Khayyam J. Math. 7 (2021), no. 1, 32-39
DOI: 10.22034/kjm.2020.117858.1095

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

A CARTESIAN CLOSED SUBCATEGORY OF TOPOLOGICAL MOLECULAR LATTICES

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Communicated by L. Hola

Abstract. A category **C** is called Cartesian closed provided that it has finite products and for each **C**-object A, the functor $(A \times -) : A \to A$ has a right adjoint. It is well known that the category **TML** of topological molecular lattices with generalized order-homomorphisms in the sense of Wang is both complete and cocomplete, but it is not Cartesian closed. In this article, we introduce a Cartesian closed subcategory of this category.

1. Introduction

A completely distributive complete lattice is called a molecular lattice. In 1992 Wang [17] introduced his important theory called topological molecular lattice as a generalization of ordinary topological spaces, fuzzy topological spaces, and L-fuzzy topological spaces in terms of closed elements, molecules, remote neighborhoods, and generalized order-homomorphims. Then many authors characterized some topological notion in such spaces, such as convergence theories of molecular nets or ideals [3, 4], separation axioms [5, 7], generalized topological molecular lattices [6, 13, 14], and so forth.

For two molecular lattices F and G with a mapping $f : F \to G$ that preserves arbitrary joins, suppose that $\hat{f}$ denotes the right adjoint of f. Then $\hat{f} : G \to F$ is defined by $\hat{f}(y) = \bigvee\{x \in F \mid f(x) \leq y\}$ for every $y \in G$. A mapping $f : F \to G$ between molecular lattices is called a generalized order-homomorphism or an **ml**-map in this article if f and its right adjoint $\hat{f}$ both preserve arbitrary joins.

Date: Received: 3 February 2019; Revised: 4 April 2020; Accepted: 17 April 2020.
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2010 *Mathematics Subject Classification.* Primary 06D10; Secondary 06F30.
Key words and phrases. Topological molecular lattices, exponentiable object, Cartesian closed category.

For a molecular lattice L, a subset τ of L is called a cotopology on L if it is closed under arbitrary meets, finite joins, and $0, 1 \in \tau$, where 0 and 1 are the smallest and the greatest elements of L, respectively. Every element of a cotopology is called a closed element. If τ is a cotopology on L, then (L, τ) is called a topological molecular lattice (briefly, **tml**). An **ml**-map $f : (L_1, \tau_1) \to (L_2, \tau_2)$ between **tml**s is said to be continuous if $b \in \tau_2$ implies $\hat{f}(b) \in \tau_1$; see [14–17]. The category of all topological molecular lattices and continuous **ml**-maps between them is denoted by **TML**, and the category of all molecular lattices and **ml**-maps between them is denoted by **MOL**. It is well known that these categories are both complete and cocomplete, and some categorical structures of them were introduced by many authors [2, 8, 10–12, 18, 19]. In the following, readers are suggested to refer to [1] for some categorical notions.

An object A of a category **C** with finite products is called exponentiable if the functor $A \times - : \mathbf{C} \to \mathbf{C}$ has a right adjoint, and it is called Cartesian closed provided that every object of **C** is exponentiable. The category **TOP** of all topological spaces is a reflective and coreflective subcategory of **TML**; see [10]. Since **TOP** is not a Cartesian closed category, it follows that the category **TML** is not Cartesian closed. Thus, it is necessary to study the Cartesian closed subcategories of **TML** or the Cartesian closed super-categories of **TML**. The Cartesian closed super-categories of **TML** were studied in [12]. Some characterizations of exponentiable objects in some categories of topological molecular lattices were introduced in [2, 11]. This article studies the Cartesian closed subcategories of **TML**. It is shown that the category of locally compactly generated **tml**s is a Cartesian closed subcategory of **TML**. Some other interesting constructions are also presented.

2. PRELIMINARIES

In this section, we first recall the definition of extra-order introduced by Li [9]. Extra-orders are useful tools to construct molecular lattices and function spaces in topological molecular lattices.

Definition 2.1. Let P be a poset and let $\prec$ be a binary relation on P.

(a) $\prec$ is called an extra-order, if it satisfies the following conditions:
  (1) $x \prec y \Rightarrow x \leq y$,
  (2) $u \leq x \prec y \leq v \Rightarrow u \prec v$.

(b) $\prec$ satisfies the interpolation property (short by INT), if $x \prec y$ implies that there exists $z \in P$ such that $x \prec z \prec y$.

Remark 2.2. If $\prec$ is an extra-order on a poset P, then there exists a largest extra-order $\bar{\prec}$ over P contained in $\prec$ satisfying INT, that is, for $x, y \in P$, $x \bar{\prec} y$ if and only if there exists a mapping $\nu : \mathbb{Q} \cap [0, 1] \to P$ such that $\nu(0) = x$, $\nu(1) = y$ and for any pair $r, s \in \mathbb{Q} \cap [0, 1]$, if $r < s$, then $\nu(r) \prec \nu(s)$, where $\mathbb{Q}$ denotes all the rational numbers.

Definition 2.3 ([9])**.** Let $\prec$ be an extra-order satisfying INT on a poset P. A subset I of P is called a lower-Dedekind $\prec$-cut, if it satisfies the following conditions:

(1) I is a lower set, that is, $\downarrow I = I$.
(2) If $x \in I$, then there exists $y \in I$ such that $x \prec y$.

The set of all lower-Dedekind $\prec$-cuts in P ordered by subset inclusion is denoted by $Low_{\prec}(P)$. The following important result is a construction of molecular lattices using extra-order.

Theorem 2.4 ([9]). *If $\prec$ is an extra-order over P satisfying INT, then $Low_{\prec}(P)$ is a molecular lattice.*

Remark 2.5 ([11]). For a complete lattice L, an extra-order $\lhd$ is defined by $a \lhd b$ if, for every subset $S \subseteq L$, $b \leq \vee S$ implies $a \leq s$ for some $s \in S$. If L is a molecular lattice, then $\lhd$ satisfies the condition INT. Also, a complete lattice L is a molecular lattice if and only if $b = \vee \lhd (b)$, where $\lhd(b) = \{a \in L \mid a \lhd b\}$.

Remark 2.6 ([11, 12]). The binary product of two topological molecular lattices (L_1, τ_1) and (L_2, τ_2) is $(L_1 \otimes L_2, \tau)$, where $L_1 \otimes L_2 := \{D \subseteq L_1 \times L_2 \mid D = \bigcup_{(x,y)\in D} \lhd(x) \times \lhd(y)\}$, and τ is generated by the subbase $\{\hat{\pi}_1(x) \mid x \in \tau_1\} \cup \{\hat{\pi}_2(y) \mid y \in \tau_2\}$, such that the projection **ml**-maps π_1 and π_2 are defined by $\pi_1(D) = \bigvee\{x \in L_1 \mid \exists y \in L_2, (x, y) \in D\}$ and $\pi_2(D) = \bigvee\{y \in L_2 \mid \exists x \in L_1, (x, y) \in D\}$.

Theorem 2.7 ([10]). ***Top*** *is a reflective and coreflective full subcategory of* **TML** *via the embedding power functor ρ :* **Top** $\to$ **TML** *defined by $\rho(X, \tau) = (\rho(X), \tau^c)$, where $\rho(X)$ is the power set of X and τ^c is the set of all closed subsets of X.*

Since the category **Top** is not Cartesian closed, by Theorem 2.7, it follows that **TML** is not a Cartesian closed category. Li [11] introduced the concept of Isbell cotopology on function spaces for the presentation of the exponentiable objects in **TML**.

For two **tml**s (L_1, τ_1) and (L_2, τ_2), let $[L_1 \to L_2]$ denote the set of all continuous sup-preserving maps. An extra-order $\prec$ over $[L_1 \to L_2]$ is defined by $f \prec g$ if $x \lhd y$ implies $f(x) \lhd g(y)$. Now, we put $[L_2^{L_1}] := Low_{\precsim}[L_1 \to L_2]$, which is a molecular lattice.

Definition 2.8 ([11]). A subset H of a **tml** (L, τ) is called Scott open if it has the following properties:

(1) H is a lower subset of τ,
(2) If $\wedge A \in H$ for a subset A of τ, then there exist finite elements $a_1, \ldots, a_n$ in A such that $a_1 \wedge \cdots \wedge a_n \in H$.

The set of all Scott open sets on τ is written by $\sigma(\tau)$. Let (L_1, τ_1) and (L_2, τ_2) be two **tml**s, let $H \in \sigma(\tau_1)$, let $x \in \tau_2$, and let $T(H, x) := \bigvee\{E \in [L_2^{L_1}] \mid \hat{f}(x) \notin H$ for every $f \in E\}$. The cotopology on $[L_2^{L_1}]$ generated by the set $\{T(H, x) \mid H \in \sigma(\tau_1), x \in \tau_2\}$ is called the Isbell cotopology.

For a **tml** (L, τ), a binary relation $\ll$ on L is defined by $a \ll b$ if, for every subset $A \subseteq \tau$, $\wedge A \leq b$ implies that there exists a finite subset D of A such that $\wedge D \leq a$.

Definition 2.9 ([11]). A **tml** (L, τ) is called locally compact if $b = \wedge\{a \in \tau \mid a \ll b\}$ for every $b \in \tau$.

Theorem 2.10 ([11])**.** *Let (M,τ) be a* **tml***. Then the following conditions are equivalent:*

1. *(M,τ) is exponentiable in* **TML** *and for every* **tml** *L, the exponential cotopology on $[L^M]$ is the Isbell cotopology.*
2. *(M,τ) is locally compact.*
3. *The mapping $ev : (L_1^M,\eta)\otimes(M,\tau)\to(L_1,\tau_1)$ defined by $ev(D)=\vee\{f(x)\mid f\in A,(A,x)\in D\}$ is a continuous* **ml***-map for every* **tml** *(L_1,τ_1).*

3. $\mathcal{C}$-GENERATED TOPOLOGICAL MOLECULAR LATTICES

For two **tml**s (L_1,τ_1) and (L_2,τ_2), we denote by $C(L_1,L_2)$ the set of all continuous **ml**-maps from L_1 to L_2. The transpose $\bar{f} : M \to [L_2^{L_1}]$ of an **ml**-map $f : M\otimes L_1 \to L_2$ is defined by $\bar{f}(a)=f_a$, where $f_a\in[L_2^{L_1}]$ is given by $f_a(x)=f(\lhd(a)\times\lhd(x))$.

By the definition of the exponentiable objects in terms of adjoint situations [1], we have a **tml** L_1 is exponentiable if and only if, for every **tml** L_2, there exists a cotopology on $[L_2^{L_1}]$ such that for any **tml** M, the association $f\leftrightarrow\bar{f}$ is a bijection from $C(M\otimes L_1,L_2)$ to $C(M,[{L_2}^{L_1}])$. If L_1 is exponentiable, then $[L_2^{L_1}]$ equipped with the exponential cotopology (that is, the Isebll cotopology) is denoted by $L_2^{L_1}$.

Lemma 3.1. *The product of two exponentiable* **tml***s is exponentiable.*

Proof. Let L_1 and L_2 be two exponentiable **tml**s. Then for arbitrary **tml**s M and N, the following bijections hold: $C(M\otimes(L_1\otimes L_2),N)\cong C((M\otimes L_1)\otimes L_2,N)$ $\cong C(M\otimes L_1,N^{L_2})\cong C(M,(N^{L_2})^{L_1})$. Moreover, $[(N^{L_2})^{L_1}]\cong[N^{L_1\otimes L_2}]$. Hence the exponential cotopology on $[(N^{L_2})^{L_1}]$ induces an exponential cotopology on $[N^{L_1\otimes L_2}]$. □

Corollary 3.2. *The product of two locally compact* **tml***s is locally compact.*

Lemma 3.3. *The coproduct of a family of exponentiable* **tml***s is exponentiable.*

Proof. Let $\{L_i\}_{i\in I}$ be a family of exponentiable **tml**s and let M be an arbitrary **tml**. By the natural bijection $\bigotimes_i[M^{L_i}]\cong[M^{\coprod_i L_i}]$, it follows that $\bigotimes_i[M^{L_i}]$ induces an exponential cotopology on $[M^{\coprod_i L_i}]$, where $\coprod_i L_i$ is the coproduct of $\{L_i\}_{i\in I}$. □

Definition 3.4. Let $\mathcal{C}$ be a fixed collection of **tml**s, which are called generating **tml**s. A probe over **tml** (L,τ) is a continuous **ml**-map from one of the generating **tml**s to L. The $\mathcal{C}$-generated cotopology $\mathcal{C}L$ on a **tml** L is the final cotopology of the probes over L, that is, the finest cotopology making all probes continuous. We say that a **tml** (L,τ) is $\mathcal{C}$-generated, if $L=\mathcal{C}L$. The category of $\mathcal{C}$-generated **tml**s and continuous **ml**-maps between them is denoted by $\mathbf{TML}_{\mathcal{C}}$.

Lemma 3.5. *The following statements hold:*

1. *Every generating* **tml** *is $\mathcal{C}$-generated.*
2. *$\mathcal{C}$-generated* **tml***s are closed under the formation of quotients.*
3. *$\mathcal{C}$-generated* **tml***s are closed under the formation of coproducts.*
4. *Every $\mathcal{C}$-generated* **tml***s is a quotient of a coproduct of generating* **tml***s.*

Proof. (1) Consider the identity probe; then the result follows.

(2) Let $f : L_1 \to L_2$ be a quotient **ml**-map and let L_1 be $\mathcal{C}$-generated. Then the composite probes $f \circ p : C \to L_2$ suffice to generate cotopology of L_2, where $p : C \to L_1$ varies over all probes of L_1.

(3) The proof is similar to (2).

(4) Let (L, τ) be a $\mathcal{C}$-generated **tml** and let I be the set of nonclosed elements of L. Then for each $i \in I$, there exists a probe $p_i : C_i \to L$, which $\hat{p}_i(i)$ is nonclosed. If we choose the probes, then an element a of L is closed if and only if $\hat{p}_i(a)$ is closed for each $i \in I$, that is, the cotopology of L is just the final cotopology of the family $\{p_i : C_i \to L\}_{i\in I}$. We can enlarge this family, if necessary, by including all constant maps from some nonempty generating **tml**s, to get sure that all elements of L are covered by probes. Let $(J_i : C_i \to S)_{i\in I}$ be the coproduct of the **tml**s $\{C_i\}_{i\in I}$. According to the universal property of coproducts, there exists a unique **ml**-map $q : S \to L$ such that $q \circ J_i = p_i$ for all $i \in I$. If we use the universal property again, then a function $f : L \to M$ is continuous if and only if $f \circ q : S \to M$ is continuous, which shows that q is a quotient **ml**-map. □

Definition 3.6. Let (L, τ) be a **tml**. Then L is called locally compactly generated, if $\mathcal{C}$ consists of locally compact **tml**s.

Corollary 3.7. *A* **tml** *is locally compactly generated if and only if it is a quotient of a locally compact* **tml***.*

Proof. Since the class of locally compact **tml**s is closed under the coproducts and by Lemma 3.5, it follows directly. □

Definition 3.8. A class $\mathcal{C}$ of generating **tml**s is called productive if each element of $\mathcal{C}$ is exponentiable, and the product of any two generating spaces is $\mathcal{C}$-generated.

Example 3.9. The class of all locally compact **tml**s is productive.

Definition 3.10. A sup-preserving map $f : L_1 \to L_2$ is called $\mathcal{C}$-continuous if $f \circ p : C \to L_2$ is continuous for each probe $p : C \to L_1$.

It is clear that every continuous **ml**-map is $\mathcal{C}$-continuous, and for **ml**-maps defined on $\mathcal{C}$-generated **tml**s, the $\mathcal{C}$-continuity coincides with continuity. The **tml**s and $\mathcal{C}$-continuous **ml**-maps between them form a category, which is denoted by $\mathbf{Map}_{\mathcal{C}}$. It is easy to show that the identity **ml**-map $CL \to L$ is an isomorphism in $\mathbf{Map}_{\mathcal{C}}$.

Lemma 3.11. *The functor* $C : \mathbf{Map}_{\mathcal{C}} \to \mathbf{TML}_{\mathcal{C}}$ *that sends a* **tml** L *to* CL *and a* $\mathcal{C}$*-continuous map to itself is an equivalence of categories.*

Proof. Since a **tml** L is $\mathcal{C}$-generated if and only if the continuity of a function defined on L is equivalent to $\mathcal{C}$-continuity, it follows that $\mathbf{TML}_{\mathcal{C}}$ is a full subcategory of $\mathbf{Map}_{\mathcal{C}}$. On the other hand, every **tml** is isomorphic in $\mathbf{TML}_{\mathcal{C}}$ to an object of $\mathbf{Map}_{\mathcal{C}}$. □

Lemma 3.12. $\mathbf{Map}_{\mathcal{C}}$ *has finite products and they are the same as in* $\mathbf{TML}_{\mathcal{C}}$*.*

Proof. It is enough to show that $L_1 \otimes L_2$ has the universal property, where (L_1, τ_1) and (L_2, τ_2) are **tml**s. Since the projection **ml**-maps π_1 and π_2 are continuous, they are $\mathcal{C}$-continuous. Let $f_i : M \to L_i$ be an arbitrary $\mathcal{C}$-continuous **ml**-map for $i = 1, 2$. Then there exists a unique **ml**-map $f : M \to L_1 \otimes L_2$ such that $f_i = \pi \circ f$. It remains to show that f is $\mathcal{C}$-continuous. It is enough to prove that $\pi_i \circ f \circ p$ is continuous for each probe $p : C \to M$. However, $\pi_i \circ f \circ p = f_i \circ p$, which is continuous. □

Recall that an element a of a lattice L is called coprime, if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$, for every $b, c \in L$. We denote by $CP(L)$ the set of all nonzero coprime elements of L. Nonzero coprime elements are also called molecules. It is well known that if L is a molecular lattice, then $CP(L)$ is a join generating base for L, that is, every element of L is a join of some elements of $CP(L)$; see [17].

Lemma 3.13. *Let $f : M \otimes L_1 \to L_2$ be a $\mathcal{C}$-continuous* **ml**-*map. Then for each $p \in CP(M)$, the* **ml**-*map $f_p : L_1 \to L_2$ defined by $f_p(a) = f(\lhd(p) \times \lhd(a))$, is $\mathcal{C}$-continuous.*

Proof. For each $p \in CP(M)$, the **ml**-map $g_p : L_1 \to M \otimes L_1$ defined by $g_p(a) = \lhd(p) \times \lhd(a)$ is $\mathcal{C}$-continuous, because it is continuous. Since $f \circ g_p = f_p$, the result follows directly. □

For two **tml**s (L_1, τ_1) and (L_2, τ_2), and a class $\mathcal{C}$ of generating **tml**s, let $[L_1 \to_{\mathcal{C}} L_2]$ denote the set of all $\mathcal{C}$-continuous sup-preserving maps from L_1 to L_2. Define an extra-order $\prec_{\mathcal{C}}$ over $[L_1 \to_{\mathcal{C}} L_2]$ by $f \prec_{\mathcal{C}} g$ if whenever $x \lhd y$, then $f(x) \lhd g(y)$. Now, we put $[L_2^{L_1}]_{\mathcal{C}} := Low_{\precsim}[L_1 \to_{\mathcal{C}} L_2]$, which is a molecular lattice.

Let $f : M \otimes L_1 \to L_2$ be a $\mathcal{C}$-continuous **ml**-map. Then we have a function $\bar{f} : M \to [{L_2}^{L_1}]_{\mathcal{C}}$ such that $\bar{f}(x) = f_x$. Suppose that any element of $\mathcal{C}$ is exponentiable. Now, we define a cotopology on $[{L_2}^{L_1}]_{\mathcal{C}}$. Each probe $p : C \to L_1$ induces a function $T_p : [{L_2}^{L_1}]_{\mathcal{C}} \to L_2^C$ defined by $T_p(g) = g \circ p$. We endow $[{L_2}^{L_1}]_{\mathcal{C}}$ with initial cotopology of the family of functions that arise in this way, obtaining a **tml** $({L_2}^{L_1})_{\mathcal{C}}$. Thus, for any **tml** L, a function $h : L \to ({L_2}^{L_1})_{\mathcal{C}}$ is continuous if and only if $T_p \circ h : L \to L_2^C$ is continuous for each probe $p : C \to L_1$.

Lemma 3.14. *Let $\mathcal{C}$ be a class of productive* **tml**s.

(1) *An* **ml**-*map $h : L \to ({L_2}^{L_1})_{\mathcal{C}}$ is continuous if and only if the* **ml**-*map $g : L \otimes C \to L_2$ defined by $g(\lhd(l) \times \lhd(c)) = (h(l))(p(c))$ is continuous, for each probe $p : C \to L_1$.*
(2) *The transpose $\bar{f} : M \to ({L_2}^{L_1})_{\mathcal{C}}$ of a function $f : M \otimes L_1 \to L_2$ is $\mathcal{C}$-continuous if and only if the function $f \circ (p \otimes q) : C_1 \otimes C_2 \to L_2$ is continuous, for all probes $p : C_1 \to M$ and $q : C_2 \to L_1$.*
(3) *A function $f : M \otimes L_1 \to L_2$ is $\mathcal{C}$-continuous if and only if the function $f \circ (p \otimes q) : C_1 \otimes C_2 \to L_2$ is continuous, for all probes $p : C_1 \to M$ and $q : C_2 \to L_1$.*

Proof. (1) For each probe $p : C \to L_1$, a function $h : L \to ({L_2}^{L_1})_{\mathcal{C}}$ is continuous if and only if $T_p \circ h : L \to L_2^C$ is continuous. Since $T_p \circ h$ is the transpose of $g : L \otimes C \to L_2$ and C is exponentiable in **TML**, we have $T_p \circ h$ is continuous if and only if g is continuous.

(2) The function $\bar{f}$ is $\mathcal{C}$-continuous if and only if $\bar{f} \circ p : C_1 \to ({L_2}^{L_1})_{\mathcal{C}}$ is continuous for each probe $p : C_1 \to M$. According to the previous part, $\bar{f} \circ p$

is continuous if and only if $g : C_1 \otimes C_2 \to L_2$ is continuous, for each probe $q : C_2 \to L_1$, where $g(\lhd(x) \times \lhd(y)) = ((\bar{f} \circ p)(x))(q(y))$. Since $((\bar{f} \circ p)(x))(q(y)) = f(\lhd(p(x)) \times \lhd(q(y))) = f \circ (p \otimes q)(\lhd(x) \times \lhd(y))$, the result follows.

(3) Since $\mathcal{C}$ is productive, $C_1 \otimes C_2$ is $\mathcal{C}$-generated. Therefore, $f \circ (p \otimes q)$ is continuous if and only if it is $\mathcal{C}$-continuous. However, $f \circ (p \otimes q)$ is $\mathcal{C}$-continuous, so it is continuous.

For converse, let $r : C \to M \otimes L_1$ be an arbitrary probe over $M \otimes L_1$. We will show that $f \circ r$ is continuous. The probes $p : C \to M$ and $q : C \to L_1$ are obtained by composing r with the projections. Since $d : C \to C \otimes C$ is continuous, then $f \circ (p \otimes q) \circ d = f \circ r : C \to L_2$ is continuous. Thus f is $\mathcal{C}$-continuous. □

Now, by Lemmas 3.11 and 3.14, we have the following main results.

Theorem 3.15. *If $\mathcal{C}$ is productive, then* $\mathbf{Map}_{\mathcal{C}}$ *is Cartesian closed and so is* $\mathbf{TML}_{\mathcal{C}}$*.*

Corollary 3.16. *The category of locally compactly generated* **tml***s is a Cartesian closed subcategory of* **TML***.*

Remark 3.17. Let 1 be the one point **tml**. Since discreet molecular lattices are 1-generated, it follows that they form a Cartesian closed category. This of course amounts to the familiar fact that the category **MOL** is Cartesian closed.

4. Conclusion

Since the category **TOP** of all topological spaces is a reflective and coreflective subcategory of the category **TML** of all topological molecular lattices, it follows that **TML** is not Cartesian closed. In this article, we presented a Cartesian closed subcategory of **TML**. We defined the concept of a locally compactly generated topological molecular lattice, and it was shown that the category of locally compactly generated topological molecular lattices is a Cartesian closed subcategory of **TML**.

Acknowledgment. We are very grateful to the referees for the careful reading of the article and for the useful comments.

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