Khayyam J. Math. 7 (2021), no. 1, 65–76
DOI: 10.22034/kjm.2020.209741.1637

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

ON $\mathcal{SIH}$ PROPERTY AND $\mathcal{SSIH}$ PROPERTY IN TOPOLOGICAL SPACES

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Communicated by L. Hola

Abstract. We further investigated the $\mathcal{SSIH}$ and $\mathcal{SIH}$ properties introduced by Das et al. recently. It is shown that the regular-closed G_δ subspace of $\mathcal{SSIH}$ (resp., $\mathcal{SIH}$) is not $\mathcal{SSIH}$ (resp., $\mathcal{SIH}$). The preservation properties of these spaces are studied under some maps. Also $\mathcal{SSIH}$ and $\mathcal{SIH}$ properties are investigated in Alexandroff spaces.

1. Introduction

The study of topological properties using ideals is not a new idea in topological spaces. The study of selection principles in topology and their relations to the game theory and Ramsey theory was started by Scheepers [23] (see also [14]). In the last two decades, it has gained enough importance to become one of the most active areas of set theoretic topology. Hence the study of covering properties is an active area for research. In covering properties, the Hurewicz property is one of the most important property. A number of the results in the literature shows that many topological properties can be described and characterized in terms of star covering properties (see [1,7,19,20]). The method of stars has been used to study the problem of metrization of topological spaces and to define several important classical topological notions. We apply such a method in the investigation of selection principles for topological spaces.

In 1925, Hurewicz [11] (see also [12]) introduced the Hurewicz property in topological spaces. This property is stronger than Lindelöf and weaker than σ-compactness. In 2004, Bonanzinga, Cammaroto, and Kočinac [2] introduced

Date: Received: 29 November 2019; Revised: 6 May 2020; Accepted: 5 June 2020.
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2010 *Mathematics Subject Classification.* Primary 54D20; Secondary 54B20.

Key words and phrases. Hurewicz space, Stone-Ćech compactification, strongly star-$\mathcal{I}$-Hurewicz, star-$\mathcal{I}$-Hurewicz property.

the star version of the Hurewicz property and also introduced relativization of the strongly star-Hurewicz property. Continuing this, in 2013, Song and Li [26] introduced the almost Hurewicz property in topological spaces. In 2016, Kočinac [16] introduced and studied the notion of mildly Hurewicz property. In 2018, Das, Chandra, and Samanta [5] introduced strongly star-$\mathcal{I}$-Hurewicz and star-$\mathcal{I}$-Hurewicz properties in topological spaces. Continuing these investigations, Tyagi, Singh, and Bhardwaj [29] (see also [30]) studied these properties and introduced star-K-$\mathcal{I}$-Hurewicz notion in topological spaces. Here we further study the strongly star-$\mathcal{I}$-Hurewicz and star-$\mathcal{I}$-Hurewicz properties and provided some examples.

This paper is organized as follows: In section 2, the definitions of the terms used in this paper are provided. In section 3, the concept of strongly star-$\mathcal{I}$-Hurewicz property is studied further and its preservation is investigated. In section 4, the hereditary property of star-$\mathcal{I}$-Hurewicz property is studied.

2. Preliminaries

Let (X, τ) or X be a topological space. We denote by $Cl(A)$ and $Int(A)$ the closure of A and the interior of A, for a subset A of X, respectively. The cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal and ω_1 the first uncountable cardinal. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. For the terms and symbols that we do not define follow [8]. The basic definitions are given.

Here, as usual, for a subset A of a space X and a collection $\mathcal{P}$ of subsets of X, $St(A, \mathcal{P})$ denotes the star of A with respect to $\mathcal{P}$, that is, the set $\bigcup\{P \in \mathcal{P} : A \cap P \neq \emptyset\}$, and for $A = \{x\}$, $x \in X$, we write $St(x, \mathcal{P})$ instead of $St(\{x\}, \mathcal{P})$.

A nonempty collection $\mathcal{I}$ of subsets of X is called an ideal if it has the following properties:

(i) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$ (hereditary).

(ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$ (finite additivity).

For more details of ideals follow [13] and [18].

Throughout the paper, $\mathcal{I}$ denotes the proper admissible ideal of subsets of natural numbers.

In this paper, $\mathcal{A}$ and $\mathcal{B}$ will be collections of the following open covers of a space X:

$\mathcal{O}$: The collection of all open covers of X.

Ω: The collection of all ω-covers of X. An open cover $\mathcal{U}$ of X is an ω-cover [10] if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in an element of $\mathcal{U}$.

Γ: The collection of all γ-covers of X. An open cover $\mathcal{U}$ of X is a γ-cover [10] if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

$\mathcal{O}^{gp}$: The collection of all groupable open covers of X. An open cover $\mathcal{U}$ of X is groupable if it can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, such that each $x \in X$ belongs to $\bigcup \mathcal{U}_n$ for all but finitely many n; see [17].

$\mathcal{I}-\mathcal{O}^{gp}$: The collection of all $\mathcal{I}$-groupable open covers of X. An open cover $\mathcal{U}$ of X is $\mathcal{I}$-groupable if it can be represented in the form $\mathcal{U}=\bigcup_{n\in\omega}\mathcal{U}_n$, where $\mathcal{U}_n$'s are finite, pairwise disjoint and each $x\in X$, $\{n\in\omega : x\notin\bigcup\mathcal{U}_n\}\in\mathcal{I}$; see [6].

Definition 2.1 ([11])**.** A space X is said to have *Hurewicz property* (in short H) if for each sequence $<\mathcal{U}_n : n\in\omega>$ of open covers of X, there is a sequence $<\mathcal{V}_n : n\in\omega>$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x\in X$ belongs to $\bigcup\mathcal{V}_n$ for all but finitely many n.

Definition 2.2 ([2])**.** A space X is said to have *star-Hurewicz property* (in short SH) if for each sequence $<\mathcal{U}_n : n\in\omega>$ of open covers of X, there is a sequence $<\mathcal{V}_n : n\in\omega>$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x\in X$ belongs to $St(\bigcup\mathcal{V}_n,\mathcal{U}_n)$ for all but finitely many n.

Definition 2.3 ([2])**.** A space X is said to have *strongly star-Hurewicz property* (in short SSH) if for each sequence $<\mathcal{U}_n : n\in\omega>$ of open covers of X, there is a sequence $<A_n : n\in\omega>$ of finite subsets of X such that each $x\in X$ belongs to $St(A_n,\mathcal{U}_n)$ for all but finitely many n.

Let $\mathcal{I}$ be a proper admissible ideal of subsets of natural numbers. Then we have following definitions.

Definition 2.4 ([4])**.** A space X is said to have *$\mathcal{I}$-Hurewicz property* (in short $\mathcal{I}H$) if for each sequence $<\mathcal{U}_n : n\in\omega>$ of open covers of X, there is a sequence $<\mathcal{V}_n : n\in\omega>$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x\in X$, $\{n\in\omega : x\notin\bigcup\mathcal{V}_n\}\in\mathcal{I}$.

Definition 2.5 ([5])**.** A space X is said to have *star-$\mathcal{I}$-Hurewicz property* (in short $S\mathcal{I}H$) if for each sequence $<\mathcal{U}_n : n\in\omega>$ of open covers of X, there is a sequence $<\mathcal{V}_n : n\in\omega>$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x\in X$, $\{n\in\omega : x\notin St(\bigcup\mathcal{V}_n,\mathcal{U}_n)\}\in\mathcal{I}$.

Definition 2.6 ([5])**.** A space X is said to have *strongly star-$\mathcal{I}$-Hurewicz property* (in short $SS\mathcal{I}H$) if for each sequence $<\mathcal{U}_n : n\in\omega>$ of open covers of X, there is a sequence $<V_n : n\in\omega>$ of finite subsets of X such that for each $x\in X$, $\{n\in\omega : x\notin St(V_n,\mathcal{U}_n)\}\in\mathcal{I}$.

Definition 2.7 ([7, 20])**.** A topological space X is strongly starcompact if for every open cover $\mathcal{U}$ of X, there exists a finite subset A of X such that $X=St(A,\mathcal{U})$.

Call a space σ-strongly starcompact if it is a union of countably many strongly starcompact spaces.

3. $SS\mathcal{I}H$ PROPERTY

From [5], we have the following theorem.

Theorem 3.1 ([5])**.** *The $SS\mathcal{I}H$ property is preserved under continuous mappings.*

Now we turn to preimages of an $SS\mathcal{I}H$ space. We shall give an example showing that the preimage of an $SS\mathcal{I}H$ space under a closed 2-to-1 continuous map does not need to be $SS\mathcal{I}H$.

Recall that a family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is said to be almost disjoint if every element of $\mathcal{A}$ is infinite and the sets $A \cap B$ are finite for all distinct elements $A, B \in \mathcal{A}$. Let $\psi(\mathcal{A}) = \mathcal{A} \cup \mathbb{N}$ for an almost disjoint family $\mathcal{A}$. A topology on $\psi(\mathcal{A})$ is defined as follows. The natural numbers are isolated and for each element $A \in \mathcal{A}$ and each finite set $F \subseteq \mathbb{N}$, $\{A\} \cup (A \setminus F)$ is a basic open neighborhood of A. The spaces constructed in this manner are called Isbell–Mrówka ψ-spaces; see [22].

Define ω^ω as the set of all functions from ω to itself. For all $f, g \in \omega^\omega$, we say $f \leq^\star g$ if and only if $f(n) \leq g(n)$ for all but finitely many n. The unbounding number, denoted by $\mathfrak{b}$, is the smallest cardinality of an unbounded subset of $(\omega^\omega, \leq^\star)$. The cardinality of continuum is denoted by $\mathfrak{c}$ and CH denotes the continuum hypothesis. It is not difficult to show that $\omega_1 \leq \mathfrak{b} \leq \mathfrak{c}$. For an ideal $\mathcal{I}$ of subsets of ω, an ideal version of the bounding number $\mathfrak{b}(\mathcal{I})$ was introduced in [9], where $\mathfrak{b}(\mathcal{I}) = \min\{|\mathcal{P}| : \mathcal{P} \subseteq \omega^\omega$ and for all $g \in \omega^\omega$, there is $f \in \mathcal{P}$ such that $\{n \in \omega : f(n) \geq g(n)\} \in \mathcal{I}^+\}$, where $\mathcal{I}^+ = \{A \subseteq \mathbb{N} : A \notin \mathcal{I}\}$. It was established in [9, 28] that $\mathfrak{b} \leq \mathfrak{b}(\mathcal{I}) \leq \mathfrak{c}$.

Theorem 3.2 ([5])**.** *Assume $\neg CH$. Then a space $X = \psi(\mathcal{A})$ has the $SS\mathcal{I}H$ property if and only if $|\mathcal{A}| < \mathfrak{b}(\mathcal{I})$.*

Let X be any topological space. The Alexandroff duplicate $A(X)$ of a space X is $X \times \{0, 1\}$. Each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $< x, 0 > \in X \times \{0\}$ is of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{< x, 1 >\})$, where U is a neighborhood of x in X.

Example 3.3. We assume that $\neg$CH and that $\mathfrak{b} = \mathfrak{c}$. There exists a closed 2-to-1 continuous map $f : X \to Y$ such that Y is an $SS\mathcal{I}H$ space, but X is not $SS\mathcal{I}H$.

Proof. An example can be obtained by a simple modification to [24, Example 2.3] and using Theorem 3.2 and [5, Proposition 4.5]. □

From the proof of [24, Theorem 2.4] and using [5, Proposition 4.5], it is easy to show the following result.

Theorem 3.4. *If X is a T_1-space and $A(X)$ is an $SS\mathcal{I}H$ space, then $e(X) < \omega_1$.*

Recall from [15] that a space X is said to have *strongly star-Menger property* (in short SSM) if for each sequence $< \mathcal{U}_n : n \in \omega >$ of open covers of X, there is a sequence $< A_n : n \in \omega >$ of finite subsets of X such that each $x \in X$ belongs to $St(A_n, \mathcal{U}_n)$ for some n.

Lemma 3.5 ([25])**.** *For the T_1-space X, $e(X) = e(A(X))$.*

Theorem 3.6. *If X is a T_1-space having the $SS\mathcal{I}H$ property with $e(X) < \omega_1$, then $A(X)$ has the SSM property.*

Proof. Let $< \mathcal{U}_n : n \in \omega >$ be a sequence of open covers of $A(X)$. For each n and each $x \in X$, choose an open neighborhood $W_{n_x} = (V_{n_x} \times \{0, 1\}) \setminus \{< x, 1 >\}$ of $< x, 0 >$ such that there exists some $U \in \mathcal{U}_n$ with $W_{n_x} \subseteq U$, where V_{n_x} is an open subset of X containing x. For each n, let $\mathcal{V}_n = \{V_{n_x} : x \in X\}$. Then $\mathcal{V}_n$ is an open cover of X and $< \mathcal{V}_n : n \in \omega >$ is a sequence of open covers of X. By applying the $SS\mathcal{I}H$ property of X, we get a sequence $< F_n : n \in \omega >$ of finite subsets of X such that for each $x \in X$, $\{n \in \omega : x \notin St(Fn, \mathcal{V}_n)\} \in \mathcal{I}$.

For each n, let $F_n^{'} = A(F_n)$. Then $F_n^{'}$ is a finite subset of $A(X)$. Thus $< F_n^{'} : n \in \omega >$ is a sequence of finite subsets of $A(X)$ and for each $x \in X$, $\{n \in \omega :< x, 0 >\notin St(F_n^{'}, \mathcal{U}_n)\} \in \mathcal{I}$. Hence $X \times \{0\} \subseteq \bigcup_{n\in\omega} St(F_n^{'}, \mathcal{U}_n)$. Let $A = A(X) \setminus \bigcup_{n\in\omega} St(F_n^{'}, \mathcal{U}_n)$. Then A is a discrete closed subset of $A(X)$. By Lemma 3.5, the set A is countable. We can enumerate A as $\{a_n : n \in \omega\}$. For each n, let $F_n^{''} = F_n^{'} \cup \{a_1, a_2, \ldots, a_n\}$. Then $F_n^{''}$ is a finite subset of $A(X)$, and hence for each $y \in A(X)$, $y \in St(F_n^{''}, \mathcal{U}_n)$ for some n. □

A function f from a topological space X to a topological space Y is said to be a perfect map (see [8]) if

(1) f is onto;
(2) f is continuous;
(3) f is a closed map;
(4) $f^{-1}(y)$ is compact in X for each $y \in Y$.

Now we give a positive result for the $SS\mathcal{I}H$ property to be inverse invariant. However the following result given in [5] using open perfect maps is not true by considering [24, Remark 2.9]. In fact, Example 3.12 given in this paper also shows a counterexample for the following theorem.

Theorem 3.7 ([5]). *The $SS\mathcal{I}H$ property is inverse invariant under perfect open maps.*

Replacement of the word "perfect" by "finite-to-one closed" makes the proof of above theorem valid.

Theorem 3.8. *The $SS\mathcal{I}H$ property is inverse invariant under finite-to-one open and closed maps.*

Proof. The proof is similar as [5, Proposition 4.6]. □

Theorem 3.9. *The $SS\mathcal{I}H$ property is inverse invariant under open bijective maps.*

Proof. Let f be an open bijective map from a space X to a space Y having the $SS\mathcal{I}H$ property and let $< \mathcal{U}_n : n \in \omega >$ be a sequence of open covers of X. Then for each n, $\mathcal{V}_n = \{f(U) : U \in \mathcal{U}_n\}$ is an open cover of Y. Since Y has the $SS\mathcal{I}H$ property, there is a sequence $< W_n : n \in \omega >$ of finite subsets of Y such that for each $y \in Y$, $\{n \in \omega : y \notin St(W_n, \mathcal{V}_n)\} \in \mathcal{I}$. Now assume that for each n, $W_n = \{f(x_1), f(x_2), \ldots, f(x_k)\}$ and $G_n = \{x_1, x_2, \ldots, x_k\}$. Then the sequence $< G_n : n \in \omega >$ witnesses the $SS\mathcal{I}H$ property of X for the sequence $< \mathcal{U}_n : n \in \omega >$.

For each $x \in X$, $f(x) \in Y$ and $\{n \in \omega : f(x) \notin St(W_n, \mathcal{V}_n)\} \in \mathcal{I}$. This implies that $f(x) \in f(U) \in \mathcal{V}_n$ and $W_n \cap f(U) \neq \emptyset$. Since f is one-to-one, $G_n \cap U \neq \emptyset$ and $x \in U$. Hence $\{n \in \omega : x \notin St(G_n, \mathcal{U}_n)\} \subseteq \{n \in \omega : f(x) \notin St(W_n, \mathcal{V}_n)\} \in \mathcal{I}$. □

Theorem 3.10. *The $SS\mathcal{I}H$ property is closed under countable unions.*

Proof. Let $\{X_k : k \in \omega\}$ be a family of subspaces having the $SS\mathcal{I}H$ property in a space X and let $< \mathcal{U}_n : n \in \omega >$ be a sequence of open covers of X. For each $k \in \omega$, consider the sequence $< \mathcal{U}_n : n \geq k >$ and define $< \mathcal{U}_n^k : n \geq k >$, where $\mathcal{U}_n^k = \{U \cap X_k : U \in \mathcal{U}_n\}$. Then $< \mathcal{U}_n^k : n \in \omega >$ is a sequence of open covers of X_k.

For each $k \in \omega$, since X_k is strongly star-$\mathcal{I}$-Hurewicz, there is a sequence $< V_{n,k} : n \geq k >$ such that for each $n \geq k$, $V_{n,k}$ is a finite subset of X_k and for each $x \in X_k$, $\{n \geq k : x \notin St(V_{n,k}, \mathcal{U}_n^k)\} \in \mathcal{I}$. For each n, let $V_n = \bigcup\{V_{n,j} : j \leq n\}$. Then each V_n is a finite subset of X. Then for each $x \in X$, $x \in \bigcup X_k$. There exists $k \in \omega$ such that $x \in X_k$. Thus, $\{n \in \omega : x \notin St(V_{n,k}, \mathcal{U}_n^k)\} \in \mathcal{I}$. Then $St(V_{n,k}, \mathcal{U}_n^k) \subseteq St(V_n, \mathcal{U}_n)$ for all $n > k$, and hence $\{n \in \omega : x \notin St(V_n, \mathcal{U}_n)\} \in \mathcal{I}$. □

Lemma 3.11. *A space X is an $SS\mathcal{I}H$ space if and only if for every sequence $< \mathcal{U}_n : n \in \omega >$ of open covers of X, there exists a sequence $< A_n : n \in \omega >$ of finite subsets of X such that for every $x \in X$, $\{n \in \omega : St(x, \mathcal{U}_n) \cap A_n \neq \emptyset\} \in \mathcal{F}(\mathcal{I})$.*

Proof. The proof is easy and thus omitted. □

The following example shows that the product of an $SS\mathcal{I}H$ space and a compact space does not need to be $SS\mathcal{I}H$.

Example 3.12. We assume that $\neg CH$ and that $\mathfrak{b} = \mathfrak{c}$. There exist an $SS\mathcal{I}H$ space X and a compact space Y such that $X \times Y$ is not $SS\mathcal{I}H$ space.

Proof. An example can be obtained by a simple modification to [24, Example 2.8] and using Theorem 3.2 and Lemma 3.11. □

Also this example shows that the preimage of an $SS\mathcal{I}H$ space under an open perfect map does not need to be an $SS\mathcal{I}H$ space. The authors do not know if in ZFC, there exist an $SS\mathcal{I}H$ space X and a compact space Y such that $X \times Y$ is not $SS\mathcal{I}H$.

However the product of two $SS\mathcal{I}H$ spaces does not need to be $SS\mathcal{I}H$. In fact, the product of two countably compact spaces does not need to be $SS\mathcal{I}H$.

Example 3.13. There exist two Tychonoff countably compact (hence $SS\mathcal{I}H$) spaces X and Y such that $X \times Y$ is not $SS\mathcal{I}H$.

Proof. An example can be obtained by a simple modification to [24, Example 2.10] and using [5, Proposition 4.5]. □

Recall that a space X is star Lindelöf [20] if for every open cover $\mathcal{U}$ of X, there exists a countable set $A \subseteq X$ such that $St(A, \mathcal{U}) = X$.

Lemma 3.14. *An $SS\mathcal{I}H$ space is SSM and hence a star-Lindelöf space.*

Proof. Let X be a space having the $SS\mathcal{I}H$ property and let $< \mathcal{U}_n : n \in \omega >$ be a sequence of open covers of X. By the $SS\mathcal{I}H$ property of X, there is a sequence $< V_n : n \in \omega >$ of finite subsets of X such that for each $x \in X$, $\{n \in \omega : x \notin St(V_n, \mathcal{U}_n)\} \in \mathcal{I}$. Now to show $X = \bigcup_{n\in\omega} St(V_n, \mathcal{U}_n)$, if possible suppose that there is $x \in X$ such that $x \notin \bigcup St(V_n, \mathcal{U}_n)$. Then $x \notin St(V_n, \mathcal{U}_n)$ for all $n \in \omega$, and hence $\{n \in \omega : x \notin St(V_n, \mathcal{U}_n)\} = \omega \in \mathcal{I}$, a contradiction. □

Now [7, Example 3.3.3] given by van Douwen et al. proved that there exist a countably compact (hence $SS\mathcal{I}H$) space X and a Lindelöf space Y such that $X \times Y$ is not star-Lindelöf. Therefore, this example shows that the product of an $SS\mathcal{I}H$ space X and a Lindelöf space Y does not need to be $SS\mathcal{I}H$, since every $SS\mathcal{I}H$ space is star-Lindelöf.

Recall that a space X is meta-Lindelöf if every open cover $\mathcal{U}$ of X has a point countable open refinement.

Theorem 3.15. *Every meta-Lindelöf space with $SS\mathcal{I}H$ property is Lindelöf.*

Proof. Let X be a meta-Lindelöf space having the $SS\mathcal{I}H$ property and let $\mathcal{U}$ be an open cover of X. Then there is a point countable open refinement $\mathcal{V}$ of $\mathcal{U}$. Since X has the $SS\mathcal{I}H$ property, there is a sequence $< A_n : n \in \omega >$ of finite subsets of X such that for each $x \in X$, $\{n \in \omega : x \notin St(A_n, \mathcal{V})\} \in \mathcal{I}$. Since $\mathcal{I}$ is a proper admissible ideal, $x \in St(A_n, \mathcal{V})$ for infinitely many n. For each n, let $\mathcal{V}_n = \{V \in \mathcal{V} : V \cap A_n \neq \emptyset\}$. Then $\mathcal{V}_n$ is a countable subset of $\mathcal{V}$. Let $\mathcal{W} = \bigcup \mathcal{V}_n$. Then $\mathcal{W}$ is a countable open cover of X. For each $V \in \mathcal{W}$, choose $U_V \in \mathcal{U}$ such that $V \subseteq U_V$. Then $\{U_V : V \in \mathcal{W}\}$ is a countable subcover of $\mathcal{U}$, which shows that X is Lindelöf. □

Recall that a space X is para-Lindelöf if every open cover $\mathcal{U}$ of X has a locally countable open refinement.

Corollary 3.16. *A para-Lindelöf space with $SS\mathcal{I}H$ property is Lindelöf.*

Corollary 3.17. *Let X be a space having the $SS\mathcal{I}H$ property. Then X is meta-Lindelöf if and only if it is Lindelöf.*

Corollary 3.18. *Let X be a space having the $SS\mathcal{I}H$ property. Then X is para-Lindelöf if and only if it is Lindelöf.*

4. $S\mathcal{I}H$ PROPERTY

Theorem 4.1. *The $S\mathcal{I}H$ property is preserved under continuous mappings.*

Proof. Let f be a continuous function from a space X having the $S\mathcal{I}H$ property onto a space Y and let $< \mathcal{U}_n : n \in \omega >$ be a sequence of open covers of Y. Since f is continuous and onto, then for each n, $f^{-1}(\mathcal{U}_n) = \{f^{-1}(U) : U \in \mathcal{U}_n\}$ is an open cover of X. Hence $< f^{-1}(\mathcal{U}_n) : n \in \omega >$ is a sequence of open covers of X. Since X has the $S\mathcal{I}H$ property, there is a sequence $< \mathcal{V}_n : n \in \omega >$ such that for each n, $\mathcal{V}_n$ is a finite subset of $f^{-1}(\mathcal{U}_n)$ and for each $x \in X$, $\{n \in \omega : x \notin St(\bigcup \mathcal{V}_n, f^{-1}(\mathcal{U}_n)\} \in \mathcal{I}$. Then for each n, $f(\mathcal{V}_n) = \{U \in \mathcal{U}_n : f^{-1}(U) \in \mathcal{V}_n\}$ is finite subset of $\mathcal{U}_n$ since $\mathcal{V}_n$ is finite. Thus the sequence $< f(\mathcal{V}_n) : n \in \omega >$ witnesses the star-$\mathcal{I}$-Hurewicz property of Y.

For each $y \in Y$, there is $a \in X$ such that $y = f(a)$. Then $\{n \in \omega : a \notin St(\bigcup \mathcal{V}_n, f^{-1}(\mathcal{U}_n))\} \in \mathcal{I}$. This implies that $a \in f^{-1}(U)$ for some $f^{-1}(U) \in f^{-1}(\mathcal{U}_n)$ and $f^{-1}(U) \cap (\bigcup \mathcal{V}_n) \neq \emptyset$. Then $f(a) \in U \in \mathcal{U}_n$ and $f(\mathcal{V}_n) \cap U \neq \emptyset$ imply that $\{n \in \omega : y = f(a) \notin St(\bigcup f(\mathcal{V}_n), \mathcal{U}_n)\} \in \mathcal{I}$. □

Now we turn to preimages of spaces having the $S\mathcal{I}H$ property. We shall give an example showing that the preimage of a space having the $S\mathcal{I}H$ property under a closed 2-to-1 continuous map does not need to have the $S\mathcal{I}H$ property.

Example 4.2. We assume that $\neg$CH and that $\mathfrak{b} = \mathfrak{c}$. Then there exists a closed 2-to-1 continuous map $f : X \to Y$ such that space Y has the $S\mathcal{I}H$ property, but X does not have the $S\mathcal{I}H$ property.

Proof. An example can be obtained by a simple modification to [24, Example 2.3] and using Theorems 3.2 and 4.1. □

From Example 4.2, it can be noted that the Alexandroff duplicate $A(X)$ of a space X having the $S\mathcal{I}H$ property does not need to have the $S\mathcal{I}H$ property.

Theorem 4.3. *The star-$\mathcal{I}$-Hurewicz property is inverse invariant under perfect open mappings.*

Proof. Let f be a perfect open mapping from a space X to a space Y having the $S\mathcal{I}H$ property and let $< \mathcal{U}_n : n \in \omega >$ be a sequence of open covers of X. Then for each $y \in Y$ and each n, there is a finite subfamily $\mathcal{U}_{n_y}$ of $\mathcal{U}_n$ such that $f^{-1}(y) \subseteq \bigcup \mathcal{U}_{n_y}$. Let $U_{n_y} = \bigcup \mathcal{U}_{n_y}$. Then

$$V_{n_y} = Y \setminus f(X \setminus U_{n_y}) \cap \{f(U) : U \in \mathcal{U}_{n_y}\}$$

is an open neighborhood of y, since f is closed and open.

For each n, let $\mathcal{V}_n = \{V_{n_y} : y \in Y\}$. Then for each n, $\mathcal{V}_n$ is an open cover of Y. Thus $< \mathcal{V}_n : n \in \omega >$ is a sequence of open covers of Y. Since Y has the $S\mathcal{I}H$ property, there is a sequence $< \mathcal{V}'_n : n \in \omega >$ such that for each n, $\mathcal{V}'_n$ is a finite subset of $\mathcal{V}_n$ and for each $y \in Y$, $\{n \in \omega : y \notin St(\bigcup \mathcal{V}'_n, \mathcal{V}_n)\} \in \mathcal{I}$.

Then for each n, $\mathcal{V}'_n = \{V_{n_{y_i}} : i \leq n'\}$. For each n, let $\mathcal{U}'_n = \bigcup_{i \leq n'} \mathcal{U}_{n_{y_i}}$. Then $\mathcal{U}'_n$ is a finite subset of $\mathcal{U}_n$. Therefore the sequence $< \mathcal{U}'_n : n \in \omega >$ witnesses the star-$\mathcal{I}$-Hurewicz property of X for $< \mathcal{U}_n : n \in \omega >$.

For $x \in X$, $f(x) \in Y$. Then $\{n \in \omega : f(x) \notin St(\bigcup \mathcal{V}'_n, \mathcal{V}_n)\} \in \mathcal{I}$. This implies that $f(x) \in V_{n_y} \in \mathcal{V}_n$ and $V_{n_y} \cap (\bigcup \mathcal{V}'_n) \neq \emptyset$. Then for some $i \leq n'$, $V_{n_y} \cap V_{n_{y_i}} \neq \emptyset$. Since $V_{n_y} \subseteq f(U)$ for each $U \in \mathcal{U}_{n_y}$, $V_{n_{y_i}} \cap f(U) \neq \emptyset$ for some $i \leq n'$. This implies that $U \cap f^{-1}(V_{n_{y_i}}) \neq \emptyset$. Since $f^{-1}(V_{n_{y_i}}) \subseteq U_{n_{y_i}}$, $U \cap U_{ny_i} \neq \emptyset$. Hence for each $U \in \mathcal{U}_{n_y}$, $U \cap (\bigcup \mathcal{U}'_n) \neq \emptyset$. There is $U \in \mathcal{U}_n$ containing x such that $U \cap (\bigcup \mathcal{U}'_n) \neq \emptyset$. □

Corollary 4.4 ([5])**.** *If a space X has the star-$\mathcal{I}$-Hurewicz property and Y is compact, then $X \times Y$ has the star-$\mathcal{I}$-Hurewicz property.*

Theorem 4.5. *The $S\mathcal{I}H$ property is closed under countable unions.*

Proof. Let $\{X_k : k \in \omega\}$ be a family of subspaces having the $S\mathcal{I}H$ property in a space X and let $< \mathcal{U}_n : n \in \omega >$ be a sequence of open covers of X. For each $k \in \omega$, consider the sequence $< \mathcal{U}_n : n \geq k >$ and define $< \mathcal{U}^k_n : n \geq k >$, where $\mathcal{U}^k_n = \{U \cap X_k : U \in \mathcal{U}_n\}$. Then $< \mathcal{U}^k_n : n \geq k >$ is a sequence of open covers of X_k. For each $k \in \omega$, since X_k is star-$\mathcal{I}$-Hurewicz, there is a sequence $< \mathcal{V}^k_n : n \geq k >$ such that for each $n \geq k$, $\mathcal{V}^k_n$ is a finite subset of $\mathcal{U}^k_n$ and for each $x \in X_k$, $\{n \geq k : x \notin St(\bigcup \mathcal{V}_n^k, \mathcal{U}_n^k)\} \in \mathcal{I}$. For each n and for each k, let $\mathcal{V}_{n,k} = \{U : U \cap X_k \in \mathcal{V}_n^k\}$ and define $\mathcal{V}_n = \bigcup \{\mathcal{V}_{n,j} : j \leq n\}$. Then each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$. For each $x \in X$, $x \in \bigcup X_k$. There exists $k \in \omega$ such that $x \in X_k$. Thus, $\{n \in \omega : x \notin St(\mathcal{V}_n^k, \mathcal{U}_n^k)\} \in \mathcal{I}$. Then $St(\mathcal{V}_n^k, \mathcal{U}_n^k) \subseteq St(\mathcal{V}_n, \mathcal{U}_n)$ for all $n > k$, and hence $\{n \in \omega : x \notin St(\mathcal{V}_n, \mathcal{U}_n)\} \in \mathcal{I}$. □

Corollary 4.6. *If X has the star-$\mathcal{I}$-Hurewicz property and Y is σ-compact, then $X \times Y$ has the star-$\mathcal{I}$-Hurewicz property.*

Theorem 4.7. *The star-$\mathcal{I}$-Hurewicz property is inverse invariant under open bijective maps.*

Proof. Let f be an open bijective map from a space X to a space Y having the $\mathcal{SIH}$ property and let $< \mathcal{U}_n : n \in \omega >$ be a sequence of open covers of X. Then for each n, $\mathcal{V}_n = \{f(U) : U \in \mathcal{U}_n\}$ is an open cover of Y. Since Y has the $\mathcal{SIH}$ property, there is a sequence $< \mathcal{W}_n : n \in \omega >$ such that for each n, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and for each $y \in Y$, $\{n \in \omega : y \notin St(\bigcup \mathcal{W}_n, \mathcal{V}_n)\} \in \mathcal{I}$. Now assume that for each n, $\mathcal{W}_n = \{f(U_{n,1}), f(U_{n,2}), \ldots, f(U_{n,k})\}$ and $\mathcal{G}_n = \{U_{n,1}, U_{n,2}, \ldots, U_{n,k}\}$. Then the sequence $< \mathcal{G}_n : n \in \omega >$ witnesses the star-$\mathcal{I}$-Hurewicz property of X for the sequence $< \mathcal{U}_n : n \in \omega >$.

Let $x \in X$. Then $f(x) \in Y$ and $\{n \in \omega : f(x) \notin St(\bigcup \mathcal{W}_n, \mathcal{V}_n)\} \in \mathcal{I}$. This implies that $f(x) \in f(U) \in \mathcal{V}_n$ and $\bigcup \mathcal{W}_n \cap f(U) \neq \emptyset$. Then $f(U_{n,i}) \cap f(U) \neq \emptyset$ for some $i \leq k$. Since f is one-to-one, $U_{n,i} \cap U \neq \emptyset$ for some $i \leq k$ and $x \in U$. Hence $\{n \in \omega : x \notin St(\bigcup \mathcal{G}_n, \mathcal{U}_n)\} \in \mathcal{I}$. □

Now we consider the hereditary property of these spaces.

Definition 4.8 ([15])**.** A space X is said to have *star-Menger property* (in short *SM*) if for each sequence $< \mathcal{U}_n : n \in \omega >$ of open covers of X, there is a sequence $< \mathcal{V}_n : n \in \omega >$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.

Theorem 4.9. *A space having the $\mathcal{SIH}$ property has the SM property.*

Proof. Let X be a space having the $\mathcal{SIH}$ property and let $< \mathcal{U}_n : n \in \omega >$ be a sequence of open covers of X. By the $\mathcal{SIH}$ property of X, there is a sequence $< \mathcal{V}_n : n \in \omega >$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \omega : x \notin St(\bigcup \mathcal{V}_n, \mathcal{U}_n)\} \in \mathcal{I}$. Now we show that $X = \bigcup_{n \in \omega} St(\bigcup \mathcal{V}_n, \mathcal{U}_n)$. If possible suppose that there is $x \in X$ such that $x \notin \bigcup_{n \in \omega} St(\bigcup \mathcal{V}_n, \mathcal{U}_n)$. Then $x \notin St(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all $n \in \omega$, and hence $\{n \in \omega : x \notin St(\bigcup \mathcal{V}_n, \mathcal{U}_n)\} = \omega \in \mathcal{I}$, a contradiction. □

Example 4.10 ([3, 21])**.** Let $X = \omega \cup \mathcal{A}$ be the Isbell–Mrówka space constructed from $\mathcal{A}$. Then

(1) X has the strongly star-Hurewicz property if and only if $|\mathcal{A}| < \mathfrak{b}$;
(2) if $|\mathcal{A}| = \mathfrak{c}$, then X does not have the star-Menger property.

The proof of the following example is taken from the proof of [27, Example 2.2].

Example 4.11. There exists a Tychonoff pseudocompact, $\mathcal{SIH}$ space having a regular-closed subspace that is not $\mathcal{SIH}$.

Proof. An example can be obtained by a simple modification to [27, Example 2.2] and using Theorem 4.9 and Example 4.10. □

For the next example, we need the following lemma.

Lemma 4.12 ([5]). *A space X is $S\mathcal{I}H$ if and only if for every sequence $< \mathcal{U}_n : n \in \omega >$ of open covers of X, there exists finite $\mathcal{O}_n \subseteq \mathcal{U}_n$ for each n such that for every $x \in X$, $\{n \in \omega : St(x, \mathcal{U}_n) \cap (\bigcup \mathcal{O}_n) \neq \emptyset\} \in \mathcal{F}(\mathcal{I})$.*

The proof of the following example is similar to the proof of [27, Example 2.4].

Example 4.13. We assume $\omega_1 < \mathfrak{b}$. There exists a Tychonoff $SS\mathcal{I}H$ (hence $S\mathcal{I}H$) space having a regular-closed G_δ-subspace that is not $S\mathcal{I}H$ (hence not $SS\mathcal{I}H$).

Proof. An example can be obtained by a simple modification to [27, Example 2.4] and using Lemma 4.12 and Example 4.10. □

For Alexandroff spaces, we have the following result.

Theorem 4.14. *If X is a T_1-space and $A(X)$ has the $S\mathcal{I}H$ property, then $e(X) < \omega_1$.*

Proof. Suppose that $e(X) \geq \omega_1$. Then there exists a discrete closed subset B of X such that $|B| \geq \omega_1$. Hence $B \times \{1\}$ is an open and closed subset of $A(X)$ and each point $< b, 1 >$ is isolated for every $b \in B$. Thus $A(X)$ does not have the $S\mathcal{I}H$ property, since every open and closed subset of a space having the $S\mathcal{I}H$ property must have the $S\mathcal{I}H$ property and $B \times \{1\}$ does not have the $S\mathcal{I}H$ property. □

Theorem 4.15. *If X is a T_1-space having the $S\mathcal{I}H$ property and $e(X) < \omega_1$, then $A(X)$ has the SM property.*

Proof. Let $< \mathcal{U}_n : n \in \omega >$ be a sequence of open covers of $A(X)$. For each n and each $x \in X$, choose an open neighborhood $W_{n_x} = (V_{n_x} \times \{0, 1\}) \setminus \{< x, 1 >\}$ of $< x, 0 >$ such that there exists some $U \in \mathcal{U}_n$ with $W_{n_x} \subseteq U$, where V_{n_x} is an open subset of X containing x. For each n, let $\mathcal{V}_n = \{V_{n_x} : x \in X\}$. Then $\mathcal{V}_n$ is an open cover of X and $< \mathcal{V}_n : n \in \omega >$ is a sequence of open covers of X. By applying the $S\mathcal{I}H$ property of X, there is a sequence $< \mathcal{F}_n : n \in \omega >$ such that for each n, $\mathcal{F}_n$ is a finite subset of $\mathcal{V}_n$ and for each $x \in X$, $\{n \in \omega : x \notin St(\bigcup \mathcal{F}_n, \mathcal{V}_n)\} \in \mathcal{I}$. Then enumerate $\mathcal{F}_n = \{V_{x_1}, V_{x_2}, \ldots, V_{x_{k_n}}\}$. Then for each $(V_{x_i} \times \{0, 1\}) \setminus \{< x, 1 >\}$, there is a member $U_i \in \mathcal{U}_n$ such that $(V_{x_i} \times \{0, 1\}) \setminus \{< x, 1 >\} \subseteq U_i$. For each n, let $\mathcal{F}_n^{'} = \{U_i : i \leq k_n\}$. Then for each n, $\mathcal{F}_n^{'}$ is a finite subset of $\mathcal{U}_n$. Thus $< \mathcal{F}_n^{'} : n \in \omega >$ is a sequence of finite subsets of $< \mathcal{U}_n : n \in \omega >$ and for each $x \in X$, $\{n \in \omega :< x, 0 >\notin St(\bigcup \mathcal{F}_n^{'}, \mathcal{U}_n)\} \in \mathcal{I}$. Hence $X \times \{0\} \subseteq \bigcup_{n \in \omega} St(\bigcup \mathcal{F}_n^{'}, \mathcal{U}_n)$. Let $A = A(X) \setminus \bigcup_{n \in \omega} St(\bigcup \mathcal{F}_n^{'}, \mathcal{U}_n)$. Then A is a discrete closed subset of $A(X)$. By Lemma 3.5 and the given hypothesis, the set A is countable, and we can enumerate A as $\{a_n : n \in \omega\}$. For each n, let $\mathcal{F}_n^{''} = \mathcal{F}_n^{'} \cup \{U_{a_1}, U_{a_2}, \ldots, U_{a_n} : a_i \in U_{a_i} \in \mathcal{U}_n\}$. Then $\mathcal{F}_n^{''}$ is a finite subset of $\mathcal{U}_n$, and hence for each $y \in A(X)$, $y \in St(\bigcup \mathcal{F}_n^{''}, \mathcal{U}_n)$. Therefore $A(X)$ has the SM property. □

Acknowledgement : The authors thank Prof. Lj.D.R. Kočinac for his kind help during the preparation of this paper. The authors are thankful to referee for valuable comments and suggestions which have improved the presentation of the paper. The first and third authors acknowledge the fellowship grant of University Grant Commission, India.

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