)^{(2n)}$ is a zero-product (Jordan zero-product) preserving map if and only if $\hat{f} \circ \varphi = 0$ on $\ker \hat{f}$.

Corollary 2.7. *Let \mathcal{V} be a non-zero vector space, let $0 \neq f \in \mathcal{V}^*$ and let $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$ be a non-zero linear map. Then the linear map $\theta^{**} : (\mathcal{V}_f)^{**} \rightarrow (\mathcal{V}_f)^{**}$ is a zero-product (Jordan zero-product) preserving map if and only if $\hat{f} \circ \theta^{**} = 0$ on $\ker \hat{f}$. (Note that \hat{f} is the canonical image of f in \mathcal{V}^{***}).*

Corollary 2.8. *Let \mathcal{V} be a non-zero vector space and let $0 \neq f \in \mathcal{V}^*$ such that, $\ker f = \ker \hat{f}$. Then a non-zero linear map $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$ is a zero-product (Jordan zero-product) preserving map if and only if the linear map $\theta^{(2n)} : (\mathcal{V}_f)^{(2n)} \rightarrow (\mathcal{V}_f)^{(2n)}$, $n \geq 0$ is a zero-product (Jordan zero-product) preserving map.*

Proof. As $\theta = \theta^{(2n)}|_{\mathcal{V}_f}$, and $f = \hat{f}|_{\mathcal{V}_f}$ the proof is obvious. \square

3. STRONGLY ZERO-PRODUCT PRESERVING MAPS ON \mathcal{V}_f

Definition 3.1. Let \mathcal{A} and \mathcal{B} be normed algebras. We say that a linear map $\theta : \mathcal{A} \rightarrow \mathcal{B}$ is a strongly zero-product preserving map if, for any two sequences $\{a_n\}_n$, $\{c_n\}_n$ in \mathcal{A} , $\theta(a_n)\theta(c_n) \rightarrow 0$ whenever $a_n c_n \rightarrow 0$.

Similarly a linear map $\theta : \mathcal{A} \rightarrow \mathcal{B}$ is a strongly Jordan zero-product preserving map if, for any two sequences $\{a_n\}_n$, $\{c_n\}_n$ in \mathcal{A} , $\theta(a_n) \circ \theta(c_n) \rightarrow 0$ whenever $a_n \circ c_n \rightarrow 0$.

Example 3.2. (1) Let \mathcal{A}, \mathcal{B} be two normed algebras. Then every continuous homomorphism $\theta : \mathcal{A} \rightarrow \mathcal{B}$ is a strongly zero-product (strongly Jordan zero-product) preserving map.

(2) Let \mathcal{V} be a normed vector space with $\dim \mathcal{V} > 1$, and let $0 \neq f$ be a continuous linear functional on \mathcal{V} . Then the linear map

$$\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f, \theta(a) = g(a)c,$$

where c is a non-zero constant element of $\ker f$ and g is a linear functional on \mathcal{V} , is a strongly zero-product (strongly Jordan zero-product) preserving map. (Note that θ is not continuous in general).

It is obvious that each strongly zero-product (strongly Jordan zero-product) preserving map is a zero-product (Jordan zero-product) preserving map. But the converse is not valid in general. Indeed there are zero-product (Jordan zero-product) preserving maps that are not strongly zero-product (strongly Jordan zero-product) preserving maps. To emphasize this fact it is useful to create an illustrative example.

Example 3.3. Let \mathcal{V} be a normed vector space with a countable basis $\beta = \{e_1, e_2, e_3, \dots\}$ such that $\|e_n\| = 1$ for all $n \geq 1$. Also let $f \in \mathcal{V}^*$ be a continuous linear functional such that $f(e_1) = 1$ and $f(e_n) = 0$ for all $n \geq 2$. So $\ker f = \text{span}\{e_2, e_3, e_4, \dots\}$. Define $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$ such that $\theta(a) = f(a)e_1 + \varphi(a)$, where $\varphi : \mathcal{V}_f \rightarrow \ker f$ is a linear map such that $\varphi(e_1) = 0$ and $\varphi(e_n) = 2^n e_2$ for all $n \geq 2$. It is obvious that $\theta(\ker f) \subseteq \ker f$. So by Theorems 2.1 and 2.2, θ is a Jordan zero-product (zero-product) preserving map. But we show that θ is not a strongly zero-product (strongly Jordan zero-product) preserving map. To this end let $a_n = \frac{e_1}{n}$ and $c_n = e_n$. So $\|a_n c_n\| = \frac{1}{n} \|e_n\| = \frac{1}{n} \rightarrow 0$. But $\lim_{n \rightarrow \infty} \|\theta(a_n)\theta(c_n)\| = \lim_{n \rightarrow \infty} \frac{2^n}{n} \|e_2\| = \lim_{n \rightarrow \infty} \frac{2^n}{n} = \infty$. This shows that θ is not a strongly zero-product preserving map. A similar argument can be applied to show that $a_n \circ c_n \rightarrow 0$ but $\|\theta(a_n) \circ \theta(c_n)\| \rightarrow \infty$. So θ is not a strongly Jordan zero-product preserving map.

Example 3.3 shows that it is worthwhile to characterize the strongly zero-product (strongly Jordan zero-product) preserving maps on \mathcal{V}_f .

Theorem 3.4. *Let \mathcal{V} be a non-zero normed vector space and let $0 \neq f$ be a continuous linear functional on \mathcal{V} . Then a linear map $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$ is a strongly zero-product preserving map if and only if there exists a linear map $\phi : \mathcal{V}_f \rightarrow \ker f$ such that the following properties hold.*

- (1) $\phi(a) = \theta(a) - f(a)\theta(e)$ for some $e \in f^{-1}(\{1\})$.
- (2) For any two sequences $\{a_n\}_n, \{c_n\}_n$ in \mathcal{V}_f , $f(\theta(e))\phi(a_n c_n) \rightarrow 0$ whenever

$$a_n c_n \rightarrow 0.$$

Proof. Let $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$ be a strongly zero-product preserving map. As θ is a zero product preserving map it follows that $\theta(\ker f) \subseteq \ker f$. Since $f \neq 0$, let $e \in f^{-1}(\{1\})$. For each $a \in \mathcal{V}_f$, $a - f(a)e \in \ker f$ so there exists a function $\phi : \mathcal{V}_f \rightarrow \ker f$ such that $\theta(a - f(a)e) = \phi(a)$, equivalently $\theta(a) - f(a)\theta(e) = \phi(a)$. Clearly ϕ is linear because θ and f are linear. Now let $\{a_n\}_n, \{c_n\}_n$ be two

sequences such that $a_n c_n \rightarrow 0$. So $\theta(a_n)\theta(c_n) \rightarrow 0$. It follows that,

$$\begin{aligned} & (f(a_n)\theta(e) + \phi(a_n))(f(c_n)\theta(e) + \phi(c_n)) \\ &= f(a_n)f(\theta(e))(f(c_n)\theta(e) + \phi(c_n)) \\ &= f(a_n c_n)f(\theta(e))\theta(e) + f(a_n)f(\theta(e))\phi(c_n) \\ &= f(a_n c_n)f(\theta(e))\theta(e) + f(\theta(e))\phi(a_n c_n) \rightarrow 0. \end{aligned}$$

On the other hand f is continuous, so $f(a_n c_n)f(\theta(e))\theta(e) \rightarrow 0$. Hence

$$f(\theta(e))\phi(a_n c_n) = \theta(a_n)\theta(c_n) - f(a_n c_n)f(\theta(e))\theta(e) \rightarrow 0.$$

The converse is obvious. \square

Theorem 3.5. *Let \mathcal{V} be a non-zero normed vector space and let $0 \neq f$ be a continuous linear functional on \mathcal{V} . Then a linear map $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$ is a strongly Jordan zero-product preserving map if and only if there exists a linear map $\phi : \mathcal{V}_f \rightarrow \ker f$ such that the following properties hold.*

- (1) $\phi(a) = \theta(a) - f(a)\theta(e)$ for some $e \in f^{-1}(\{1\})$.
- (2) For any two sequences $\{a_n\}_n, \{c_n\}_n$ in \mathcal{V}_f , $f(\theta(e))\phi(a_n \circ c_n) \rightarrow 0$ whenever

$$a_n \circ c_n \rightarrow 0.$$

Proof. An argument quite similar to the proof of Theorem 3.4 can be applied. \square

Theorem 3.6. *Let \mathcal{V} be a non-zero normed vector space and let $0 \neq f$ be a continuous linear functional on \mathcal{V} . Then a linear map $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$ is a strongly zero-product preserving map if and only if one of the following conditions holds.*

- (1) $f \circ \theta = 0$
- (2) $\theta(\ker f) \subseteq \ker f$ and θ is continuous.

Proof. Let $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$ be a strongly zero product preserving map. As θ is a zero-product preserving map, by Theorem 2.2, $\theta(\ker f) \subseteq \ker f$. If $f \circ \theta \neq 0$ then there exists $a \in \mathcal{V}_f$ such that $f \circ \theta(a) \neq 0$. For each sequence $\{c_n\}_n$ in \mathcal{V}_f , satisfying $c_n \rightarrow 0$, since $a c_n \rightarrow 0$ so $\theta(a)\theta(c_n) \rightarrow 0$. Equivalently $f(\theta(a))\theta(c_n) \rightarrow 0$. Hence $\theta(c_n) \rightarrow 0$. So θ is continuous. For the converse, the condition $f \circ \theta = 0$ clearly implies that θ is a strongly zero-product preserving map. Let θ be continuous and $\theta(\ker f) \subseteq \ker f$. So there exists a linear map $\phi : \mathcal{V}_f \rightarrow \ker f$ such that $\phi(a) = \theta(a) - f(a)\theta(e)$, $a \in \mathcal{V}_f$ and $e \in f^{-1}(1)$. As θ and f are continuous so ϕ is continuous. Now let $\{a_n\}_n, \{c_n\}_n$ be two sequences in \mathcal{V}_f satisfying, $a_n c_n \rightarrow 0$. So

$$\begin{aligned} & \theta(a_n)\theta(c_n) \\ &= (f(a_n)\theta(e) + \phi(a_n))(f(c_n)\theta(e) + \phi(c_n)) \\ &= f(a_n)f(\theta(e))(f(c_n)\theta(e) + \phi(c_n)) \\ &= f(a_n c_n)f(\theta(e))\theta(e) + f(\theta(e))\phi(a_n c_n) \rightarrow 0. \end{aligned}$$

This shows that θ is a strongly zero-product preserving map. \square

Theorem 3.7. *Let \mathcal{V} be a non-zero normed vector space and let $0 \neq f$ be a continuous linear functional on \mathcal{V} . Then a linear map $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$ is a strongly Jordan zero-product preserving map if and only if one of the following conditions holds.*

(1) $f \circ \theta = 0$

(2) $\theta(\ker f) \subseteq \ker f$ and θ is continuous.

Proof. An argument quite similar to the proof of Theorem 3.6 can be applied. \square

Corollary 3.8. *Let \mathcal{V} be a non-zero normed vector space and let $0 \neq f$ be a continuous linear functional on \mathcal{V} . Then a linear map $\theta : \mathcal{V}_f \rightarrow \mathcal{V}_f$ is a strongly zero-product preserving map if and only if it is a strongly Jordan zero-product preserving map.*

Proof. By Theorems 3.6 and 3.7 the result is obvious. \square

4. POLYNOMIALS WITH COEFFICIENTS IN \mathcal{V}_f

From now on let \mathcal{V} be a non-zero vector space over \mathbb{C} and let $f \in \mathcal{V}^*$ be a non-zero linear functional on \mathcal{V} . It is obvious that f is an algebraic homomorphism on \mathcal{V}_f , indeed for $a, b \in \mathcal{V}_f$, $f(ab) = f(f(a)b) = f(a)f(b)$. In this section we characterize the form of a polynomial with coefficients in \mathcal{V}_f . Also we show that the algebraic fundamental theorem for polynomial equations over an arbitrary algebra is not valid in general.

Proposition 4.1. *Let \mathcal{V} be a non-zero vector space, let $f \in \mathcal{V}^*$ be a non-zero linear functional, and let $P = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$ be a polynomial over \mathcal{V}_f . Then $P = (f(a_n)f(x)^{n-1} + f(a_{n-1})f(x)^{n-2} + \dots + f(a_1))x + a_0$.*

Proof. Since $a_i x^i = a_i x^{i-1} x = f(a_i x^{i-1})x = f(a_i)f(x^{i-1})x = f(a_i)f(x)^{i-1}x$, $i \geq 1$, it follows that $P = (f(a_n)f(x)^{n-1} + f(a_{n-1})f(x)^{n-2} + \dots + f(a_1))x + a_0$. \square

It is clear that the polynomial $P = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$ is of precise degree $n \geq 1$ if and only if $f(a_n) \neq 0$. If $P = a_0$ and $a_0 \neq 0$ then P is of precise degree $n = 0$.

Theorem 4.2. *Let \mathcal{V} be a non-zero vector space, let $f \in \mathcal{V}^*$ be a non-zero linear functional, and let $P = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$ be a polynomial over \mathcal{V}_f of precise degree $n \geq 0$, with $a_0 \neq 0$. Then P has precisely n zeros in \mathcal{V}_f .*

Proof. Since the case $n = 0$ is trivial, we prove the theorem for $n \geq 1$. Let $P = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0 = 0$, that is equivalent to $(f(a_n)f(x)^{n-1} + f(a_{n-1})f(x)^{n-2} + \dots + f(a_1))x + a_0 = 0$. Since $a_0 \neq 0$ so $(f(a_n)f(x)^{n-1} + f(a_{n-1})f(x)^{n-2} + \dots + f(a_1)) \neq 0$. It follows that

$$x = \frac{-a_0}{f(a_n)f(x)^{n-1} + f(a_{n-1})f(x)^{n-2} + \dots + f(a_1)}.$$

On the other hand $P(x) = 0$ implies that $f(P(x)) = 0$. Equivalently, $f(a_n)f(x)^n + f(a_{n-1})f(x)^{n-1} + \dots + f(a_1)f(x) + f(a_0) = 0$. Let $f(x) = z$ so $f(a_n)z^n + f(a_{n-1})z^{n-1} + \dots + f(a_1)z + f(a_0) = 0$ is a usual complex polynomial

equation of precise degree $n \geq 1$ over \mathbb{C} (note that since P is of precise degree n , $f(a_n) \neq 0$). Let z_1, z_2, \dots, z_n be the solutions to the last equation. So for $i = 1, 2, \dots, n$,

$$x_i = \frac{-a_0}{f(a_n)z_i^{n-1} + f(a_{n-1})z_i^{n-2} + \dots + f(a_1)}$$

are the solutions to the polynomial equation $P = 0$. \square

In contrast to the algebraic fundamental theorem we present the following result.

Theorem 4.3. *Let \mathcal{V} be a non-zero vector space, let $f \in \mathcal{V}^*$ be a non-zero linear functional, and let $P = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1$ be a polynomial over \mathcal{V}_f of precise degree $n \geq 2$. Then P has infinitely many zeros whenever $\dim(\mathcal{V}) \geq 2$.*

Proof. Since $P = (f(a_n)f(x)^{n-1} + f(a_{n-1})f(x)^{n-2} + \dots + f(a_1))x$, so $P = 0$ implies $x = 0$ or $f(a_n)f(x)^{n-1} + f(a_{n-1})f(x)^{n-2} + \dots + f(a_2)f(x) + f(a_1) = 0$. Let $z = f(x)$ and let z_i , $i = 1, 2, \dots, n-1$ be the solutions to the equation $f(a_n)z^{n-1} + f(a_{n-1})z^{n-2} + \dots + f(a_2)z + f(a_1) = 0$. Since f is surjective so $z_i = f(x_i)$, for some $x_i \in \mathcal{V}_f$, $i = 1, 2, \dots, n-1$. For each $t \in \ker f$, $z_i = f(x_i) = f(x_i + t)$. It follows that $X = \bigcup_{i=1}^{n-1} (x_i + \ker f)$ is the set of solution to the equation $f(a_n)f(x)^{n-1} + f(a_{n-1})f(x)^{n-2} + \dots + f(a_2)f(x) + f(a_1) = 0$, when x be considered as an unknown. Where $x_i + \ker f = \{x_i + t \mid t \in \ker f\}$. Since $\dim(\ker f) \geq 1$, so $\ker f$ is an uncountable set. Hence $X \cup \{0\}$ is the set of solution to the equation $P = 0$, that is uncountable. This shows that P has infinitely many zeros in \mathcal{V}_f . \square

Remark 4.4. The previous theorem shows that the algebraic fundamental theorem for polynomial equations over an arbitrary algebra is not valid in general.

Acknowledgement. The author would like to thank the referee for careful reading of the paper and giving some useful suggestions.

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