



**SOME INTEGRAL INEQUALITIES FOR
 α -, m -, (α, m) -LOGARITHMICALLY CONVEX FUNCTIONS**

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ABSTRACT. In this paper, the authors establish some Hermite-Hadamard type inequalities by using elementary inequalities for functions whose first derivative absolute values are α -, m -, (α, m) -logarithmically convex.

1. INTRODUCTION AND PRELIMINARIES

In this section, we will present definitions and some results used in this paper.

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequalities:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

hold. This double inequality is known in the literature as the Hermite-Hadamard inequality for convex functions (see [1]- [8]).

Definition 1.1. Let I be an interval in \mathbb{R} . Then $f : I \rightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y). \tag{1.1}$$

for all $x, y \in I$ and $t \in [0, 1]$.

The concepts of α -, m - and (α, m) -logarithmically convex functions were introduced as follows.

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Definition 1.2. [1] A function $f : [0, b] \rightarrow (0, \infty)$ is said to be m -logarithmically convex if the inequality

$$f(tx + m(1-t)y) \leq [f(x)]^t [f(y)]^{m(1-t)} \quad (1.2)$$

holds for all $x, y \in [0, b]$, $m \in (0, 1]$, and $t \in [0, 1]$.

Obviously, if putting $m = 1$ in Definition 1.2, then f is just the ordinary logarithmically convex on $[0, b]$.

Definition 1.3. [8] A function $f : [0, b] \rightarrow (0, \infty)$ is said to be α -logarithmically convex if

$$f(tx + (1-t)y) \leq [f(x)]^{t^\alpha} [f(y)]^{(1-t)^\alpha} \quad (1.3)$$

holds for all $x, y \in [0, b]$, $\alpha \in (0, 1]$ and $t \in [0, 1]$.

Clearly, when taking $\alpha = 1$ in Definition 1.3, then f becomes the ordinary logarithmically convex on $[0, b]$.

Definition 1.4. [1] A function $f : [0, b] \rightarrow (0, \infty)$ is said to be (α, m) -logarithmically convex if

$$f(tx + m(1-t)y) \leq [f(x)]^{t^\alpha} [f(y)]^{m(1-t)^\alpha} \quad (1.4)$$

holds for all $x, y \in [0, b]$, $(\alpha, m) \in (0, 1] \times (0, 1]$, and $t \in [0, 1]$.

Clearly, when taking $\alpha = 1$ in Definition 1.4, then f becomes the standard m -logarithmically convex function on $[0, b]$, and, when taking $m = 1$ in Definition 1.4, then f becomes the α -logarithmically convex function on $[0, b]$.

In [3], the following theorem which was obtained by Dragomir and Agarwal contains the Hermite-Hadamard type integral inequality.

Theorem 1.5. [3, Theorem 2.2] Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the interior of I , $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \quad (1.5)$$

Theorem 1.6. [3, Theorem 2.3] Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and $p > 1$. If the new mapping $|f'(x)|^{p/(p-1)}$ is convex on $[a, b]$, then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2(p+1)^{1/p}} \left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right]^{(p-1)/p}. \end{aligned} \quad (1.6)$$

The aim of this paper is to establish some integral inequalities of Hermite-Hadamard type for α -, m -, (α, m) -logarithmically convex functions.

2. HADAMARD TYPE INEQUALITIES

In order to prove our main theorems, we need the following lemma [7].

Lemma 2.1. [7] *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{b-a}{2} \int_0^1 \int_0^1 [f'(ta + (1-t)b) - f'(sa + (1-s)b)](s-t) dt ds. \end{aligned} \quad (2.1)$$

A simple proof of this equality can be also done integrating by parts in the right hand side (see [7]).

The next theorems gives a new result of the upper Hermite-Hadamard inequality for α -, m -, (α, m) -logarithmically convex functions.

Theorem 2.2. *Let $I \supset [0, \infty)$ be an open interval and let $f : I \rightarrow (0, \infty)$ be a differentiable function on I such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|$ is (α, m) -logarithmically convex on $[0, \frac{b}{m}]$ for $(\alpha, m) \in (0, 1]^2$, then*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \begin{cases} \frac{(b-a)}{3} \left| f' \left(\frac{b}{m} \right) \right|^m, & \eta = 1 \\ \frac{(b-a)}{2} \left| f' \left(\frac{b}{m} \right) \right|^m \frac{-\alpha^2 \ln^2 \eta - 2\alpha \ln \eta + 2\eta^{\alpha-2}}{\alpha^3 \ln^3 \eta}, & \eta < 1 \end{cases} \end{aligned} \quad (2.2)$$

where $\eta = |f'(a)| / |f'(\frac{b}{m})|^m$.

Proof. By Lemma 2.1 and since $|f'|$ is an (α, m) -logarithmically convex on $[0, \frac{b}{m}]$, then we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} \int_0^1 \int_0^1 |(f'(ta + (1-t)b)) - (f'(sa + (1-s)b))| |s-t| dt ds \\ & \leq \frac{b-a}{2} \left[\int_0^1 \int_0^1 |s-t| |f'(a)|^{t\alpha} \left| f' \left(\frac{b}{m} \right) \right|^{m(1-t\alpha)} dt ds \right] \\ & \quad + \frac{b-a}{2} \left[\int_0^1 \int_0^1 |s-t| |f'(a)|^{s\alpha} \left| f' \left(\frac{b}{m} \right) \right|^{m(1-s\alpha)} dt ds \right] \end{aligned}$$

Let $0 < k \leq 1$, $0 \leq m \leq 1$, and $0 < n \leq 1$. Then

$$k^{m^n} \leq k^{nm}. \quad (2.3)$$

When $\eta = 1$, by (2.3), we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\begin{aligned} &\leq \frac{b-a}{2} \left| f' \left(\frac{b}{m} \right) \right|^m \left[\int_0^1 \int_0^1 |s-t| dt ds + \int_0^1 \int_0^1 |s-t| dt ds \right] \\ &= \frac{b-a}{3} \left| f' \left(\frac{b}{m} \right) \right|^m \end{aligned}$$

When $0 < \eta < 1$, by (2.3), we get

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ &\leq \frac{b-a}{2} \left| f' \left(\frac{b}{m} \right) \right|^m \left[\int_0^1 \int_0^1 |s-t| \eta^{\alpha t} dt ds + \int_0^1 \int_0^1 |s-t| \eta^{\alpha s} dt ds \right] \\ &= \frac{b-a}{2} \left| f' \left(\frac{b}{m} \right) \right|^m \left[\frac{-\alpha^2 \ln^2 \eta - 2\alpha \ln \eta + 4\eta^\alpha + \alpha^2 \eta^\alpha \ln^2 \eta - 2\alpha \eta^\alpha \ln \eta - 4}{2\alpha^3 \ln^3 \eta} \right. \\ &\quad \left. + \frac{-\alpha \ln \eta + 2\eta^\alpha - \alpha \eta^\alpha \ln \eta - 2}{2\alpha^2 \ln^2 \eta} \right] \end{aligned}$$

which completes the proof. \square

Corollary 2.3. Let $I \supset [0, \infty)$ be an open interval and let $f : I \rightarrow (0, \infty)$ be a differentiable function on I such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|$ is m -logarithmically convex on $[0, \frac{b}{m}]$ for $m \in (0, 1]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \begin{cases} \frac{(b-a)}{3} \left| f' \left(\frac{b}{m} \right) \right|^m, & \eta = 1 \\ \frac{(b-a)}{2} \left| f' \left(\frac{b}{m} \right) \right|^m \frac{-\ln^2 \eta - 2\ln \eta + 2\eta - 2}{\ln^3 \eta}, & \eta < 1 \end{cases}$$

where η is same as Theorem 2.2.

Corollary 2.4. Let $I \supset [0, \infty)$ be an open interval and let $f : I \rightarrow (0, \infty)$ be a differentiable function on I such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|$ is α -logarithmically convex on $[0, b]$ for $\alpha \in (0, 1]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \begin{cases} \frac{(b-a)}{3} |f'(b)|, & \eta = 1 \\ \frac{(b-a)}{2} |f'(b)| \frac{4\eta^\alpha - 4\alpha \ln \eta - 2\alpha^2 \ln^2 \eta - 4}{2\alpha^3 \ln^3 \eta}, & \eta < 1 \end{cases}$$

where $\eta = |f'(a)| / |f'(b)|$.

Theorem 2.5. Let $I \supset [0, \infty)$ be an open interval and let $f : I \rightarrow (0, \infty)$ be a differentiable function on I such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|^q$ is an (α, m) -logarithmically convex on $[0, \frac{b}{m}]$ for $(\alpha, m) \in (0, 1]^2$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} &\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \tag{2.4} \\ &\leq \begin{cases} (b-a) |f' \left(\frac{b}{m} \right)|^m \left(\frac{2}{(p+1)(p+2)} \right)^{\frac{1}{p}}, & \eta = 1 \\ (b-a) |f' \left(\frac{b}{m} \right)|^m \left(\frac{2}{(p+1)(p+2)} \right)^{\frac{1}{p}} \times \left(\frac{\eta(\alpha q, \alpha q) - 1}{\ln \eta(\alpha q, \alpha q)} \right)^{\frac{1}{q}}, & \eta < 1 \end{cases} \end{aligned}$$

where $\eta(\alpha, \alpha)$ is same as Theorem 2.2.

Proof. Since $|f'|^q$ is an (α, m) -logarithmically convex on $[0, \frac{b}{m}]$, from Lemma 2.1 and the well known Hölder inequality, we have

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{b-a}{2} \int_0^1 \int_0^1 |(f'(ta + (1-t)b)) - (f'(sa + (1-s)b))| |s-t| dt ds \\
 & \leq \frac{b-a}{2} \int_0^1 \int_0^1 |s-t| |f'(a)|^{t\alpha} \left| f' \left(\frac{b}{m} \right) \right|^{m(1-t\alpha)} dt ds \\
 & \quad + \frac{b-a}{2} \int_0^1 \int_0^1 |s-t| |f'(a)|^{s\alpha} \left| f' \left(\frac{b}{m} \right) \right|^{m(1-s\alpha)} dt ds \\
 & \leq \frac{b-a}{2} \left| f' \left(\frac{b}{m} \right) \right|^m \left(\int_0^1 \int_0^1 |s-t|^p dt ds \right)^{\frac{1}{p}} \\
 & \quad \times \left[\left(\int_0^1 \int_0^1 \eta^{qt\alpha} dt ds \right)^{\frac{1}{q}} + \left(\int_0^1 \int_0^1 \eta^{qs\alpha} dt ds \right)^{\frac{1}{q}} \right]
 \end{aligned} \tag{2.5}$$

If $\eta = 1$, by (2.3), we obtain

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq (b-a) \left| f' \left(\frac{b}{m} \right) \right|^m \left(\int_0^1 \int_0^1 |s-t|^p dt ds \right)^{\frac{1}{p}} \\
 & = (b-a) \left| f' \left(\frac{b}{m} \right) \right|^m \left(\frac{2}{(p+1)(p+2)} \right)^{\frac{1}{p}}
 \end{aligned}$$

If $\eta < 1$, by (2.3), we obtain

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{b-a}{2} \left| f' \left(\frac{b}{m} \right) \right|^m \left(\int_0^1 \int_0^1 |s-t|^p dt ds \right)^{\frac{1}{p}} \\
 & \quad \times \left[\left(\int_0^1 \int_0^1 \eta^{qt\alpha} dt ds \right)^{\frac{1}{q}} + \left(\int_0^1 \int_0^1 \eta^{qs\alpha} dt ds \right)^{\frac{1}{q}} \right] \\
 & = (b-a) \left| f' \left(\frac{b}{m} \right) \right|^m \left(\frac{2}{(p+1)(p+2)} \right)^{\frac{1}{p}} \times \left(\frac{\eta(\alpha q, \alpha q) - 1}{\ln \eta(\alpha q, \alpha q)} \right)^{\frac{1}{q}}
 \end{aligned} \tag{2.6}$$

which completes the proof. \square

Corollary 2.6. Let $I \supset [0, \infty)$ be an open interval and let $f : I \rightarrow (0, \infty)$ be a differentiable function on I such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|^q$ is

an m -logarithmically convex on $[0, \frac{b}{m}]$ for $m \in (0, 1]$ and $p = q = 2$, then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq (b-a) \left| f' \left(\frac{b}{m} \right) \right|^m \sqrt{\frac{1}{6}} \times \begin{cases} 1, & \eta = 1 \\ \left(\frac{\eta(2,2)-1}{\ln \eta(2,2)} \right)^{\frac{1}{2}}, & \eta < 1 \end{cases} \end{aligned}$$

Corollary 2.7. Let $I \supset [0, \infty)$ be an open interval and let $f : I \rightarrow (0, \infty)$ be a differentiable function on I such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|$ is α -logarithmically convex on $[0, b]$ for $\alpha \in (0, 1]$, then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq (b-a) |f'(b)| \left(\frac{2}{(p+1)(p+2)} \right)^{\frac{1}{p}} \times \begin{cases} 1, & \eta = 1 \\ \left(\frac{\eta(\alpha q, \alpha q)-1}{\ln \eta(\alpha q, \alpha q)} \right)^{\frac{1}{q}}, & \eta < 1 \end{cases} \end{aligned}$$

where $\eta = |f'(a)| / |f'(b)|$.

Theorem 2.8. Let $I \supset [0, \infty)$ be an open interval and let $f : I \rightarrow (0, \infty)$ be a differentiable function on I such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|^q$ is (α, m) -logarithmically convex on $[0, \frac{b}{m}]$ for $(\alpha, m) \in (0, 1]^2$, and then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{3} \left| f' \left(\frac{b}{m} \right) \right|^m \\ & \times \begin{cases} 1, & \eta = 1 \\ \frac{3}{2} \left(\frac{1}{3} \right)^{1-\frac{1}{q}} \left[\left(\frac{2\varphi-2}{[\ln \varphi]^3} - \frac{\varphi+1}{[\ln \varphi]^2} - \frac{1-\varphi}{2 \ln \varphi} \right)^{\frac{1}{q}} + \left(\frac{\varphi-1}{[\ln \varphi]^2} - \frac{\varphi+1}{2 \ln \varphi} \right)^{\frac{1}{q}} \right] & \eta < 1 \end{cases} \end{aligned}$$

where $\eta(\alpha, \alpha)$ is same as Theorem 2.2, and $\varphi = \eta(\alpha q, \alpha q)$.

Proof. Since $|f'|^q$ is an (α, m) -logarithmically convex on $[0, \frac{b}{m}]$, for $q \geq 1$, from Lemma 2.1 and the well known power mean integral inequality, we get

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} \int_0^1 \int_0^1 |(f'(ta + (1-t)b)) - (f'(sa + (1-s)b))| |s-t| dt ds \\ & \leq \frac{b-a}{2} \left(\int_0^1 \int_0^1 |s-t| dt ds \right)^{1-\frac{1}{q}} \left(\int_0^1 \int_0^1 |s-t| |f'(ta + (1-t)b)|^q dt ds \right)^{\frac{1}{q}} \\ & \quad + \frac{b-a}{2} \left(\int_0^1 \int_0^1 |s-t| dt ds \right)^{1-\frac{1}{q}} \left(\int_0^1 \int_0^1 |s-t| |f'(sa + (1-s)b)|^q dt ds \right)^{\frac{1}{q}} \\ & \leq \frac{b-a}{2} \left| f' \left(\frac{b}{m} \right) \right|^m \left(\int_0^1 \int_0^1 |s-t| dt ds \right)^{1-\frac{1}{q}} \left(\int_0^1 \int_0^1 |s-t| \eta^{qt\alpha} dt ds \right)^{\frac{1}{q}} \\ & \quad + \frac{b-a}{2} \left| f' \left(\frac{b}{m} \right) \right|^m \left(\int_0^1 \int_0^1 |s-t| dt ds \right)^{1-\frac{1}{q}} \left(\int_0^1 \int_0^1 |s-t| \eta^{qs\alpha} dt ds \right)^{\frac{1}{q}} \end{aligned}$$

When $\eta = 1$, by (2.3), we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \left| f' \left(\frac{b}{m} \right) \right|^m \left(\int_0^1 \int_0^1 |s-t| dt ds \right)^{\frac{1}{q}} \\ & \quad + \frac{b-a}{2} \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \left| f' \left(\frac{b}{m} \right) \right|^m \left(\int_0^1 \int_0^1 |s-t| dt ds \right)^{\frac{1}{q}} \\ & = \frac{b-a}{3} \left| f' \left(\frac{b}{m} \right) \right|^m \end{aligned}$$

When $\eta < 1$, by (2.3), we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \left| f' \left(\frac{b}{m} \right) \right|^m \left(\int_0^1 \int_0^1 |s-t| \eta^{\alpha qt} dt ds \right)^{\frac{1}{q}} \\ & \quad + \frac{b-a}{2} \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \left| f' \left(\frac{b}{m} \right) \right|^m \left(\int_0^1 \int_0^1 |s-t| \eta^{\alpha qs} dt ds \right)^{\frac{1}{q}} \\ & = \frac{b-a}{2} \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \left| f' \left(\frac{b}{m} \right) \right|^m \\ & \quad \times \left\{ \left[\frac{2\eta(\alpha q, \alpha q) - 2}{[\ln(\eta(\alpha q, \alpha q))]^3} - \frac{\eta(\alpha q, \alpha q) + 1}{[\ln(\eta(\alpha q, \alpha q))]^2} - \frac{1 - \eta(\alpha q, \alpha q)}{2 \ln(\eta(\alpha q, \alpha q))} \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\frac{\eta(\alpha q, \alpha q) - 1}{[\ln(\eta(\alpha q, \alpha q))]^2} - \frac{\eta(\alpha q, \alpha q) + 1}{2 \ln(\eta(\alpha q, \alpha q))} \right]^{\frac{1}{q}} \right\}, \end{aligned}$$

which completes the proof. \square

Corollary 2.9. Let $I \supset [0, \infty)$ be an open interval and let $f : I \rightarrow (0, \infty)$ be a differentiable function on I such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|^q$ is m -logarithmically convex on $[0, \frac{b}{m}]$ for $m \in (0, 1]$, then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \begin{cases} \frac{b-a}{3} \left| f' \left(\frac{b}{m} \right) \right|^m, & \eta = 1 \\ \frac{(b-a)}{2} \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \left| f' \left(\frac{b}{m} \right) \right|^m \left\{ \left[\frac{2\eta(q, q) - 2}{[\ln \eta(q, q)]^3} - \frac{\eta(q, q) + 1}{[\ln \eta(q, q)]^2} - \frac{1 - \eta(q, q)}{2 \ln \eta(q, q)} \right]^{\frac{1}{q}} \right. \\ \quad \left. + \left[\frac{\eta(q, q) - 1}{[\ln \eta(q, q)]^2} - \frac{\eta(q, q) + 1}{2 \ln \eta(q, q)} \right]^{\frac{1}{q}} \right\}, & \eta < 1 \end{cases} \end{aligned}$$

Corollary 2.10. Let $I \supset [0, \infty)$ be an open interval and let $f : I \rightarrow (0, \infty)$ be a differentiable function on I such that $f' \in L(a, b)$ for $0 \leq a < b < \infty$. If $|f'(x)|$

is α -logarithmically convex on $[0, b]$ for $\alpha \in (0, 1]$, then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{3} |f'(b)| \\ & \leq \begin{cases} 1, & \eta = 1 \\ \frac{3}{2} \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \left\{ \left[\frac{2\eta(\alpha q, \alpha q) - 2}{[\ln(\eta(\alpha q, \alpha q))]^3} - \frac{\eta(\alpha q, \alpha q) + 1}{[\ln(\eta(\alpha q, \alpha q))]^2} - \frac{1 - \eta(\alpha q, \alpha q)}{2 \ln(\eta(\alpha q, \alpha q))} \right]^{\frac{1}{q}} \right. \\ \quad \left. + \left[\frac{\eta(\alpha q, \alpha q) - 1}{[\ln(\eta(\alpha q, \alpha q))]^2} - \frac{\eta(\alpha q, \alpha q) + 1}{2 \ln(\eta(\alpha q, \alpha q))} \right]^{\frac{1}{q}} \right\}, & \eta < 1 \end{cases} \end{aligned}$$

where $\eta = |f'(a)| / |f'(b)|$.

Theorem 2.11. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be differentiable on I° , $a, b \in I$, with $a < b$ and $f' \in L([a, b])$. If $|f'|$ is an (α, m) -logarithmically convex $[0, \frac{b}{m}]$ for $(\alpha, m) \in (0, 1]^2$ and $\mu_1, \mu_2, \tau_1, \tau_2 > 0$ with $\mu_1 + \tau_1 = 1$ and $\mu_2 + \tau_2 = 1$, then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{2} \left| f' \left(\frac{b}{m} \right) \right|^m \quad (2.7) \\ & \times \begin{cases} \frac{2\mu_1^3}{(2\mu_1+1)(\mu_1+1)} + \frac{2\mu_2^3}{(2\mu_2+1)(\mu_2+1)} + \tau_1 + \tau_2, & \eta = 1 \\ \frac{2\mu_1^3}{(2\mu_1+1)(\mu_1+1)} + \frac{2\mu_2^3}{(2\mu_2+1)(\mu_2+1)} + \tau_1 \frac{\eta(\frac{\alpha}{\tau_1}, \frac{\alpha}{\tau_1}) - 1}{\ln \eta(\frac{\alpha}{\tau_1}, \frac{\alpha}{\tau_1})} + \tau_2 \frac{\eta(\frac{\alpha}{\tau_2}, \frac{\alpha}{\tau_2}) - 1}{\ln \eta(\frac{\alpha}{\tau_2}, \frac{\alpha}{\tau_2})}, & \eta < 1 \end{cases} \end{aligned}$$

where $\eta(\alpha, \alpha)$ is same as Theorem 2.2.

Proof. Since $|f'|^q$ is an (α, m) -logarithmically convex on $[0, \frac{b}{m}]$, from Lemma 2.1, we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \quad (2.8) \\ & \leq \frac{(b-a)}{2} \int_0^1 \int_0^1 |(f'(ta + (1-t)b)) - (f'(sa + (1-s)b))| |s-t| dt ds \\ & \leq \frac{(b-a)}{2} \left[\int_0^1 \int_0^1 |s-t| |f'(a)|^{t\alpha} \left| f' \left(\frac{b}{m} \right) \right|^{m(1-t\alpha)} dt ds \right] \\ & \quad + \frac{(b-a)}{2} \left[\int_0^1 \int_0^1 |s-t| |f'(a)|^{s\alpha} \left| f' \left(\frac{b}{m} \right) \right|^{m(1-s\alpha)} dt ds \right] \\ & = \frac{(b-a)}{2} \left| f' \left(\frac{b}{m} \right) \right|^m \left[\int_0^1 \int_0^1 |s-t| \eta^{t\alpha} dt ds + \int_0^1 \int_0^1 |s-t| \eta^{s\alpha} dt ds \right] \end{aligned}$$

for all $t \in [0, 1]$. Using the well known inequality $rt \leq \mu r^{\frac{1}{\mu}} + \tau t^{\frac{1}{\tau}}$, on the right side of (2.8), we get

$$\begin{aligned} & \int_0^1 \int_0^1 |s-t| \eta^{t\alpha} dt ds + \int_0^1 \int_0^1 |s-t| \eta^{s\alpha} dt ds \quad (2.9) \\ & \leq \mu_1 \int_0^1 \int_0^1 |s-t|^{\frac{1}{\mu_1}} dt ds + \tau_1 \int_0^1 \int_0^1 \eta^{\frac{t\alpha}{\tau_1}} dt ds \end{aligned}$$

$$+ \mu_2 \int_0^1 \int_0^1 |s-t|^{\frac{1}{\mu_2}} dt ds + \tau_2 \int_0^1 \int_0^1 \eta^{\frac{s\alpha}{\tau_2}} dt ds$$

When $\eta = 1$, by (2.3), we get

$$\begin{aligned} & \int_0^1 \int_0^1 |s-t| \eta^{t\alpha} dt ds + \int_0^1 \int_0^1 |s-t| \eta^{s\alpha} dt ds \\ & \leq \frac{2\mu_1^3}{(2\mu_1+1)(\mu_1+1)} + \frac{2\mu_2^3}{(2\mu_2+1)(\mu_2+1)} + \tau_1 + \tau_2 \end{aligned} \quad (2.10)$$

When $\eta < 1$, by (2.3), we get

$$\begin{aligned} & \int_0^1 \int_0^1 |s-t| \eta^{t\alpha} dt ds + \int_0^1 \int_0^1 |s-t| \eta^{s\alpha} dt ds \\ & \leq \mu_1 \int_0^1 \int_0^1 |s-t|^{\frac{1}{\mu_1}} dt ds + \tau_1 \int_0^1 \int_0^1 \eta^{\frac{t\alpha}{\tau_1}} dt ds \\ & \quad + \mu_2 \int_0^1 \int_0^1 |s-t|^{\frac{1}{\mu_2}} dt ds + \tau_2 \int_0^1 \int_0^1 \eta^{\frac{s\alpha}{\tau_2}} dt ds \\ & \leq \mu_1 \int_0^1 \int_0^1 |s-t|^{\frac{1}{\mu_1}} dt ds + \mu_2 \int_0^1 \int_0^1 |s-t|^{\frac{1}{\mu_2}} dt ds \\ & \quad + \tau_1 \int_0^1 \int_0^1 \eta^{\frac{\alpha t}{\tau_1}} dt ds + \tau_2 \int_0^1 \int_0^1 \eta^{\frac{\alpha s}{\tau_2}} dt ds \\ & = \frac{2\mu_1^3}{(2\mu_1+1)(\mu_1+1)} + \frac{2\mu_2^3}{(2\mu_2+1)(\mu_2+1)} \\ & \quad + \tau_1 \frac{\eta\left(\frac{\alpha}{\tau_1}, \frac{\alpha}{\tau_1}\right) - 1}{\ln \eta\left(\frac{\alpha}{\tau_1}, \frac{\alpha}{\tau_1}\right)} + \tau_2 \frac{\eta\left(\frac{\alpha}{\tau_2}, \frac{\alpha}{\tau_2}\right) - 1}{\ln \eta\left(\frac{\alpha}{\tau_2}, \frac{\alpha}{\tau_2}\right)} \end{aligned} \quad (2.11)$$

from (2.8)-(2.11), which completes the proof. \square

Corollary 2.12. *Under the assumptions of Theorem 2.11, and $\mu = \mu_1 = \mu_2 > 0$, $\tau = \tau_1 = \tau_2 > 0$ with $\mu + \tau = 1$, then we have*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{2} \left| f' \left(\frac{b}{m} \right) \right|^m \times \begin{cases} \frac{4\mu^3}{(2\mu+1)(\mu+1)} + 2\tau, & \eta = 1 \\ \frac{4\mu^3}{(2\mu+1)(\mu+1)} + 2\tau \frac{\eta\left(\frac{\alpha}{\tau}, \frac{\alpha}{\tau}\right) - 1}{\ln \eta\left(\frac{\alpha}{\tau}, \frac{\alpha}{\tau}\right)}, & \eta < 1 \end{cases} \end{aligned}$$

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