MAPS STRONGLY PRESERVING THE SQUARE ZERO OF $\lambda$-LIE PRODUCT OF OPERATORS

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Communicated by A. Jiménez-Vargas

Abstract. Let $\mathcal{A}$ be a standard operator algebra on a Banach space $\mathcal{X}$ with $\dim \mathcal{X} \geq 2$. In this paper, we characterize the forms of additive maps on $\mathcal{A}$ that strongly preserve the square zero of $\lambda$-Lie product of operators. That is, if $\phi: \mathcal{A} \rightarrow \mathcal{A}$ is an additive map satisfying

$$[A, B]^2_{\lambda} = 0 \Rightarrow [\phi(A), B]^2_{\lambda} = 0,$$

for every $A, B \in \mathcal{A}$ and for a scalar number $\lambda$ with $\lambda \neq -1$, then it is shown that there exists a function $\sigma: \mathcal{A} \rightarrow \mathbb{C}$ such that $\phi(A) = \sigma(A)A$ for every $A \in \mathcal{A}$.

1. Introduction

In the last decade, many mathematicians have studied preserving problems. In particular, maps preserving a certain property of products of elements are considered; see [2–11]. We recall some of them which are related to our purpose.

Let $\mathcal{A}$ be a Banach algebra, let $A, B \in \mathcal{A}$, and let $\lambda$ be a scalar. Then $AB + \lambda BA$ is said to be the $\lambda$-Lie product of $A$ and $B$ and is denoted by $[A, B]_{\lambda}$. The $\lambda$-Lie product is said to be the Jordan product or the Lie product, whenever $\lambda = 1$ or $\lambda = -1$, respectively. The Lie product of $A$ and $B$ is denoted by $[A, B]$. The triple Jordan product of $A$ and $B$ is defined by $ABA$. These products play a rather important role in mathematical physics.

Taghavi et al. [10] considered the maps strongly preserving the $\eta$-Lie product on an algebra $\mathcal{A}$, that is a map $\phi: \mathcal{A} \rightarrow \mathcal{A}$ satisfying $\phi(A)\phi(P) + \eta\phi(P)\phi(A) = AP + \eta PA$, for every $A \in \mathcal{A}$, some idempotent $P \in \mathcal{A}$, and some scalar $\eta$.

Date: Received: 1 December 2019; Revised: 19 June 2020; Accepted: 21 June 2020.

2010 Mathematics Subject Classification. Primary 46J10; Secondary 47B48.

Key words and phrases. Preserver problem, Standard operator algebra, $\lambda$-Lie product, Lie product.
Let $\mathcal{B}(\mathcal{X})$ be the Banach algebra of all bounded linear operators on a Banach space $\mathcal{X}$. In [6], the authors characterized unital surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of product of operators, in both directions. Wang et al. [11] characterized linear surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of either products of operators or triple Jordan product of operators. Also Fang [5] characterized linear surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of Jordan product of operators.

We recall that a standard operator algebra $\mathcal{A}$ on a Banach space $\mathcal{X}$ is a norm closed subalgebra of $\mathcal{B}(\mathcal{X})$ that contains the identity and all finite rank operators.

We say that a map $\phi : \mathcal{A} \to \mathcal{A}$ strongly preserves the square zero of $\lambda$-Lie product of operators, whenever

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0$$

for every $A, B \in \mathcal{A}$.

In this paper, we characterize the forms of additive maps that strongly preserve the square zero of $\lambda$-Lie products of operators. Our main result is the following theorem.

**Theorem 1.1.** Assume that $\mathcal{A}$ is a standard operator algebra on a Banach space $\mathcal{X}$ with $\dim \mathcal{X} \geq 2$. Let $\phi : \mathcal{A} \to \mathcal{A}$ be an additive map that satisfies

$$[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,$$

for every $A, B \in \mathcal{A}$ and for a scalar $\lambda$ with $\lambda \neq -1$. Then there exists a function $\sigma : \mathcal{A} \to \mathbb{C}$ such that $\phi(A) = \sigma(A)A$ for every $A \in \mathcal{A}$.

2. Proof of main result

First we recall some notations. We assume that $\mathcal{X}$ is a Banach space and $\mathcal{A}$ is a standard operator algebra on $\mathcal{X}$. We denote by $\mathcal{X}^*$, the dual space of $\mathcal{X}$. For every nonzero $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, the symbol $x \otimes f$ stands for the rank one linear operator on $\mathcal{X}$ defined by $(x \otimes f)y = f(y)x$ for any $y \in \mathcal{X}$. Note that every rank one operator in $\mathcal{B}(\mathcal{X})$ can be written in this way. We denote by $F_1(\mathcal{X})$ the set of all rank one operators in $\mathcal{B}(\mathcal{X})$. The rank one operator $x \otimes f$ is idempotent if and only if $f(x) = 1$ and is nilpotent if and only if $f(x) = 0$.

**Proposition 2.1.** Let $A \in \mathcal{A}$, let $x \in \mathcal{X}$, let $f \in \mathcal{X}^*$ such that $f(x) \neq 0$, and let $\lambda \neq 0, -1$. Then $[A, x \otimes f]_\lambda^2 = 0$ if and only if one of the following statements occurs:

(i) $Axf(Ax) = -\lambda x f(A^2 x)$ and $Axf(x) = -\lambda x f(Ax)$.

(ii) $fA = 0$.

**Proof.** First assume that $Axf(Ax) = -\lambda x f(A^2 x)$ and $Axf(x) = -\lambda x f(Ax)$ hold. Hence

$$[A, x \otimes f]_\lambda^2 = (Ax \otimes f + \lambda x \otimes fA)^2$$

$$= f(Ax)Ax \otimes f + \lambda f(x)Ax \otimes fA + \lambda^2 f(Ax)x \otimes fA + \lambda f(A^2 x)x \otimes f$$

$$= -\lambda x f(A^2 x) \otimes f - \lambda^2 x f(Ax) \otimes fA + \lambda^2 f(Ax)x \otimes fA + \lambda f(A^2 x)x \otimes f$$

$$= 0.$$
Now if \( fA = 0 \), then
\[
[A, x \otimes f]_\lambda^2 = (Ax \otimes f + \lambda x \otimes fA)^2 \\
= (Ax \otimes f)^2 = f(Ax)Ax \otimes f = 0.
\]

Conversely, assume that \( [A, x \otimes f]_\lambda^2 = 0 \). For an operator \( B \), it is clear that
\[
B^2 = 0 \iff (B(Bx) = 0, \text{ for all } x \in \mathcal{X}) \iff \text{Im}B \subseteq \ker B.
\]
This together with the assumptions implies
\[
[A, x \otimes f]_\lambda^2 = 0 \iff \text{Im}[A, x \otimes f]_\lambda \subseteq \ker[A, x \otimes f]_\lambda.
\]
Let \( fA \neq 0 \). If \( fA \) and \( f \) are linearly independent, then \( \text{Im}[A, x \otimes f]_\lambda = \text{span}\{Ax, x\} \), and so
\[
\text{span}\{Ax, x\} \subseteq \ker(Ax \otimes f + \lambda x \otimes fA),
\]
which implies
\[
(Ax \otimes f + \lambda x \otimes fA)(Ax) = Ax f(Ax) + \lambda xf(A^2x) = 0,
\]
\[
(Ax \otimes f + \lambda x \otimes fA)(x) = Ax f(x) + \lambda xf(Ax) = 0,
\]
which are the asserted relations. If \( fA \) and \( f \) are linearly dependent, then there exists a nonzero scalar \( a \) such that \( fA = af \), and so
\[
[A, x \otimes f]_\lambda = Ax \otimes f + \lambda x \otimes fA = (Ax + ax) \otimes f.
\]
Thus \( \text{Im}[A, x \otimes f]_\lambda = \text{span}\{Ax + \lambda ax\} \), and so
\[
\text{span}\{Ax + \lambda ax\} \subseteq \ker(Ax \otimes f + \lambda x \otimes fA) = \ker((Ax + ax) \otimes f),
\]
which implies
\[
((Ax + \lambda ax) \otimes f)(Ax + \lambda ax) = 0
\]
\[
\Rightarrow (Ax + \lambda ax)[f(Ax) + \lambda af(x)] = 0
\]
\[
\Rightarrow (Ax + \lambda ax)af(x)(1 + \lambda) = 0.
\]
Since \( f(x) \neq 0 \) and \( \lambda \neq -1 \), we obtain \( Ax + \lambda ax = 0 \). This together with \( fA = af \)
implies
\[
Ax f(Ax) = -\lambda axf(Ax) = -\lambda f(Af)(Ax) = -\lambda xf(A^2x)
\]
and
\[
Ax f(x) = -\lambda ax f(x) = -\lambda xf(Ax),
\]
and these complete the proof.

□

In the following lemmas, assume that \( \phi : A \rightarrow A \) is a map that satisfies
\[
[A, B]_\lambda^2 = 0 \Rightarrow [\phi(A), B]_\lambda^2 = 0,
\]
for every \( A, B \in A \) and for a scalar number \( \lambda \) with \( \lambda \neq 0, -1 \).

**Lemma 2.2.** For every \( A \in A \), \( \text{ker} A \subseteq \text{ker} \phi(A) \).
Proof. If \( x \in \ker A \), then
\[
[A, x \otimes f]^2 = (Ax \otimes f + \lambda x \otimes f A)^2
= (\lambda x \otimes f A)^2 = \lambda^2 f(Ax) x \otimes f A = 0,
\]
for every \( f \in \mathcal{X}^* \), and so \([\phi(A), x \otimes f]^2 = 0\). Applying Proposition 2.1, we have
\[
\phi(A) x f(\phi(A)) = -\lambda x f(\phi(A)^2) x)
\]
and
\[
\phi(A) x f(x) = -\lambda x f(\phi(A)) x)
\]
or \( f\phi(A) = 0 \), for every \( f \in \mathcal{X}^* \) such that \( f(x) \neq 0 \). We show \( \phi(A)x = 0 \).
First let relations (2.1) and (2.2) hold and let \( f(x) = 1 \). From (2.2), we obtain
\[
\phi(A)x = -\lambda x f(\phi(A)) x)
\]
and since \( f\phi(A) = 0 \), then \( f(\phi(A))x = 0 \). Therefore, \( \phi(A)x = 0 \).

Next assume that \( \phi \) is additive.

**Lemma 2.3.** For every rank one operator \( A \), \( \phi(A) = 0 \) or \( \phi(A) = \kappa(A)A \), where \( \kappa : \mathcal{A} \to \mathbb{C} \) is a function. 

**Proof.** Let \( A = x \otimes f \), for some \( x \in \mathcal{X} \) and \( f \in \mathcal{X}^* \). From Lemma 2.2, we have
\[
\ker x \otimes f \subseteq \ker \phi(x \otimes f),
\]
which implies that \( \ker f \subseteq \ker \phi(x \otimes f) \) and since \( \ker f \) is a hyperspace of \( \mathcal{X} \), \( \ker \phi(x \otimes f) = \mathcal{X} \) or \( \ker \phi(x \otimes f) = \ker f \). Therefore \( \phi(x \otimes f) \) is a zero operator or there exists a vector \( y \) such that \( \phi(x \otimes f) = y \otimes f \). We divide the rest of the proof into two cases:

**Case 1.** Let \( f(x) \neq 0 \) and let \( g \) be a functional such that \( g(x) = 0 \). We have
\[
[x \otimes f, x \otimes g]^2 = [f(x)x \otimes g + \lambda g(x)x \otimes f]^2
= [f(x)x \otimes g]^2 = 0
\]
and then
\[
[\phi(x \otimes f), x \otimes g]^2 = [y \otimes f, x \otimes g]^2 = 0,
\]
which implies
\[
[f(x)y \otimes g + \lambda g(y)x \otimes f]^2 = f(x)y \otimes g + \lambda^2 g(y)f(x)x \otimes f
+ \lambda g(y)f(x)f(y)x \otimes g = 0.
\]

Since \( f(x) \neq 0 \), we obtain
\[
y \otimes g(y)g = x \otimes (-\lambda^2 g(y)f - \lambda g(y)f(g(y)).
\]
This implies that \( x \) and \( y \) are linearly dependent or \( g(y) = 0 \). If \( g(y) = 0 \) and \( x \) and \( y \) are linearly independent, we get a contradiction, since in this case by \( \dim \mathcal{X} \geq 2 \), there exists a functional \( g \) such that \( g(x) = 0 \) but \( g(y) = 1 \).
Therefore $x$ and $y$ are linearly dependent and then there is a scalar $\kappa(A)$ such that $\phi(A) = \kappa(A)A$.

Case 2. Let $f(x) = 0$. There exists a linear functional $h$ such that $h(x) = 1$ and then by Case 1, we have

$$\phi(x \otimes (f + h)) = kx \otimes (f + h),$$

where $k = \kappa(x \otimes (f + h))$. On the other hand, the additivity of $\phi$ together with Case 1 implies

$$\phi(x \otimes (f + h)) = \phi(x \otimes f) + \phi(x \otimes h) = \phi(x \otimes f) + tx \otimes h,$$

where $t = \kappa(x \otimes h)$. Thus

$$\phi(x \otimes f) = kx \otimes (f + h) - tx \otimes h = x \otimes (kf + kh - th).$$

This together with $\phi(x \otimes f) = y \otimes f$ implies that $x$ and $y$ are linearly dependent and this completes the proof.

\textbf{Proof of Theorem 1.1.} We divide the proof into two cases.

Case 1. Let $\lambda = 0$. First we show $\ker A \subseteq \ker \phi(A)$ for every $A \in \mathcal{A}$. Assume $Ax = 0$. This together with the assumption yields $(\phi(A)x \otimes f)^2 = 0$ for every $f \in \mathcal{X}^*$. Thus $\phi(A)x = 0$ or $f(\phi(A)x) = 0$, for every $f \in \mathcal{X}^*$. Since $f$ is arbitrary, in the second case, we obtain $\phi(A)x = 0$, too. Thus by the first paragraph of the proof of Lemma 2.3, for every $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, we have $\phi(x \otimes f) = 0$ or there exists a vector $y$ such that $\phi(x \otimes f) = y \otimes f$. If $f(x) \neq 0$, then $[(x \otimes f)(x \otimes g)]^2 = 0$, for every functional $g$ such that $g(x) = 0$. This implies that

$$[(\phi(x \otimes f))(x \otimes g)]^2 = 0$$

$$\Rightarrow [(y \otimes f)(x \otimes g)]^2 = 0$$

$$\Rightarrow (f(x)y \otimes g)^2 = 0 \Rightarrow g(y) = 0.$$

Hence $x$ and $y$ are linearly dependent. If $f(x) = 0$, then by Case 2 in the proof of Lemma 2.3, we obtain that $x$ and $y$ are linearly dependent, too. Therefore, $\phi(x \otimes f) = 0$ or $\phi(x \otimes f) = kx \otimes f$ for some scalar $k$.

Let $A \in \mathcal{A} \setminus \mathcal{F}_1(\mathcal{X})$ and let $x \in \mathcal{X}$. We know $(Ax \otimes f)^2 = 0$, for every $f \in \mathcal{X}^*$ with $f(Ax) = 0$. Thus $(\phi(A)x \otimes f)^2 = 0$ and then $\phi(A)x = 0$ or $f(\phi(A)x) = 0$, which implies that $Ax$ and $\phi(A)x$ are linearly dependent for every $x \in \mathcal{X}$. Hence by [1, Theorem 2.3], there exists a scalar number $k$ such that $\phi(A) = kA$. This together with the previous discussion implies that there exists a function $\sigma : \mathcal{A} \to \mathbb{C}$ such that $\phi(A) = \sigma(A)A$ for every $A \in \mathcal{A}$.

Case 2. Let $\lambda \neq 0$. Let $A \in \mathcal{A} \setminus \mathcal{F}_1(\mathcal{X})$ and let $x \in \mathcal{X}$. There exists a linear functional $f$ such that $f(x) = 1$. Set $P = Ax \otimes f$. It is clear that $(A - P)x = 0$, and so Lemma 2.2 implies

$$(\phi(A) - \phi(P))x = 0 \Rightarrow \phi(A)x = \phi(P)x.$$

By Lemma 2.3, we have $\phi(P) = 0$ or $\phi(P) = \kappa(P)P$. If $\phi(P) = 0$, then $\phi(A)x = 0$. In the second case, if $\phi(P) = \kappa(P)P$, then $\phi(A)x = \kappa(P)Px = \kappa(P)Ax$. However, in both cases, $\phi(A)x$ and $Ax$ are linearly dependent, for every $x \in \mathcal{X}$, and so by [1, Theorem 2.3], there exists a scalar number $k$ such that $\phi(A) = kA$. This
together with Lemma 2.3 follows that there exists a function $\sigma : \mathcal{A} \rightarrow \mathbb{C}$ such that $\phi(A) = \sigma(A)A$ for every $A \in \mathcal{A}$. \hfill \Box

**Acknowledgement.** The author is thankful to the referee for the careful reading of the paper and for the valuable comments and suggestions.

**References**


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